AN

ELEMENTARY TREATISE

ON

ALGEBRA

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PART I

BY

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PREFACE.

The present Treatise on Algebra is in many respects a new ɔook of its kind. It contains many theorems which, though ıseful to junior students, are not generally found in the ext books in common use. My object has been from the ɔeginning to draw the attention of the student to a theoretical study of the subject, which should properly be the aim of every elementary book. Hence many important suggestions ıave been made in the shape of Remarks, Foot-notes, etc. n the treatment of the whole subject, I have followed what o me seemed to be the most approved method of writing ext-books, *viz*, to illustrate every article by examples, to give others for exercise, and where possible, to direct the student o work out the same example by different methods. At the end of each Chapter, Questions for examination have been given which, while testing the knowledge of the student in he portion he has gone through, will naturally lead him to ubjects treated later on.

The four fundamental operations of Addition, Subtraction, Multiplication and Division have been placed consecutively in order to enable the student to have a fair acquaintance with he first four rules of Algebra, before beginning the more omplex Chapters on Formulæ and Factors. The treatment of his portion has moreover been very copious, as regards both he theory and the practice of the subject, while such matter s presents difficulty to the beginner has been carefully xcluded. It is hoped therefore that the book in its earlier ortions will be found to serve pre-eminently as a beginner's and-book.

The Chapter on Factors, justly considered as of the highest importance to beginners, has been greatly developed The tentative process of factorisation, itself of great practical value, has been supplemented, in the case of expressions of the second degree, by the general method adapted from the Theory of Quadratics The resolution into factors of some special expressions of higher degree and of homogeneous symmetrical expressions by *the help of the* Remainder Theorem, has been given in Chapter XXII It is thus believed that as far as the range of this treatise permits, the theory of factorisation has been presented in a tolerably complete form

I have introduced, after Division, a section on Simple Equations, with a view to make Algebra interesting to the student at an early stage of his progress This appears to me, moreover, to be the most natural order, for without some knowledge of Equations, the theory of Divisors and Multiples and of Fractions cannot be properly understood Hence Simple Equations have been treated in two sections, in the second of which, inserted after Fractions, the theory of the subject has been more fully dwelt upon.

In the second section on Equational Problems, instead of merely giving illustrative examples, as is ordinarily done, I have tried to explain the general principles on which the solution of particular kinds of problems depends [See §§ 231-235 inclusive]

In Chapter XXII, many useful theorems with their application, together with some other interesting matters have been given, which, it is hoped, will make the present treatise interesting to the student of Algebra

The examples are numerous, many of which are original while the rest have been taken from Cambridge and other

Examination Papers. The book thus contains many new examples, besides some old ones which have been inserted on account of their special interest

In the Appendix, solutions of a number of examples, chiefly selected from the body of the work, have been given, in illustration of what is commonly called algebraical *'artifices.'* The Appendix also contains a set of Miscellaneous Examples for exercise, arranged in the form of Papers.

At the end, Examination Papers of the Indian Universities, together with an Addendum containing solutions of difficult questions set at the several Universities, have been given.

It is thus hoped that the present edition will be of still greater service to students than its predecessors, especially, to those who begin the subject for the first time.

Any suggestions towards improvement from gentlemen engaged in the cause of education, if kindly communicated, will be most thankfully received.

CALCUTTA, *December 21, 1894.*

PREFACE TO THE THIRTEENTH EDITION.

In this edition the book has been thoroughly revised and much useful matter has been added, especially in the Chapters on *Formulæ* and *Factors.* It is, therefore, believed that the work in its present form will be found particularly interesting to the beginner

S. C. BASU.

CALCUTTA, *December 19, 1903.*

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[Portions marked with an asterisk may be omitted on a first reading]

INTRODUCTION

DEFINITIONS

1 **Origin of the Science** It is now generally admitted that the Hindus were the inventors of the Science of Algebra It was introduced into Europe about the beginning of the thirteenth century, from the works of the Arabic writers "who certainly derived their knowledge from the Hindus" Hence the name Algebra "is the European corruption of the first words of an Arabic phrase, which may be thus written, *al jebr e al mokabalah* meaning *restoration and reduction*"

2. **Quantity, unit, measure** In Algebra, **Quantity** denotes a magnitude that can be *exactly measured* by a *smaller magnitude* of its *own kind*. This smaller magnitude is called its **unit**, and is represented by the figure 1 (read *one* or *unity*) Thus the quantity "five rupees" contains *five times* its unit *one rupee*; "eight yards" means *eight times* its unit *one yard*, "thirty-six square feet" denotes an area *thirty-six times* as much as the unit *one square foot*; and so on

It thus appears that every quantity is *measured* by the *number of times* it contains its unit. This *number*, which indicates *how often* a quantity contains its unit, is called the **measure** of that quantity, thus in the above examples 5 is the measure of the sum of money viz "5 rupees"; 8 is the measure of the length viz "8 yards"; and so on Therefore the number which is the measure of a quantity represents that quantity. In Algebra we are concerned with only these numbers and therefore the letters which are symbols of quantities always represent numbers whether *whole* or *fractional* [see § 29] A whole number is called an **Integer**. the corresponding adjective is *integral*.

Sometimes the word quantity is used to denote an *Expression* [§ 16]

3 **Definition** ALGEBRA is the science that treats of numbers.

In Arithmetic (which also treats of numbers), numbers are represented by figures having *definite values*, but in Algebra numbers are denoted by letters, each of which can stand for *any number* whatever Hence, conclusions in Algebra are *general*, but those in Arithmetic are *particular* Algebra may, therefore, be termed general or universal Arithmetic *

* Sir ISAAC NEWTON calls it Universal Arithmetic

4 Symbols The letters used in Algebra, to denote numbers, are usually called SYMBOLS Generally the small letters of the alphabet are employed, as $a, b, c, \ldots m, n, \ldots x, y, z$ Sometimes capital letters and letters of the Greek alphabet are used, as $A, B, C, \ldots \alpha, \beta, \gamma \ldots$

5 Letters with accents and suffixes To preserve symmetry of operations, it is often found convenient to use the same letters with *accents* or *suffixes*, as

$a', b'', c''' \ldots$ (read *a dash, b two-dash, c three dash*, ..),

$a_1, b_2, c_3 \ldots$ (read *a one, b two, c three*,...)

Note It should be carefully remembered that $a', b'', c''', \ldots$ or $a_1, b_2, c_3, \ldots$ have no necessary connection (as to value) with $a, b, c, \ldots$

6 Knowns or Constants Quantities which have determinate values are called Known or Constant Quantities, or simply CONSTANTS They are generally represented by the *first letters* of the alphabet, as $a, b, c, d, \ldots$

7 Unknowns or Variables Quantities whose values have to be determined, are called Unknown Quantities or UNKNOWNS We shall also call them Variable Quantities or simply VARIABLES They are usually represented by the *last letters* of the alphabet, as $x, y, z, u, \ldots$

8 Sign + This sign, which is read *plus*, signifies that the quantity before which it stands, is to be *added* Thus $a+b$ means that the quantity b is to be added to the quantity a if a represent 10 and b represent 3, then $a+b$ represents $10+3$ or 13 Similarly $a+m+z$ signifies that z is to be added to the sum of a and m, *i e*, m is to be added to a, and then to the sum z is to be added, if a stand for 5, m for 8, and z for 1, then $a+m+z$ stands for $5+8+1$ or 14

9 Sign − This sign, which is read *minus*, implies that the quantity before which it stands is to be *subtracted* Thus $a-b$ signifies that the quantity b is to be subtracted from the quantity a if a represent 6 and b represent 4, $a-b$ represents $6-4$ or 2 Similarly $a-m-z$ signifies that the quantity m is to be subtracted from a and then z is to be subtracted from the result if a stand for 8, m for 4 and z for 3, then $a-m-z$ stands for $8-4-3$ or 1

So also $a-p+q$ signifies that p is to be first subtracted from a, and then q added to the result and $a+p-q$ signifies that p is to be added to a, and then q subtracted from the result

Note From §§ 8 and 9, it is seen that *in Additions and Subtractions, the order of the operations is from left to right*

10. Sign $\sim$. This sign, placed between two quantities, signifies that their *difference* is to be taken. Thus $x \sim y$ signifies either $x-y$ or $y-x$ according as x or y is the greater. If x represent 2 and y represent 5, $x \sim y$ represents 3. This sign is used when we *do not know* which of the two quantities is the *greater*.

11. Sign $\pm$. This sign, which is read *plus or minus*, signifies that the quantity before which it stands, is to be *added* or *subtracted*. Thus $a \pm b$ signifies that b is to be added to or subtracted from, a. If a stand for 8 and b for 3, $a \pm b$ stands for 8 ± 3, *i.e.*, for 11 or 5.

12. Sign $\times$. This sign, which is read *into*, signifies that of the two quantities between which it is placed, the first is to be *multiplied* by the second. Thus $a \times b$ means that the quantity a is to be multiplied by the quantity b. If a represent 8 and b represent 4, $a \times b$ represents 8×4 or 32. Similarly $a \times b \times c$ signifies that a is to be multiplied by b and then the result is to be multiplied by c.

In Algebra, this sign is *often* omitted between a figure and a letter, or between two letters. Thus $5 \times a$ is the same as $5a$; $a \times b$ is the same as ab; $3 \times x \times y \times z$ is $3xyz$.

Sometimes a *point* is used instead of this sign, thus $a \times b$ is represented by $a \cdot b$, $1 \cdot 2$ and 1×2 signify the same thing. [See § 18, Note 2.]

The sign $\times$, or its equivalent *point*, cannot be omitted between *numeral* figures, for in Arithmetic 23 means, not 2×3, but $20+3$. If, however, it is meant that 2 is to be multiplied by 3, then it must be written either 2×3 or $2 \cdot 3$.

Remark. To distinguish a *decimal point* from a *point* as a sign of multiplication, the former is placed *higher up*. Thus $2 \cdot 3$ means $2\frac{3}{10}$ and $2 . 3$ means 2×3 or 6.

13. Sign $\div$. This sign, which is read *divided by* or *by*, signifies that of the two quantities between which it is placed, the former is to be *divided* by the latter. Thus $a \div b$ signifies that a is to be divided by b; if a stand for 84 and b for 7, $a \div b$ stands for $84 \div 7$ or 12. Similarly $a \div b \div c$ means that a is to be divided by b, and then the result is to be divided by c.

So also $a \div b \times c$ means that a is to be divided by b and then the result multiplied by c; and $a \times b \div c$ means that a is to be multiplied by b, and then the result divided by c.

Note. From §§ 12 and 13, it is seen that *in Multiplications and Divisions, the order of the operations is from left to right.*

14. Signs of Operation. The four signs $+$, $-$, $\times$ and $\div$ are called the *signs of operation*.

15 Sign = This sign, which is read *is equal to* or *equals*, signifies that the quantities between which it is placed, are *equal*. Thus $x=a$ signifies that x *is equal to* a

16 Expression, Term Any collection of algebraic symbols connected by the signs of operation, is called an ALGEBRAICAL EXPRESSION or simply an EXPRESSION Thus $a+b-m+fd$ is an expression in which a, b, m, f, d are symbols An expression is sometimes called a QUANTITY [See § 2]

The parts of an expression, connected by the signs + or −, are called its TERMS Thus the expression $3a^2+b+1-4cx^3$ contains 4 *terms* $3a^2$, b, 1 and $4cx^3$, the expression $a-b\times c-2d+1-c\times 5f$ contains 3 terms a, $b\times c-2d$ and $1-c\times 5f$, and so on

17 Particular Names of Expressions When an expression consists of *one* term, it is called a Simple Expression, or a **Monomial**, as a, $-5b$, $3px^4$, &c When an expression consists of *two* terms, it is called a Binomial Expression, or briefly a **Binomial**, as $a+b$, $ax-by$ &c When an expression consists of *three* terms, it is called a Trinomial Expression, or briefly a **Trinomial**, as $a+b+c$, $ax+by-3z$, &c When an expression consists of *several* terms, it is called a Multinomial or a Polynomial Expression, or briefly a **Polynomial**, as $a+b+c+d+e+f$, $ax^4+bx^3+cx^2+dx+e$, &c A Polynomial is sometimes termed *Compound Quantity* or *Compound Expression*

18 Numerical Value We have seen [§ 3] that symbols in Algebra may stand for *any number whatever* An expression therefore will have a *numerical value* when the value of each of its symbols is given in *numbers* To find this value, we *substitute* the given numbers for the letters and proceed as in Arithmetic.

Examples (1)

Ex 1. If $x=8$, $y=5$ and $z=3$, find the numerical value of $x+y+z$

$x+y+z=8+5+3=16$

Ex 2 If $a=4$, $b=9$, find the numerical values of $7a$ and ab

$7a=7\times 4=28$,

$ab=4\times 9=36$

Ex 3 If $a=25$, $b=10$, find the numerical value of $a+2b$

$a+2b=25+2\times 10=25+20=45$

Ex 4 If $x=5$, $y=11$, find the numerical value of $3x+y-10$

$3x+y-10=3\times 5+11-10=15+11-10=16$

Ex 5 If $a=25$, $b=10$, find the numerical value of $1+a-b+ab$.

$1+a-b+ab=1+25-10+25\times 10=16+250=266$

Ex. 6 If $a=13\frac{1}{2}$, $x=18$, $y=3$, find the numerical value of $a-\frac{x}{y}$.

$$a-\frac{x}{y}=13\frac{1}{2}-\frac{18}{3}=13\frac{1}{2}-6=7\frac{1}{2}.$$

Ex. 7 If $a=25$, $x=\frac{4}{5}$, find the numerical value of $\frac{5}{a}+x$.

$$\frac{5}{a}+x=\frac{5}{25}+\frac{4}{5}=\frac{1}{5}+\frac{4}{5}=\frac{1+4}{5}=\frac{5}{5}=1.$$

If $a=6$, $b=5$, $c=4$, $d=3$, $e=2$, $f=1$, find the numerical value of

8. $2a+3$. 9. $5c+d$. 10. $b+c+d$. 11. $3a+b-1$.

12. $a-c$. 13. $14f-2a$. 14. $5c-d+2f$. 15. $ab-7c$.

16. $6ae-cd-30f$. 17. $12b-2c-3d$. 18. $3a-2b-5f+6e$.

19. $10c+10d-5e-5f$. 20. $6abc-3bcd-6ef$.

21. $9abf+3aef-2cd$. 22. $abc+bcd+cde+def$.

23. $\frac{b}{a}+\frac{a}{c}$. 24. $\frac{2}{a}+\frac{1}{d}-\frac{1}{2f}$. 25. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$.

If $a=\frac{1}{2}$, $b=\frac{1}{4}$, $c=\frac{3}{5}$, find the numerical value of

26. $5a+2b$. 27. $3c-\frac{2}{5}a$. 28. $\frac{a}{b}+\frac{b}{c}-\frac{1}{3}b$. 29. $4ab-\frac{1}{3}c+2b$.

Note 1 The following properties of 0 should be carefully remembered — (i) *When any one of the several numbers multiplied together is 0, the result is 0*; (ii) *When 0 is divided by any number, the result is 0*; (iii) *The addition or subtraction of 0 cannot affect the value of a Quantity.*

Examples (ii)

If $a=3$, $b=7$, $c=2$, $d=0$, find the numerical value of

Ex 1. $2a+b+d$ and $2a+b-d$.

The addition or subtraction of d which $=0$, does not affect the value, therefore each of the quantities

$$=2a+b=2\times3+7=6+7=13$$

Ex. 2. $6abc$; $12bad$; $3bcd$, $18acd$

$$6abc=6\times3\times7\times2=252$$

Each of the others is 0, for d which $=0$, occurs in each of them.

If $a=3$ $b=7$, $c=2$, $d=0$, find the numerical value of

Ex 3 $5b+2d-17c$

$$5b+2d-17c=5\times7+2\times0-17\times2=35+0-34=1$$

Ex. 4 $\dfrac{ad}{bc}+\dfrac{5bc}{14}-\dfrac{ac}{3b}$

$$\frac{ad}{bc}+\frac{5bc}{14}-\frac{ac}{3b}=\frac{0}{7\times2}+\frac{5\times7\times2}{14}-\frac{3\times2}{3\times7}=0+5-\frac{2}{7}=5-\frac{2}{7}=4\tfrac{5}{7}$$

5 $12bc$ 6 $18ac$ 7 $13ad$ 8 $15abd$ 9 $2abc$

10 $\dfrac{a+d}{b+c}$ 11. $\dfrac{15c-3b}{4d+9c}$ 12 $\dfrac{2b+4c}{3b+a-2}$

If $a=1$, $b=7$, $c=0$, $x=10$, $y=6$, $z=5$, find the value of

13 $8ab$ 14 $16bx$ 15 $17cy$ 16 $\frac{3}{4}ay$ 17 $\frac{1}{10}xyz$

18 $\frac{3}{10}acx$ 19 $\frac{1}{35}bxz$ 20 $\frac{16}{105}abxy$ 21 $3ax-z$

22 $7xy-ab$ 23 $56ay-3bxz$ 24 $\frac{5}{9}abc-3xy$

25 $\dfrac{12xz}{21}-\dfrac{2by}{3x}$

If $a=3$, $b=7$, $c=2$, $d=0$, find the numerical value of

26 $2a-8d$ 27 $13b+10d-a$ 28 $18c+a+5d$

29 $3ab-2cd+5b$ 30 $4abd+9bc-2abc$ 31 $\dfrac{5d}{2a}$, $\dfrac{cd}{ab}$, $\dfrac{2bd}{5ac}$

32 $\dfrac{2c}{3a}-\dfrac{5d}{c}+\dfrac{1}{2}$ 33 $\dfrac{6a-2b+4d}{5a+3d-2c}$ 34 $\dfrac{2ab+bc-5cd}{3ac-4ab+6bc}$

Note 2 The preceding examples are simple, but there, as elsewhere, the *order of the operations* is from left to right [§§ 9, 13, Notes] Hence to find the *numerical value* of an expression, we find (1) the value of *each term* by proceeding from left to right, and then (2) the value of the *whole expression* by proceeding also from left to right Thus to find the value of

$$15\times8-12+36-4\times14-6-72\times5-8-3$$

The expression has 3 terms, viz

$$15\times8-12,\quad 36-4\times14-6,\quad \text{and } 72\times5-8-3$$

First term $=15\times8-12=120-12=10$,

second term $=36-4\times14-6=9\times14-6=126-6=21$,

third term $=72\times5-8-3=360-8-3=45-3=15$

Value reqd $=10+21-15=31-15=16$

REMARK The case in which the sign of multiplication is omitted between two or more quantities deserves *special notice* Here the omission

of the sign and the consequent closeness of the quantities cause the result to be regarded as a single quantity. Thus $a \div b \times c$ means that a is to be *first* divided by b and then the result is to be multiplied by c [§ 13], but $a \div bc$ means that b and c are to be multiplied *first* and then a is to be divided by the result. Hence if $a=36$, $b=3$ and $c=4$, $a \div b \times c = 36 \div 3 \times 4 = 12 \times 4 = 48$, but $a \div bc = 36 \div 12 = 3$. Similarly $a \div m \times n \times p$ and $a \div mnp$ give different results; and so on.

Examples (iii).

Find the value of

1. $20 \div 4 \times 3$. 2. $35 \div 7 \div 5$. 3. $27 \div 6 \times 4$. 4. $18 \div 8 \div 4$.

5. $5 \times 3 \div 15 \div 4 + 6 \div 2 \times 5 \div 3$.

6. $2 \times 6 \div 4 + 9 \div 2 \times 5 \div 4 - 15 \div 3 \times 2 \div 5$.

If $a=12$, $b=2$, $c=3$, $d=1$, find the numerical value of

7. $a \div b \div c$. 8. $c \times b \div c$. 9. $ab \div c$. 10. $a \div c \times d$.

11. $a \div cd$. 12. $a + d \div b$. 13. $a \div b \times c$. 14. $a \div b \div c \times d$.

15. $4a \div b \div cd$. 16. $c \div b + d \div b - 2c$. 17. $a \times b \div c + a \div b \times c$.

18. $a \div 3b + a \div 3 \times b + 1$. 19. $2b \times c \div 3d \times 4c - a$.

20. $3a \div 2b \div c \times 5d \div 5 \times d$. 21. $a \times b \div c + a \div b \times c - a \div bc$.

22. $6ab \times 4b \div c \div ad + 5ad \div bc - 3ac \div 2b \div 36 \times 5d$.

23. $8a \div 3b \times 5c \div 6d \div 2 - 12a \div 8b \div 32c \times 16c - 9d + a \div 2bcd$.

Examples (iv).

If $a=4$, $b=8$, $p=10\frac{1}{2}$, $q=\frac{5}{8}$, $r=1\frac{3}{4}$, find the value of

1. $10 + 2q - b$. 2. $3p - a - 8q$. 3. $\frac{b}{a} - \frac{q}{p} + 1$.

4. $a + \frac{b}{p} - \frac{1}{q}$. 5. $15a + 10b - 2p + 4q$. 6. $\frac{a}{b} - \frac{b}{c} + \frac{82}{p} - q$.

7. $\frac{2a}{3b} + \frac{2p}{b} - \frac{13\frac{3}{5} \times aq}{5p} - \frac{r}{12a}$.

If $r = \cdot02$, $s = 3\cdot5$, $t = \cdot005$, find the value of

8. $5r + 3s - 10t$. 9. $r - 15t + 2s$. 10. $4s - 3r + t - r$.

---

19. Product, Factor, Coefficient. When two or more quantities are multiplied together the result is called the CONTINUED PRODUCT, or simply the PRODUCT of the quantities. Thus $2abc$ is the product of 2, a, b, and c.

Each of the quantities multiplied together to form a product is called a FACTOR of the product. Thus 2, a, b, and c, are each a factor of $2abc$.

When a product is considered as divided into two sets of factors, each set is called the COEFFICIENT (that is, the *co factor*) of the other Thus in $2abc$, 2 is the co efficient of abc, $2a$ is the co-efficient of bc, and $2ab$ is the co efficient of c, in xyz, x is the co efficient of yz, y of xz, and z of xy When the co efficient is expressed in *number*, it is called a *numerical co efficient*, and is *generally* placed *first*, thus 2 is the numerical coefficient in $2abc$ The numerical coefficient may be either *integral* or *fractional*, thus $3, \frac{1}{2}, \frac{5}{3}$ are coefficients in $3a, \frac{1}{2}x$ and $\frac{5}{3}mn$ respectively When the coefficient is expressed in *letter*, it is called a *literal coefficient*, thus m is the literal coefficient in ma

Note Where there is no numerical factor, the coefficient may be supposed to be *unity* Thus in a and xyz the coefficient of a and of xyz is unity

20 Power, Index, Exponent If a quantity be multiplied by *itself* any number of times, the *product* is called a POWER of that quantity Thus aa is called the *second power* or the *square* of a, aaa is called the *third power* or the *cube* of a, $aaaa$ is called the *fourth power* of a, and so on The quantity *itself* is called its *first power*, thus the *first power* of a is a

For the sake of convenience aa is written a^2, aaa is written a^3, $aaaa$ is written a^4, and generally $aaaa$. to n factors is written a^n. The small figure or letter placed above a quantity and to its right *to indicate how often that quantity is to be taken as a factor in a power*, is called the **Index** or **Exponent** of that power Thus 2, 3, 4, n are the *indices* or *exponents* of a^2, a^3, a^4 and a^n respectively

The first power of a being a or a^1, the *index of the first power of a quantity is unity*

In the above examples, a^2 is read "a raised to the *second power*," or "*a squared*", a^3 is read "a raised to the *third power*," or "*a cubed*", a^4 is read, "a raised to the *fourth power*," or briefly, "a to the *fourth*", and a^n is read "a to the nth," or "a nth"

21 Bracket, Vinculum The signs ( ), { }, [ ] are called BRACKETS They are used to enclose the terms of an expression which are meant to be taken *collectively* Thus in $a+(b+c)$, the terms b and c are enclosed in a bracket, for their sum is meant to be added to a, similarly $m(a+b)$ means that m to be multiplied by the sum of a and b, $(a+b)-x^2$ means that the sum of a and b is to be divided by x^2, $(ab)^2$ means that a is to be multiplied by b, and the product is to be squared, and so on

Sometimes instead of brackets a *line* is placed over the terms which are meant to be taken as a whole, thus $a+\overline{b+c}$ is the same as $a+(b+c)$ The line is then termed a VINCULUM

The three kinds of Brackets are sometimes conveniently called *Parentheses*, *Braces* and *Crotchets* respectively

Examples.

If $a=3, x=2, y=4$, find the value of

Ex. 1. $4^5=4\times4\times4\times4\times4=1024$

Ex. 2. $(15)^a=(15)^3=15\times15\times15=3375$

Ex. 3 $7^{2a}=7^{2\times3}=7^6=7\times7\times7\times7\times7\times7=117649$

Ex. 4 $4a^4=4\times3^4=4\times3\times3\times3\times3=324.$

Ex 5. $(5ax)^x=(5\times3\times2)^2=(30)^2=30\times30=900.$

6. 5^3. 7. 8^x. 8 $(28)^2$. 9 $(12)^a$. 10 $(50)^y$. 11 3^{ax}.

If $a=2, b=0, c=8, x=3, y=4$, find the value of

12 a^5 13. $3c^3$. 14. $14y^3$ 15. $12ax^5$ 16. $24bx^8$.

17. $16c^2x^4$. 18 $\frac{3}{8}a^2xy^3$. 19 $\frac{11}{16}x^2y^3$ 20 $\frac{30}{37}a^3b^2x^4$

21 $\frac{5}{32}a^4cy^5$. 22. $\frac{1}{9}x^3c^3y^5$ 23 $\frac{5}{12}a^7b^3x^5$.

If $a=5, b=1, c=4, m=2, n=3, r=7, x=6$, find the value of

24. a^m. 25. m^x 26 $3x^a$. 27 $5b^a$. 28. $4c^a$. 29. $(2ab)^x$.

30. $(4a^2x)^c$. 31 $(3ar)^{mb}$ 32. $3m^2x^c$ 33 $\frac{3}{4}xa^m$. 34. $\frac{5}{18}nx^n$.

35. $\frac{3}{7}r^2m^r$. 36. $\frac{4}{9}(a^2x^3)^c$. 37 $\frac{14}{225}a^3c^mn^x$. 38 $b^rc^2r^bx^n$.

If $a=4, x=r=2, y=3$, find the value of

39. $\left(\frac{y}{x}\right)^5$ 40. $\frac{5}{6}\left(\frac{y}{a}\right)^r$. 41 $\left(\frac{5a}{30}\right)^y$. 42. $\left(\frac{3y}{5a}\right)^x$

43 $\frac{3}{4}\left(\frac{3x^2}{4y^2}\right)^a$.

22. Square Root, Cube Root The SQUARE ROOT of a given quantity is that quantity whose *square*, or *second power*, gives the proposed quantity. Thus 2 is a square root of 4, for $2^2=4$, a is a square root of a^2; &c The square root of a quantity, say a, is written $\sqrt[2]{a}$, or more commonly $\sqrt{a}$, hence $\sqrt{4}=2$, $\sqrt{9}=3$, &c. $\sqrt{a}$ is read "*square root of a*," or more commonly "*root a*"

The CUBE ROOT of a given quantity is that quantity whose *cube* or *third power*, gives the proposed quantity. Thus 3 is a cube root of 27, for $3^3=27$; a is a cube root of a^3; &c The cube root of a quantity, say a, is represented by $\sqrt[3]{a}$; hence $\sqrt[3]{8}=2$ $\sqrt[3]{125}=5$, &c $\sqrt[3]{a}$ is read "*cube root of a*"

The sign $\sqrt{}$, by means of which the root of a given quantity is expressed, is called the RADICAL SIGN, and is a corruption of the initial letter *r* of the word *radix*.

Note From § 21, we at once see that $\sqrt{a+b}$ is equivalent to $\sqrt{(a+b)}$, the line over $a+b$ in the first case, serving as a *vinculum*.

Hence there is a distinction between $\sqrt{a+b}$ and $\sqrt{a}+b$, for $\sqrt{a+b}$ means the square root of the sum of a and b, whereas $\sqrt{a}+b$ means that b is to be added to the square root of a. So also $\sqrt{3x}$ means the square root of the product $3x$, but $\sqrt{3}x$ means that the square root of 3 is to be multiplied by x. Thus *where there is no bracket or vinculum, the radical sign refers only to the quantity before which it is placed.* Hence to express the root of a number, bracket or vinculum is unnecessary; thus $\sqrt{35}$ is sufficient to express $\sqrt{(35)}$, or $\sqrt{\overline{35}}$.

Examples

If $a=12$ and $x=9$, find the value of

Ex 1. $\sqrt{(3ax)}=\sqrt{(3\cdot12\cdot9)}=\sqrt{(3\cdot3\cdot4\cdot3\cdot3)}=3\cdot2\cdot3=18$.

Ex 2. $\sqrt[3]{(24x^4)}=\sqrt[3]{(3\cdot8\cdot9\cdot9\cdot9\cdot9)}=\sqrt[3]{(27\cdot8\cdot9\cdot9\cdot9)}=3\cdot2\cdot9=54$.

Ex 3. If $x=4$ and $c=5$, find the value of $\sqrt{x^c}$.

$$\sqrt{x^c}=\sqrt{4^5}=\sqrt{(4\cdot4\cdot4\cdot4\cdot4)}=2\cdot2\cdot2\cdot2\cdot2=32.$$

Ex 4. If $x=8$, $a=2$, $b=32$, find the value of $\sqrt[3]{(2bx^a)}$.

$$\sqrt[3]{(2bx^a)}=\sqrt[3]{(2\cdot32\cdot8^2)}=\sqrt[3]{(64\cdot8\cdot8)}=\sqrt[3]{\overline{(4\cdot4\cdot4)(4\cdot4\cdot4)}}=4\cdot4=16.$$

If $a=12$, $b=15$, $x=1$, $y=9$, find the value of

5. $\sqrt{(16xy)}$. 6. $\sqrt{(12a^2y)}$. 7. $\frac{2}{5}\sqrt{(5ab^3)}$. 8. $3x\sqrt{(a^2xy)}$.

9. $3\sqrt{(20aby^3)}$. 10. $\sqrt[3]{(24y)}$. 11. $2\sqrt[3]{(80ab^2)}$.

12. $2b\sqrt[3]{(3y^4)}$. 13. $\frac{4}{3}\sqrt[3]{(4a^2y^2)}$. 14. $\frac{8}{15}\sqrt[3]{(15b^2xy)}$.

If $a=2$, $b=4$, $c=1$, $d=9$, $x=8$, $y=3$, $z=0$, find the value of

15. $\sqrt{\left(\frac{4b^3}{ax}\right)}$. 16. $\sqrt{\left(\frac{3ay}{8cd}\right)}$. 17. $\sqrt{\left(\frac{3dx^3}{b^2y}\right)}$.

18. $\sqrt{\left(\frac{1}{2dx}\right)}$. 19. $\frac{1}{\sqrt{(3d^2y)}}$. 20. $\frac{1}{\sqrt{(6a^3x^2y)}}$.

21. $\frac{8a^2z}{\sqrt{(8c^2d^3)}}$. 22. $\sqrt[3]{\left(\frac{64x}{9y}\right)}$. 23. $\sqrt[3]{\left(\frac{2bx}{27dy}\right)}$.

24. $\sqrt[3]{\left(\frac{16by^4}{27c^4}\right)}$. 25. $\sqrt[3]{\left(\frac{1}{125ab}\right)}$. 26. $\sqrt[3]{\left(\frac{1}{8cx^2}\right)}$.

Ex 37. Find the values of $\sqrt{4x+y}$, $\sqrt{4x}+y$ and $\sqrt{4}x+y$, when $x=16$ and $y=36$.

$$\sqrt{4x+y}=\sqrt{4\times16+36}=\sqrt{100}=10.$$

$$\sqrt{4x}+y=\sqrt{4}\times\sqrt{16}+36=\sqrt{64}+36=8+36=44.$$

$$\sqrt{4}x+y=2\times16+36=32+36=68.$$

If $a=9$, $b=1$, $c=25$, $d=8$, $x=16$, $y=36$, find the value of

28. $c\sqrt{x+a}$	29. $c\sqrt{x}+a$.	30. $\sqrt{cx}+\sqrt{a}$.
31. $\sqrt{(ax+bc)}$.	32. $\sqrt{ax}+\sqrt{(bc)}$.	33. $\sqrt{(ax)}+\sqrt{bc}$.
34. $\sqrt{a}x+\sqrt{b}c$.	35. $\sqrt[3]{3(b+d)}$	36. $\sqrt[3]{3}a(b+d)$.
37. $2\sqrt[3]{d}(x-a)$	38. $x\sqrt[3]{3a+b+y}$.	39. $2\sqrt[3]{4x+y+c}(2a-x)^3$.

23 Like and Unlike Terms Terms which do not differ at all or differ only in their numerical coefficients, are called LIKE TERMS Thus a, $3a$, $5a$, $\frac{1}{2}a$, are like terms; so also are $2a^2b^3$, a^2b^3, and $\frac{1}{4}a^2b^3$. When this is not the case they are said to be UNLIKE TERMS Thus $3a$, b and $\frac{1}{3}d$ are unlike terms

24 [§ 17]

25 Other Signs The sign $>$, which is read *is greater than*, indicates that of the two quantities between which it is placed, the former is *greater* than the latter Thus $a>b$ denotes that a is *greater* than b

The sign $<$, which is read *is less than* indicates that of the two quantities between which it is placed, the former is *less* than the latter Thus $a<b$ denotes that a is *less* than b

The sign $\therefore$ signifies *hence* or *therefore*

The sign signifies *since* or *because*

26 Peculiar meaning of the word "sign" The word "*sign*" when used alone, *denotes the two signs* $+$ and $-$, *and no other signs* Hence "*the sign of a term*" means either the sign $+$ or the sign $-$, which is prefixed to it, and the phrase "*to change the signs*" means "to change *all* the $+$ signs into $-$, and *all* the $-$ signs into $+$."

Again the phrase "*like signs*" when used with reference to two signs, means signs *both* of which are $+$ or *both* $-$, and "*unlike signs*" means signs *one* of which is $+$ and the *other* $-$

Similarly when used with reference to several signs, "like signs" means signs *all* of which are $+$ or *all* $-$; and "unlike signs" means signs *some* of which are $+$ and the *rest* $-$

27 Examination upon the Introduction.

1 Define "Algebra" To whom is the invention of this science ascribed?

2 What symbols are used in Algebra to denote numbers?

3 Define "Quantity" and "Integer" How are quantities represented in Algebra?

4. Define *unit* and *measure* Can you measure an area of 5 sq yds by means of the unit one yard? If not, what unit should you employ?

5 What is understood by $x+y$ and $x-y$ respectively?

6 What is meant by $a\times b-c$, $a-b-c$, and $a-b\times c$ respectively?

7 Distinguish between 5 6 and 5 6 Find their difference.

8 How may the product of 10, x and y be represented?

9 If x stand for 4 and y for 5, how will 45 be represented by means of these symbols?

10 Mention some of the properties of 0

11 Define the terms—*Product*, *Factor*, *Coefficient*, *Power*, and *Index* Distinguish between $3a$ and a^3, and find their difference when $a=2$ What are the *factors* of $3a(b+c)$?

12 Define the *square root* and the *cube root* of a number What is the use of brackets? What are meant by $2x^2$ and $(2x)^2$, and what is the difference between them when $x=3$

13 Distinguish between $\sqrt{a+b}$ and $\sqrt{a}+b$ If $x=4$, find the difference between $\sqrt{9}x$ and $\sqrt{9x}$

14 Define an *algebraical expression*, a *term*, *like terms*, *unlike terms*, a *binomial*, a *trinomial*, a *polynomial* and a *compound expression* Give examples of each Is $8a^3bc^2x^4y$ a compound expression? If not, what is it?

15 What is the order of the operations in simplifying an *expression*? Find the value of $5+8\times 2-4+18-(3\times 2)-18-3\times 2$

16 What is the peculiar meaning of the word *sign* when used alone? What are *like signs* and *unlike signs*?

28 Examples

If $a=25$ and $b=10$, find the value of

1. $ab-a-b$ 2. $(a+b)(a-b)+2ab$.

3 $5ab+3a-2b-3b-8$ 4 $10a-b+b-a-ab-25$

If $a=20$, $b=12$, $x=5$ and $y=3$, find the value of

5. $ax\pm by$ 6. $ab\pm xy$

7 $5ax-3by+ay-10$ 8 $2(a+b)x-3a-x$

9 $(a+x)(b-y)-(a+b)(x-y)$ 10. $axy-b-abx-ax+by(1+y)$.

If $a=6$, $b=5$, $c=4$, $d=3$, $e=2$, $f=1$ and $g=0$, find the value of

11 $5a-d+4b-c-10d-e+2g$

12 $4ab-3bc+6cd+5ef-2dg$. 13 $ab+5bc-4de+5fg$

14 $4ac-3bf+5dg-ad+4ce$ 15 $5ab-8cg+15cde-4aef$.

If $a=6$, $b=5$, $c=4$, $d=3$, $e=2$, $f=1$ and $g=0$, find the value of

16. $22abg-19cd+33ab-13cdef$ 17 $\frac{ab}{cd}+\frac{bd}{ae}-\frac{cde}{abf}$.

If $a=2$, $b=3$, $c=5$, $d=0$, find the value of

18 a^2+b^2+2ab. 19 $a^3+b^3+c^3+3abc$ 20 $a^3b+b^3c+c^3d+d^3a$.

21 $a^4+b^4+c^4+d^4+4abcd$ 22 $3a^3+4bc^2-4abc$

23 $10a^4c^3-5b^3cd+2a^3bc^2-3ad^3+27c^3$.

24. $\frac{1}{5}ac^2+\frac{1}{2}b^2c-\frac{3}{4}abc-a^3+b^3$

25. $\frac{3}{10}a^3c+2a^5d-80d^4+\frac{1}{2}c^4-\frac{5}{8}b^3a^3$.

26. $\frac{2}{3}abc+\frac{3}{8}bc^3d-15a^3c+\frac{4}{5}bc^3-\frac{6}{7}d^2a^3$.

If $a=5$, $b=2$, $c=3$, $d=1$, find the value of

27. $a^2+b^2+c^2+d^2$. 28 $a^3+b^3+c^3+d^3$. 29 $a^4+b^4+c^4+d^4$

30 $(a+b)^2+(c+d)^2$ 31 $(a-d)^2-(c-b)^2$. 32 $(a^2-c^2)^3-(b^2-d^2)^3$

33. $(a+b+c)^2-a^2-b^2-c^2$ 34 $(a+b+c)^3-a^3-b^3-c^3$

35. $(ad+bc)^2+(ac-bd)^2$. 36. $(2a-b^2)^3+(3c-bc)^3$.

37 $\frac{4}{5}(2c^2-ab)^3+\frac{5}{12}(a^2d^4-c^2)^3$

If $a=2$, $b=\frac{1}{2}$, $c=\frac{1}{3}$ and $d=\frac{1}{4}$, find the value of

38 $5ac+14bc-9cd$. 39 $7ad-3bd+5ac-bcd$.

40 $3a^2b+4a^3-3ab^2+4b^3-8d^3$. 41 $d^4-4a^3b+9b^2c^2-48cd^4$.

42. $3(ab-6cd)^2+4(ad-bc)^2$ 43 $\frac{a}{b}-\frac{1+b}{2-c}+\frac{2d}{3-b}$.

44 $\frac{3c^2}{a^2}+\frac{b^3}{d^2}-\frac{3-b^2}{2(1+2c^2)}$

If $a=5$, $b=3$, $c=0$, $x=4$ and $y=2$, shew that

45 $ab(a+b)+3ac(a+c)+2xy(x+y)+17$
$=ax(a+x)+cy(c+y)+ay(a+y)-17$.

46 $\frac{1}{2}ab+\frac{1}{3}(ax+by)+\frac{1}{2}xy+4\frac{5}{12}$
$=\frac{1}{2}ax+\frac{1}{3}(a+b)(x+y)+\frac{1}{2}by-4\frac{5}{12}$

47 Shew that $5x^2-32x+12=0$, whether $x=6$ or $=\frac{2}{5}$

48. Shew that $4x^3-15x^2+17x-6=0$, whether $x=1$, or $=2$, or $=\frac{3}{4}$.

If $a=4$, $b=3$, $c=5$, $d=0$, find the value of

49. $6\sqrt{(a^2+b^2)}-(c-bd)^2$ 50 $\sqrt{(5ac^3)}+\frac{(3a^2-2bc)^2}{\sqrt{(3b^3)}}$.

51. $\frac{2}{3}\sqrt{(a^2+b^2+d^2)}-\frac{\sqrt{(c^2-b^2-d^2)}}{3ac}$.

52. $\frac{\sqrt{(3b^2-ac+9)}}{ad+bc}+\sqrt[3]{\{(c-b)(a-d)\}}$

CHAPTER I

FUNDAMENTAL NOTIONS.

29 Unit We have seen [§ 2] that if *one* square foot be the unit, an area of 36 square feet will be represented by the number 36 Similarly if *two* square feet be the unit, the same area will be represented by 18, if *three* square feet be the unit, it will be represented by 12, and so on Thus by taking *different* units we may represent one and the same quantity by *different* numbers The following examples will illustrate the truth of our remark

Examples

Ex 1. If R5 be the unit of measure, what will be the measure of R100 ?

Every R5 is represented by *unity*, ∴ the number that represents R100 is $100 \div 5 = 20$

Ex 2 If a rupee be the unit, what will be the measure of 250 half rupees ?

Every unit = 2 half rupees, therefore the required number $= 250 \div 2 = 125$

Note From these two examples we see that if the unit be of the *same* or *higher* denomination, we *divide*

Ex 3 If half-a-rupee be the unit, what number will represent R75 ?

Here each rupee = 2 units, ∴ required number $= 75 \times 2 = 150$

Note Hence when the unit is of a *lower* denomination, we *multiply*.

Ex 4 If 2 feet and 2 yards be respectively the unit, how will 18 yards 2 feet be represented ?

(1) Here 18 yds 2 ft = 56 ft, therefore the number that represents 18 yds 2 ft is $56 \div 2 = 28$

(2) Again, 18 yds 2 ft $= 18\frac{2}{3}$ yds, therefore the number that represents 18 yds. 2 ft is $18\frac{2}{3} \div 2 = 9\frac{1}{3}$

Note Hence the unit and the proposed quantity *must be reduced to the same denomination*, if not already so reduced

Ex 5 What are the units when R100 is represented by 25 and 24 respectively ?

Since 25 measures 100, it is clear that the unit is contained 25 times in 100, ∴ unit required $= 100 \div 25 = 4$

Similarly, in the second case, the unit is contained 24 times,

$$\therefore \text{ unit required} = 100 \div 24 = 4\tfrac{1}{6}.$$

Ex. 6. If an hour and a quarter be represented by 20, what is the unit?

The required unit is contained 20 times in 1 hr 15 min, or 75 min,

$$\therefore \text{ unit required} = 75 \div 20 = 3\tfrac{3}{4}.$$

Ex 7 If 40 secs be taken as the unit, what time will 40 represent?

Each unit of time = 40 secs, therefore 40 will represent 40 times 40 secs = 1600 secs. = 26 min 40 secs.

8 If 4 rupees be the unit, what number will measure 18 rupees?

9 If 2s. be the unit, what will be the measures of £15 and 20 half-crowns?

10 If £1 be the unit, how will 200 six-pences be represented?

11 If 3s 4d be the unit, what sum will be represented by 18?

12 If £8 3s 4d be represented by 20, what is the unit?

13 If 4s 3d be the unit, what will be the measure of £4 5s?

14 If 150 miles be measured by 25, what number will measure 102 miles?

15 If an area of 20 sq. ft be represented by $3\frac{1}{3}$, how will one square yard be represented?

16 If ₨16 4a. be denoted by 20, what will be the measure of ₨26?

17 If the unit be 5 yards, what distance will 352 represent?

18 If 164 sq yds be represented by $20\frac{1}{2}$, what will be the measure of the unit in feet?

19 If $5\frac{1}{4}$ be the measure of 189 sq. ft, how many sq yds are there in 5 times the unit?

20 If $4\frac{1}{4}$ be the measure of ₨12 12a, what will be the measure of ₨12, supposing the new unit to be $\frac{1}{4}$ of the old unit?

30 Meaning of the Symbol a We know that 2 means 2 units This is expressed algebraically thus

$2 = 1 + 1$ where unity is written *two* times.

Similarly $3 = 1 + 1 + 1$ where unity is written *three* times,

$4 = 1 + 1 + 1 + 1$ where unity is written *four* times,

$a = 1 + 1 + 1 + \ldots\ldots$ where *unity* is to be written a times

Thus a means a *units, whatever the unit may be*

31 Positive and Negative Quantities Many quantities of the *same class* may have *two* characters directly *opposite* to each other

Thus a sum of money may be either a *gain* or a *loss*, it may be an *income* or an *expenditure*, &c Gain and loss are quantities of the *same class*, inasmuch as each is a *sum of money*, but they are of *opposite nature* or *character*, inasmuch as a gain *increases* our stock while a loss *decreases* it, and if our gain be just as much as our loss, the effect on our stock will be *nothing* So also, income and expenditure are of the same class, being both of them *money*, but *contrary* in character, inasmuch as an income *adds* to our assets while an expenditure *diminishes* them

Again, a distance measured in one direction may have a character *contrary* to that measured in an *opposite* direction For if a person first walks a certain distance towards the *east* and then walks *back* the same distance towards the *west*, his position with regard to the starting point will be the same as before, or in other words, his walking a certain distance, and his walking back the same distance in the *opposite* direction taken together, will produce no effect on his journey for the place he intends to go to

This will be made clearer by the following Geometrical illustration Suppose AB to be a straight road in which O is a fixed point

A O C B

Suppose a person intends to go from O to B, distant 8 miles He first walks to C, distant 3 miles, and then walks back 3 miles towards A It is evident he will now be at O, that is, his position on the road will be the same as before Hence his walking the distances OC and CO, *i e*, 6 miles, will not affect his journey for B, for he will have still the whole distance 8 miles to walk to reach B

Thus a distance in one direction and that in the opposite direction are quantities of the *same class*, inasmuch as each is a *length*, but opposite in character inasmuch as each affects the other in a *directly opposite way*

We might give many other examples For instance, the distance of a place *north* of the equator and that of a place *south* of the equator are quantities of the same class but of opposite character, so also a date *before* a particular epoch, *e g* the birth of Christ, and another *after* that epoch are quantities of the same class but contrary in character, and so on

Hence *Positive and Negative Quantities are magnitudes of the same class but of opposite character*

32 Representation of Positive and Negative Quantities by means of the signs + and – From the nature of the

signs $+$ and $-$, it seems clear that whatever quantity we are considering, $+a$ will always denote what increases that quantity by a units and $-a$ will always denote what decreases that quantity by a units

If we are speaking of a man's *gains*, calculated in rupees, $+20$ will denote an amount that increases his gains by 20 rupees, *i e*, $+20$ will denote a *gain* of 20 rupees, and -20 will denote an amount that *decreases* his *gains* by 20 rupees, *i e*, -20 will represent a *loss* of 20 rupees If on the other hand, we are speaking of a man's *losses*, $+20$ will represent an amount that increases his losses by 20 rupees, *i e*, $+20$ will now denote a *loss* of 20 rupees, and -20 will represent an amount that decreases his losses by 20 rupees, *i e*, -20 will now denote a *gain* of 20 rupees.

Again, if we are considering a certain distance (measured in miles) to the *north* of a given point, $+20$ will denote an addition of 20 miles to that distance, *i e*, $+20$ will indicate a distance of 20 miles *northwards*, while -20 will denote a subtraction of 20 miles from the same distance, *i e*, -20 will indicate a distance of 20 miles *southwards*, *i e*, in the *opposite* direction If on the other hand we consider a distance *south* of the same point, $+20$ will now denote a distance of 20 miles measured *southwards*, while -20 will represent 20 miles measured *northwards* And so on

It is thus clear that the signs $+$ and $-$ will thoroughly serve our purpose to mark respectively *positive* and *negative* quantities Hence if a and b are quantities of the same class, $+a$ will denote a *positive* quantity and $-b$ will denote a *negative* quantity. The sign $+$, as a mark of positive quantities, is however often omitted, thus a means $+a$ Hence when *no sign* stands before a quantity, the sign $+$ is to be understood

It thus appears that the signs $+$ and $-$ serve *two distinct* purposes.—*First*, they are used to indicate the operations of addition and subtraction: and *secondly*, they are used respectively to mark *positive* and *negative* quantities, and then they are called *positive* and *negative signs* The positive and negative signs are also called *signs of affection*, as they mark the *quality* of *quantities* before which they stand.

33 [Incorporated with § 32]

34. Positive Quantity once chosen our choice must remain unchanged throughout the same investigation. We have seen above [§ 32] that we may represent a gain of £20, by $+20$, and a loss of £20 by -20, or we may represent a loss of £20 by $+20$ and a gain of £20 by -20 that is, it is perfectly optional to call any quantity we please positive, and then a quantity of opposite character shall be called negative.

But though we are at liberty to call what quantity we like positive, yet, having once made our choice, *we must throughout the same question, call that quantity positive and the quantity of opposite kind negative*

35 Meaning of $-a$ We have seen that a negative quantity produces a decrease Now, to decrease a given quantity by 2 units is the same as to decrease it first by unity and then again by unity Hence $-2=-1-1$, where 1 is subtracted *two* times
Similarly $-3=-2-1=-1-1-1$, where 1 is subtracted *three* times,

$-4=-3-1=-1-1-1-1$, where 1 is subtracted *four* times, and so on Therefore

$-a=-1-1-1-1-$ where 1 is to be subtracted a times

Thus from the above and the explanation of a negative quantity, it seems clear that $-a$ may stand alone just as $+a$ does, and *indicate that a units are to be subtracted from some other algebraical expression*, or *that $-a$ is a units of a character opposite to that denoted by $+a$*, and *if $-a$ is a final result, the last meaning is the only meaning that can be assigned to it*

36 Absolute value The magnitude of a quantity considered without reference to its sign, is called its ABSOLUTE VALUE Thus the absolute value of $+3$ and of -3 is 3, of $+a$ and of $-a$ is a, and so on

Hence from § 31, it is clear that *the sum of a positive and a negative quantity having the same absolute value is* 0, that is, $+a-a=0$, or $-a+a=0$

37 Examination upon Chapter I

1. Explain what is meant by the symbols a and $-a$

2 What is the nature of positive and negative quantities? Are these quantities essentially different from each other?

3 Can you call *any* quantity positive? If so, what quantity should you then call negative?

4 How are positive and negative quantities distinguished?

5 What double purpose is served by the signs + and −? What are positive and negative signs? What is a "sign of affection," and why is it so called?

6 What is the absolute value of a quantity? Give examples Explain $+a-a=0$

7 A person first gains 25 rupees and then loses 30 rupees, how much does he *gain* altogether? And how much does he *lose* altogether?

8. A man walks 3 miles towards the North and then 5 miles towards the South, how far is he now from the starting point?

9. A clock which loses 2 min. a day, indicates at noon 3 min past 12; what time will it indicate at noon of the third day?

10. If a debt of 50 rupees be represented by 30, what will -21 represent?

11. If a distance of 230 ft east of a point be represented by 15, what number will represent a distance of 230 yds. west of the point?

12. AB is a straight line in which O is a given point. Mark off on it the distances $-4, 1, 0, -3$ and 2.

CHAPTER II.

FUNDAMENTAL LAWS—ADDITION AND SUBTRACTION

ADDITION.

38. Definitions. ADDITION is the process of finding the single quantity which is equal to several quantities put together. These several quantities are called ADDENDS or SUMMANDS, and the single quantity is called their SUM.

39. Addition of a Positive Quantity and of a Negative Quantity. We know that a positive quantity always produces an *increase* in a given quantity, therefore *to add a positive quantity is the same as to add its absolute value.*

For example, if a person *has* 8 rupees and then he *earns* 3 rupees, he will then have altogether the sum of $+8$ and $+3$ rupees with him. Thus

$$+8+(+3)=+8+3$$

Hence generally $\qquad +a+(+b)=+a+b \qquad$ (A)

Again, since a negative quantity produces a *decrease*, *to add a negative quantity is the same as to subtract its absolute value.*

For example, if a person *has* 8 rupees out of which he *spends* 3 rupees, then he has $8-3$ rupees, *i.e.*, he has then with him the difference of 8 and 3 rupees. Hence, if we call the *earnings* $+8$ rupees, and the *expenditure* -3 rupees, the *sum* of $+8$ and -3 will be the same as the *difference* of $+8$ and $+3$. Thus

$$+8+(-3)=+8-3$$

Hence generally $\qquad +a+(-b)=+a-b \qquad$ (B)

In establishing (A) and (B), we have supposed $+a$ to be positive; but a little consideration will shew that whatever be the character

of the quantity to which $+b$ or $-b$ may be added, *b will always retain its own sign in the result* Hence

$$-a+(+b)=-a+b,$$

and
$$-a+(-b)=-a-b$$

Thus for the addition of any term we have the following RULE .— *Place the term with its sign unchanged after the expression to which it is to be added*

Note 1 The quantities $a+b, a-b, -a+b$, and $-a-b$ cannot be simplified any more, and hence they are considered as *final results algebraically* But they may have *simple* numerical values when we give numerical values to a and b

Thus if $a=2$ and $b=3$, we have

(1) $a+b=2+3=1+1+1+1+1$ [§ 30] $=+5$ [§ 30]

(2) $a-b=2-3=2-2-1$ [§ 35] $=0-1$ [§ 36, Cor] $=-1$

(3) $-a+b=-2+3=-2+2+1$ [§ 30] $=0+1$ [§ 36, Cor] $=+1$.

(4) $-a-b=-2-3=-1-1-1-1-1$ [§ 35] $=-5$ [§ 35]

From these results we obtain the following **three inferences** for the addition of two **numbers** —

(I) From (1), we see that *to add two positive numbers, we add their absolute values and prefix the sign + to the result*

(II) From (4), we see that *to add two negative numbers, we add their absolute values and prefix the sign − to the result*

(III) From (2) and (3), we see that *to add a positive and a negative number together, we subtract the absolute value of the less from the absolute value of the greater, and prefix the sign of the greater to the result*

Note 2 Any expression in arithmetical **numbers** may now be simplified Thus to simplify $3+1-7-5+6$

Since the order of the operations is from left to right [§ 19, Note 2],

$$3+1-7-5+6=4-7-5+6 \text{ [Inf I]} =-3-5+6 \text{ [Inf III]}$$
$$=-8+6 \text{ [Inf II]} =-2 \text{ [Inf III]}$$

Examples

Ex 1 Find the sum of $+7$ and -6, of -5 and -9, of $+10$ and -8, and of -11 and $+8$

(1) Required sum $=+7+(-6)=+7-6=+1$

(2) Required sum $=-5+(-9)=-5-9=-14$

(3) Required sum $=+10+(-8)=+10-8=+2$

(4) Required sum $=-11+(+8)=-11+8=-3$

Ex. 2. Find the sum of a, $-b$, and $-c$, when $a=3$, $b=4$ and $c=5$

$$\text{Required sum } = a+(-b)+(-c) = a-b-c = 3-4-5$$
$$= -1-5 = -6$$

Ex. 3. Simplify $-3+4-5-6$

$$-3+4-5-6 = +1-5-6 = -4-6 = -10$$

4 Find the sum of $+8$ and $+5$; of $+8$ and -5; of -8 and $+5$; and of -8 and -5

5 Find the sum of 1, 3, 8 and 9, of -1, -3, -8 and -9; of -5, -4, $+7$ and -2, and of 3, -8, -1, $+10$ and -4.

6. Find the sum of $-2a$, $3b$ and $-4c$, when $a=8$, $b=9$ and $c=6$

7 Simplify $-12+3-8-7+20$, and $-3-1+12-4+1$.

8 Add together $3x$, $-2y$, z and $8w$, when $x=3$, $y=8$, $z=-4$ and $w=1$

9. If $a=1$, $b=2$, $c=3$ and $d=4$, find the sum of $3a$, $-4b$, $5c$ and $-6d$

10. If $a=2$, $b=3$ and $c=1$, find the value of $-ab^2+bc^2-2ca^2$.

11. Find the value of $4a-8b+9c-3$, when $a=1$, and $b=c=2$.

12 Simplify $-xyz+2xy-3yz+4xz$, when $x=3$, $y=5$ and $z=4$.

13 Find the sum of $8x$, $-4y$, $-5z$ and $3x$, if $x=5$, $y=3$ and $z=2$.

14. Find the value of $a^3-2a^2b+3ab^2-4b^3$, when $a=b=4$

15 Find the value of $5ax^2-2b^2cx-3a^2x^2+2abc^2$, when $a=2$, $b=4$, $c=8$ and $x=3$

16 If $x=2$, $y=3$, find the sum of $-x^2y$, $2xy^3$, $-5x^2y$ and $-6x^2y^2$.

17. If $x=3$ and $y=2$, find the value of $x^4-3x^3y-1xy^4+4x^3y^3-2y^6$

40 Algebraic sum. In Arithmetic *sum* means the aggregate of *positive quantities only*; but in Algebra it means (1) the *sum* of quantities *all* of which are positive or *all* of which are negative, and (2) the sum of quantities *some* of which are positive and the *rest* negative Thus an **Algebraical sum** may be *positive* or *negative* [§ 39] Thus the algebraic sum of $+8$ and $+3$ is $+11$, of -8 and -3 is -11, of -8 and $+3$ is -5, of $+5$, -3, $+4$ and -7 is -1, and so on

SUBTRACTION.

41. Definition. SUBTRACTION may be defined as the *inverse* of addition For to find the result of *subtracting* b from a is the same as to find a quantity which must be *added* to b to produce a Thus if the result of subtracting b from a is $a-b$, the quantity $a-b$ is such that $(a-b)+b=a$

The quantity from which another quantity is to be subtracted is called the MINUEND, and this latter is called the SUBTRAHEND, and the quantity which remains after the operation has been performed is called the REMAINDER or DIFFERENCE Thus a is the minuend, b is the subtrahend and $a-b$ is the remainder

42 Subtraction of a Positive Quantity and of a Negative Quantity By definition, subtraction is the inverse of addition, therefore the *subtraction* of a positive quantity must produce a *decrease*, for its *addition* produces an *increase* Therefore when we subtract a positive quantity, we subtract *its absolute value* Thus the result of subtracting $+3$ from $+8$ is $+8-3$, that is,

$$+8-(+3)=+8-3$$

Hence generally $$+a-(+b)=+a-b \qquad \text{(C)}$$

Again, as subtraction is the inverse of addition, the *subtraction* of a negative quantity must produce an *increase*, for its *addition* produces a *decrease* Therefore when we subtract a negative quantity, we *add its absolute value* Thus the result of subtracting -3 from $+8$ is $+8+3$, that is

$$+8-(-3)=+8+3$$

Hence generally $$+a-(-b)=+a+b \qquad \text{(D).}$$

The results (C) and (D) give, for the subtraction of a term, the following RULE —*Place the term with its sign changed after the expression from which it is to be subtracted, and proceed as in Addition*

Examples.

Ex. 1 Subtract $+5$ from $+11$, -4 from $+7$, $+6$ from -8 and -3 from -4

(i) $+11-(+5)=+11-5=+6$ [*Inference* III, § 39]

(ii) $+7-(-4)=+7+4=+11$ [*Inference* I, § 39]

(iii) $-8-(+6)=-8-6=-14$ [*Inference* II, § 39]

(iv) $-4-(-3)=-4+3=-1$ [*Inference* III, § 39].

Ex 2 Subtract $-a$ from $-b$, and simplify the difference when $a=8$ and $b=13$

$$-b-(-a)=-b+a=-13+8=-5 \text{ [\textit{Inference} III, § 39]}$$

Ex 3 From the sum $12-31+17$ take the sum $-5+8-1$

Now $12-31+17=-19+17=-2$, and $-5+8-1=+3-1=+2$

required difference$=-2-(+2)=-2-2=-4$

4 Subtract -12 from $+3$, -15 from -8, and $+3$ from -1.

5 From $+18$ take $+12$, from -15 take $+10$; from $+9$ take -9; and from -11 take -5

6 From a take $-b$, and find the value of the result when $a=-5$ and $b=-1$.

7. Subtract b from $-a$ and find the value of the remainder when $a=3$ and $b=-6$

8 From $a-b$ subtract $-c$ and simplify the result when $a=1$, $b=2$ and $c=5$

9 Find the difference between the sum of 5, -4 and -6, and the sum of -18, -3 and 9

43 The Law of Signs It is easy to see from results (A) of § 39 and (D) of § 42, that when the *same* sign is prefixed to a bracket as well as to the term within the bracket, the sign of the term will be $+$ when the bracket is removed; and from results (B) and (C), that when the signs are *different* the sign of the term will be $-$, when the bracket is removed We have thus an important law called the Law of Signs which we briefly enunciate thus.—*Like signs produce* $+$, *and unlike signs produce* $-$.

That is, $+(+5)=+5$; $-(-3)=+3$; $+(-4)=-4$, $-(+6)=-6$

Examples

Ex 1 Find the value of $-8-(-7)-(+3)+(-6)$

Required value $=-8+7-3-6=-1-3-6=-10$.

Ex 2 Find the value of $a-b-c$, when $a=-3, b=-5$, and $c=-8$.

$a-b-c=+(-3)-(-5)-(-8)=-3+5+8=+2+8=+10$.

Ex 3 Simplify $3a-(-b)+(-c)$, when $a=5$, $b=-11$, $c=12$

$$3a-(-b)+(-c)=3a+b-c=3\times5-11-12$$
$$=15-11-12=4-12=-8$$

4 Find the values of $7-(-2)+(-4)$ and $-3-(-2)+(+1)$

5. Simplify $4-(+3)-(-2)-(+5)$ and $-(-3)+(-4)-(-1)+(-2)$

If $a=-1$, $b=-2$, $c=-3$, $x=-4$, $y=-5$, $z=6$, find the value of

6. $-a-b+c$.

7 $-(-a)+b+(-c)$

8. $a-(-x)-(+b)+(-y)$

9 $-a-(-b)+(-c)-3z$

10 $(-a)-x-(+y)-(-z)$

11 $-(-x)-(-y)-(+2z)-3$.

12 $8-(+a)+(-b)-(+c)$.

13 $4z-(+x)-(-y)+(-c)$.

44. Algebraic Difference The difference between $+a$ and $+b$ is $+a-b$, which is *arithmetically* intelligible so long as $a>b$,

but when $a<b$, this difference is *negative*, *e.g.*, $3-4=-1$; also the difference between $+a$ and $-b$ is $+a+b$ from (D) Thus the algebraic difference of two positive quantities is the *difference* between their absolute values and may be negative, also the algebraic difference of $+a$ and $-b$ is the *sum* of their absolute values For example

the algebraic difference of $+6$ and $+2$ is $+6-(+2)=+6-2=+4$,

$+6$ and -2 is $+6-(-2)=+6+2=+8$,

-6 and $+2$ is $-6-(+2)=-6-2=-8$,

-6 and -2 is $-6-(-2)=-6+2=-4$.

Geometrical Illustration

Let AB be a straight line (see figure § 31) in which O is a fixed point, and let C be any other point Suppose the distances measured *towards* B to be *positive*, and therefore the distances measured *towards* A are *negative*

Let $CA=7$, $OC=3$ and $OB=8$ Thus the absolute values of OA OC and OB are 7, 3 and 8 respectively, and their algebraic values are respectively -7, $+3$ and $+8$

Hence the algebraic difference of the distances of B and C from O

$$=OB-OC=+8-(+3)=+8-3=+5,$$

the algebraic difference of the distances OC and OB

$$=OC-OB=OC-(OC+CB)=+3-(+3+5)$$
$$=+3-3-5=-5,$$

and the algebraic difference of the distances OB and OA

$$=OB-OA=+8-(-7)=+8+7=+15.$$

Examples

1 Find the algebraic difference between 6 and -7; -6 and 7, 6 and 7, -6 and -7

2 Find the algebraic difference between $8m$ and $3m$, $8m$ and $-3m$, $-8m$ and $3m$, $-8m$ and $-3m$

45 Definition of "greater than" and of "less than" The introduction into Algebra of negative quantities requires an extended definition of the phrases *greater than* and *less than*. We therefore define them as follows —

a is said to be greater than b when their algebraic difference $a-b$ is positive, and less than b, when this difference is negative, a and b being any quantities whatever

Thus
$$\begin{aligned}
+4&>+3, \text{ for } +4-(+3)=+4-3\ [\S\ 43]=+1.\\
-3&>-4, \text{ for } -3-(-4)=-3+4\ [\S\ 43]=+1.\\
+2&>-8, \text{ for } +2-(-8)=+2+8\ [\S\ 43]=+10\\
-5&<-2, \text{ for } -5-(-2)=-5+2\ [\S\ 43]=-3.\\
-6&<+1, \text{ for } -6-(+1)=-6-1\ [\S\ 43]=-7.
\end{aligned}$$

Hence the quantities

$-6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4,\ +5,\ +6,$ are in *ascending order of magnitude* i.e., each is *greater than* any one that precedes it.

Examples

1. Which is the greater -5 or -2; -1 or -3; -1 or $+1$?
2. Which is the less -3 or -4, 0 or -1; -100 or 0?
3. Which is the greater -1 or $+1$, and by how much?
4. Which is the greater, the sum of -3 and 2, or the sum of 5 and -7?
5. Is the sum of -3, 2 and -4 less than the difference of -2 and -1? If so, by how much?
6. Which is the less, the difference of -6 and -4, or the sum of 8, -12 and 5, and by how much?

Addition and Subtraction of Polynomials

46. In an expression the terms may be bracketed in any manner. It is easy to see that

$$\begin{aligned}
(1+1+1+1+1+1+1)&=(1+1+1)+(1+1+1+1)\\
&=(1+1)+(1+1+1+1+1)\\
&=(1+1)+(1+1)+(1+1+1)=\&c
\end{aligned}$$

For supposing each of the brackets to represent a group, say, of men, and replacing each bracket by the number it represents [§ 30], we see that a group of 7 men, is equal to two groups of 3 and 4 men, or equal to another two groups of 2 and 5 men, or, equal to three groups of 2, 2 and 3 men, and so on

Similarly we see that

$$\begin{aligned}
a+b+c+d+e+f&=a+(b+c)+(d+e)+f\\
&=(a+b)+(c+d+e)+f\\
&=(a+b+c)+d+(e+f)=\&c
\end{aligned}$$

So also, exactly as in the second example, we see that

$$\begin{aligned}
a+(-b)+c+(-d)+(-e)+f&\\
&=a+\{(-b)+c\}+\{(-d)+(-e)\}+f\\
&=\{a+(-b)\}+\{c+(-d)+(-e)\}+f\\
&=\{a+(-b)+c\}+(-d)+\{(-e)+f\}=\&c.
\end{aligned}$$

now by the Law of Signs [§ 43], we see that

$$\text{the second step } = a+\{-b+c\}+\{-d-e\}+f,$$
$$\text{the third step } = \{a-b\}+\{c-d-e\}-f,$$
$$\text{and the fourth step } = \{a-b+c\}-d+\{-e+f\},$$

also that $a+(-b)+c+(-d)+(-e)+f=a-b+c-d-e+f$,

$$\begin{aligned} a-b+c-d-e+f &= a+(-b+c)+(-d-e)+f \\ &= (a-b)+(c-d-e)+f \\ &= (a-b+c)-d+(-e+f) = \&c \end{aligned}$$

Hence *the terms of an expression may be grouped in any manner*. This principle is called the algebraical **Law of Association**

47 In an expression, the terms may be written down in any order From definition

$$\begin{aligned} 5 &= 1+1+1+1+1 \text{ [§ 30]} \\ &= (1+1)+(1+1+1) \text{ [§ 46]} \\ &= (1+1+1)+(1+1) \text{ [§ 46]}, \end{aligned}$$

hence $(1+1)+(1+1+1)=(1+1+1)+(1+1)$,

i e, $+2+3=+3+2$

Similarly it may be shewn that

$$2+3+8=8+3+2=3+2+8=\&c,$$
$$a+b+c+d+\&c=c+b+a+d+\&c=d+b+a+c+\&c=\&c,$$
$$a+b-c=b+a-c=-c+a+b=-c+b+a=\&c,$$

and so on

Hence *the order in which additions may be made is indifferent*; that is, we can arrange the terms of an expression in *any manner*. This principle is known as the algebraical **Law of Commutation**

48 Addition of several terms Suppose we have to add together a, $-b$, c, d, $-e$, and $-f$

We first add $-b$ to a by the rule of § 39 and we get $a-b$; we then add c to $a-b$ by the same rule and we get $a-b+c$, we next add d to $a-b+c$ and we get the result $a-b+c+d$, and so on. Thus the required sum $=a-b+c+d-e-f$

Hence *to add any number of terms, we place them in a line with their signs unchanged, the expression thus formed being the sum required*

We may, however, arrange the terms of this sum so as to suit our convenience. For taking the sum just found, we see that

$$\begin{aligned} a-b+c+d-e-f &= a+c+d-b-e-f \text{ [§ 47]} \\ &= (a+c+d)+(-b-e-f) \text{ [§ 46]} \end{aligned}$$

Here we put the positive terms together and the negative terms together The practical nature of this arrangement will be seen from the following numerical examples

$$(1)\quad 8-7-5+9-3+11=8+9+11-7-5-3 \text{ [§ 47]}$$
$$=(8+9+11)+(-7-5-3) \text{ [§ 46]}$$
$$=(+28)+(-15) \text{ [§ 39, } \textit{Notes}\text{]}$$
$$=+28-15 \text{ [§ 43]}=+13$$

$$(2)\quad 6+1-5-2+4-8=6+1+4-5-2-8 \text{ [§ 47]}$$
$$=(6+1+4)+(-5-2-8) \text{ [§ 46]}$$
$$=(+11)+(-15) \text{ [§ 39, } \textit{Notes}\text{]}$$
$$=-11-15 \text{ [§ 43]}=-4$$

Thus for the addition of several terms, whether positive or negative, we have the following RULE.—*Place the terms in a line with their signs unchanged, group the positive terms together, and also the negative terms together, find the sum of each group; take the difference of these two sums; and affix to the result the sign of the greater sum.*

Examples

Ex 1 Add together a, $-b$, c, $-d$ and e, and find the value of the sum when $a=-3$, $b=2$, $c=4$, $d=-1$, and $e=5$

According to the Rule, the required sum $=a-b+c-d+e$

This sum cannot be further simplified and is *algebraically final*, but as numerical values are given to the letters, it may be still more simplified, as shewn below

The required value $=-3-2+4-(-1)+5$

$$=-3-2+4+1+5, \text{ by the Law of Signs [§ 43]}$$
$$=(4+1+5)+(-3-2)$$
$$=(+10)+(-5)$$
$$=+10-5=+5.$$

Here we take the difference of 10 and 5, which is 5, and affix the sign $+$ which is the sign of the greater sum $+10$.

Ex 2 Find the sum of x, y, $-z$, u and $-w$; also the value of the sum when $x=1$, $y=-2$, $z=-3$, $u=4$ and $w=9$.

The required sum $=x+y-z+u-w$.

The required value $=1+(-2)-(-3)+4-(+9)$

$$=1-2+3+4-9 \text{ [§ 43]}$$
$$=(1+3+4)+(-2-9)$$
$$=(+8)+(-11)=+8-11 \text{ [§ 43]}=-3.$$

Here we take as before the difference of 11 and 8 which is 3, and affix the sign $-$ which is the sign of the greater sum -11

3 Simplify $-11+8-3+6-9+7-1$

4 Simplify $5-6+8-10-20+41-7$

5 Add together m, n, $-p$, q, $-r$ and $-s$, and find the value of the sum when $m=-3$, $n=2$, $p=-7$, $q=8$, $r=-10$ and $s=1$

6 Find the sum of a, $-n$, $-x$, y, $-p$ and $-z$, and find its value when $a=6$, $n=-2$, $p=4$, $x=-5$, $y=3$ and $z=-1$

7 Find the value of $-(-a)+(-b)-(+c)-(-d)$, when $a=-2$, $b=-3$, $c=8$ and $d=-4$

49 Addition and subtraction of an algebraical expression By definition, an *algebraical expression* is a mere collection of its *terms*, thus the expression $a-b+c-d$ may be looked upon as the *algebraical sum* [§ 40] of its terms $+a$, $-b$, $+c$ and $-d$

Therefore to add the *whole* expression $a-b+c-d$ is the same as to *add its terms* $+a$, $-b$, $+c$ and $-d$ in succession Thus if E be an expression to which $a-b+c-d$ is to be added, we have

$$E+(a-b+c-d)=E+(+a)+(-b)+(+c)+(-d)$$
$$=E+a-b+c-d\ [\S\ 43] \qquad (A)$$

Hence *to add an expression, affix its terms in succession to the expression to which it is to be added, with their signs unchanged*

Similarly to subtract the *whole* expression $a-b+c-d$ is the same as to *subtract its terms* $+a$, $-b$, $+c$, and $-d$ in succession Thus if E be an expression from which $a-b+c-d$ is to be subtracted, we get

$$E-(a-b+c-d)=E-(+a)-(-b)-(+c)-(-d)$$
$$=E-a+b-c+d\ [\S\ 43] \qquad (B)$$

Hence *to subtract an expression, affix its terms in succession to the expression from which it is to be subtracted, with their signs changed*

Corollary From (A), it follows that

(i) $a+(b+c)=a+b+c$, (ii) $a+(b-c)=a+b-c$,

and from (B), it follows that

(iii) $a-(b+c)=a-b-c$, (iv) $a-(b-c)=a-b+c$

50 Examination upon Chapter II.

1 Define the terms—*addition*, *addends*, *sum* and *algebraical sum* Illustrate the last by an example.

2 Prove that $+4+(+3)=+4+3$, and that $+4+(-3)=+4-3$.

3. State the rule for the addition of a term What inferences can you draw from the rule when the addends are numbers instead of letters?

4 Define—*subtraction, minuend, subtrahend, remainder* and *algebraic difference* Illustrate the last by examples

5 What is the algebraic sum of 5 -9 and 1? What of a, b, $-c$ and -3? What is the algebraic difference of 8 and -5, of $-5a$ and $6a$; and of $-18xy$ and $-10xy$?

6. Prove that $+4-(+3)=+4-3$, and that $+4-(-3)=+4+3$.

7. State the rule for the subtraction of a term

8 State and explain the Law of Signs for Addition and Subtraction, and explain fully what is meant by $a>b$.

9 Arrange the numbers

$$3,\ -2,\ 1,\ -5,\ -7,\ 2 \text{ and } -1,$$

(1) in *ascending* order of magnitude, and (2) in *descending* order.

10 State the rule for the addition of several terms, whether positive or negative

11. Explain the principle which enables us to group the terms of an expression in any manner. Illustrate it by an example

12. Prove that in Addition the order of addends is indifferent

13 State and illustrate the rule for adding any number of terms whether positive or negative

14 Prove that $a+(b-c)=a+b-c$, and that $a-(b-c)=a-b+c$

CHAPTER III.

ADDITION

51 Addition of Like Terms. We have seen [§ 39] that to add a term to an expression we place it after the expression with its sign *unchanged*, and to add an expression to another we affix its terms in succession to the latter with their signs *unchanged* [§ 49] Thus the sum of $4a$ and $3a$ is $4a+3a$; of $-4a$ and $-2a$ is $-4a-2a$, of $5a$ and $-2a$ is $5a-2a$; of $3a$ and $-5a$ is $3a-5a$, and of $4a-2b$ and $3a+4b$ is $4a-2b+3a+4b$.

In each of these examples, the sum may be further simplified by *collecting the like terms together* and unless this is done the result is not considered as *final* Thus we have

$$4a+3a=(a+a+a+a)+(a+a+a)=a+a+a+a+a+a+a=7a,$$

where 7 is the sum of the coefficients 4 and 3,

$4a-2a=(-a-a-a-a)+(-a-a)=-a-a-a-a-a-a$
$=-6a$, where -6 is the sum of -4 and -2,

$5a-2a=(a+a+a+a+a)+(-a-a)=a+a+a+a+a-a-a=3a$,
where 3 is the algebraic sum of the coefficients 5 and -2,

$3a-5a=(a+a+a)+(-a-a-a-a-a)=a+a+a-a-a-a-a$
$-a=-2a$, where -2 is the algebraic sum of the coefficients 3 and -5.

Hence for the addition of Like Terms, we have the following RULE —*Find the algebraic sum of the coefficients, and affix the common letter or letters*

REMARK The algebraic sum of the coefficients may be found by the Rule of § 48

Examples (i).

Ex 1. Find the sum of x, $5x$ and $8x$

Required sum $=x+5x+8x=14x$, (where 14 is the sum of the coefficients)

Ex 2 Required the sum of $-a$, $-3a$, $-4a$ and $-11a$

Sum required $=-a-3a-4a-11a=-19a$

Find the sum of

3 $3x$, $10x$, x, $18x$, $24x$ 4 $5a$, a, $3a$, $7a$, $10a$

5 m, $3m$, $8m$, $7m$, $5m$ 6 $12n$, $18n$, n, $2n$, $4n$

7 $2r$, r, $5r$, $4r$, $6r$, $15r$ 8 y, $6y$, $9y$, $12y$, $8y$, $10y$

9 $-5c$, $-c$, $-8c$, $-28c$ 10 $-3x$, $-5x$, $-x$, $-6x$, $-10x$

11 $-4y$, $-6y$, $-2y$, $-y$, $-8y$ 12 $-6ab$, $-7ab$, $-2ab$, $-ab$, $-9ab$

13 $-xy^2$, $-3xy^2$, $-7xy^2$, $-xy^2$, $-4xy^2$

14 $-max$, $-3max$, $-25max$, $-4max$, $-19max$, $-21max$

15 $2ab$, $\frac{1}{2}ab$, $20ab$, ab, $15ab$ 16 $\frac{3}{5}xyz$, xyz, $\frac{1}{2}xyz$, $\frac{3}{4}xyz$, $\frac{18}{20}xyz$, xyz

17 $-\frac{3}{4}mn$, $-mn$, $-4mn$, $-\frac{10}{7}mn$, $-14mn$

18 $\frac{1}{2}pq$, $\frac{2}{5}pq$, $\frac{3}{7}pq$, $\frac{9}{10}pq$ 19 ax, bx, cx, dx 20 ax, x, $5x$, px, $3x$

Examples (ii)

Ex 1 Add together $3a$, $-2a$, $-7a$, $4a$, $-a$ and $6a$

The sum $=3a-2a-7a+4a-a+6a$ [§ 48], and the sum of the coefficients $=+13-10=+3$, required sum $=+3a$, or $3a$

Find the sum of

2. $3a, -4a, 5a, -a, -2a$ 3. $7x, 2x, -x, 4x, -10x$

4. $8m, 20m, -3m, -15m, m, 17m$

5. $xy, -18xy, 5xy, 16xy, -xy, 12xy$.

[In the following examples the addends are placed in a column.]

6.		7.		8.		9.	
	$-abc$		$16rs$		$5axy$		$\frac{2}{3}cz$
	$50abc$		$-74rs$		$\frac{2}{5}axy$		$-\frac{1}{4}cz$
	$27abc$		$24rs$		$-3axy$		$-\frac{5}{10}cz$
	$-76abc$		$31rs$		$-\frac{1}{10}axy$		cz
	abc		$3rs$				$-\frac{49}{30}cz$

10.		11.		12.		13.	
	$\frac{3}{2}mn$		$35a$		$8z$		$-6by$
	$-\frac{1}{3}mn$		$-17a$		$30z$		$20by$
	$\frac{5}{6}mn$		$-a$		$57z$		$-18by$
	$-\frac{7}{12}mn$		$20a$		$-108z$		$-42by$
	$-\frac{3}{8}mn$		$-45a$		$21z$		$-11by$
					$-25z$		by

14.		15.		16.		17.	
	$\frac{3}{8}ab$		ax^2		$2ax$		$3x^2y$
	$\frac{1}{4}ab$		$-bx^2$		$-ax$		$-x^2y$
	$-ab$		cx^2		$4px$		ax^2y
	$\frac{5}{6}ab$		$-dx^2$		$3ax$		$5x^2y$
	$-\frac{4}{5}ab$		fx^2		$-qx$		$3ax^2y$
	$-2ab$						

52; 53. [Incorporated with § 51.]

54. Addition of Unlike Terms From § 48, we see that these terms are added when we merely place them in a line with their signs unchanged.

Examples

Ex. 1. Add together a, b, c and f.

Sum $= a + b + c + f$, or $= a + f + c + b$ or $=$ &c.

Ex. 2. Add together $5x$, $3y$, $-2z$ and xyz.

Sum $= 5x + 3y - 2z + xyz$, or $= xyz - 2z + 3y + 5x$, or $=$ &c.

Add together

3.		4.		5.		6.	
	$43ab$		$12abc$		$3a^2$		$5ab$
	$5az$		$-axy$		$10b^2$		$-3xy$
	$20xy$		$20pqr$		$-18c^2$		$8x^2$
	$18pq$		$15mr$		$25ab$		$-2a^2c$
			$29df$		$-bc$		$4abc$
					$17ca$		

Note When some of the terms are *like*, the sum may be *simplified* by collecting them together

Thus to find the sum of $3a$, $2b$, $-4c$, $5a$, $3c$ and $-2a$

The required sum $=3a+2b-4c+5a+3c-2a$

$=3a+5a-2a+2b-4c+3c$ [§ 47]

$=6a+2b-c$ [§ 51]

7	8	9	10	11
$2a$	$15xy$	$10ax$	mn	$5abc$
$4b$	$3yz$	$7by$	$-np$	$6abx$
$-c$	$-10xy$	$4cz$	$5mn$	$-2abc$
$4a$	$5xz$	$-11ax$	$8nq$	$7xyz$
$7d$	$-6xy$	$-7cz$	$-10np$	$-4abx$
$-3a$	$3xz$	$2ax$	$2nq$	$-abc$
$-2b$		$-9by$		

55 Addition of Polynomials. We have seen that a polynomial is added when we affix its terms in succession with their signs unchanged [§ 49] For example, $-3a+5b-7c$ is added to $8a-4b+3c$ when we simply subjoin it to $8a-4b+3c$ Thus the required sum $=8a-4b+3c-3a+5b-7c$, (which should be simplified as in § 54, *Note*)

Examples

Ex 1 Add together $3a+5b$, $a-2b$ and $4a+b$

The required sum $=3a+5b+a-2b+4a+b$

$=3a+a+4a+5b-2b+b$ [§ 47]

$=8a+4b$ [§ 51]

Note A more practical way of finding the sum would be, however, to place the given expressions one under another, so that *like* terms may fall under *like* terms, and then collect them as usual Thus

$$\begin{array}{r} 3a+5b \\ a-2b \\ 4a+\ b \\ \hline 8a+4b \end{array}$$

$8a+4b$, the same result as before

Ex 2 Add $13x+2y-3z$, $5y-18x+4$, $15x+10z-y$, and $z-6x-12y$

Req sum $=13x+2y-3z+5y-18x+4+15x+10z-y+z-6x-12y$

$=13x-18x+15x-6x+2y+5y-y-12y-3z+10z+z+4$

$=4x-6y+8z+4$,

or it may be found, thus

$$\begin{array}{rrrr} 13x+ & 2y- & 3z & \\ -18x+ & 5y & & +4 \\ 15x- & y+ & 10z & \\ -6x- & 12y+ & z & \\ \hline 4x- & 6y+ & 8z & +4. \end{array}$$

Ex. 3 Add together $6ax+3by-2$, $4by+3cz+5$, and $8cz-2af-2fx$.

Required sum $=6ax+3by-2+4by+3cz+5+8cz-2af-2fx$

$=6ax-2fx+3by+4by+3cz+8cz-2af-2+5$

$=6ax-2fx+7by+11cz-2af+3$

or thus

$$\begin{array}{rrrrrr} 6ax & +3by & & & & -2 \\ & +4by & +3cz & & & +5 \\ & -2fx & & +8cz & -2af & \\ \hline 6ax & -2fx & +7by & +11cz & -2af & +3 \end{array}$$

Ex. 4 Add together $7a-2b+4c$, $-5a+3b-3c$, $8a-2c$ and $6a+7b+c$.

$$\begin{array}{rrr} 7a & -2b & +4c \\ -5a & +3b & -3c \\ 8a & & -2c \\ 6a & +7b & +c \\ \hline 16a & +8b & +0 \end{array}$$

The last term in the sum is 0, for in the third column the *algebraic sum* of the coefficients of c is 0. It is usual to omit the term 0, and the sum is written thus $16a+8b$

Find the sum of

5. $\begin{array}{r} x-y \\ \underline{x+y} \end{array}$ 6. $\begin{array}{r} 29+c \\ \underline{29-c} \end{array}$ 7. $\begin{array}{r} 2a+18 \\ \underline{3a-18} \end{array}$ 8. $\begin{array}{r} 2ax+3by \\ \underline{3ax-2by} \end{array}$

9. $\begin{array}{r} \frac{2}{3}xy-\frac{1}{4}ab \\ \underline{\frac{5}{4}ab-\frac{5}{3}xy} \end{array}$ 10. $\begin{array}{r} 4a-5b+6c \\ 8a-10b+13c \\ \underline{26a+17b-15c} \end{array}$ 11. $\begin{array}{r} ab+2cd-3 \\ 2ab-5cd+16 \\ \underline{-4ab+cd-13} \end{array}$

12. $\begin{array}{r} 2a+3b-4c \\ -3a+4b-c \\ 4a+7b+7c \\ a-b-4c \\ \underline{-5a+2b-6c} \end{array}$ 13. $\begin{array}{r} 3x-2y+z \\ -5x+8y-4z \\ 6x-5y-5z \\ -2x+3y \\ \underline{-2x-y+8z} \end{array}$ 14. $\begin{array}{r} 3x^2+3xy \\ -5x^2-xy \\ 7x^2+8xy \\ -2x^2-11xy \\ \underline{11x^2+xy} \end{array}$

Find the sum of

15
$$\begin{array}{r} 2ax - 4by + 3cz \\ 13ax - 9by + 7cz \\ -5ax + 7by - 14cz \\ 2ax - by + cz \\ \underline{-21ax + 13by - 4cz} \end{array}$$

16
$$\begin{array}{r} 7a - 3b + 4c - 2d + 7 \\ -8a + 4b - 6c + 2d - 11 \\ 13a + 3b - 5c + 4d - 4 \\ 2a - b + c \qquad + 11 \\ \underline{a \qquad\qquad + d - 3} \end{array}$$

17
$$\begin{array}{r} 2a^2 + 2ax - 3y^2 \\ a^2 + ax + 7y^2 \\ -7a^2 + 7ax - y^2 \\ 5a^2 + 10ax - 6y^2 \\ \underline{6a^2 + 5ax + 5y^2} \end{array}$$

18
$$\begin{array}{r} x^2 - 4ab + \frac{3}{4}a^2 \\ \frac{3}{2}x^2 - \frac{1}{2}ab + \frac{9}{4}a^2 \\ 2x^2 - \frac{1}{2}ab + \frac{3}{4}a^2 \\ 6x^2 - 12ab + 3a^2 \\ \underline{\frac{5}{3}x^2 - \frac{2}{3}ab + \frac{5}{3}a^2} \end{array}$$

19.
$$\begin{array}{r} 5x^2 - 2xy + 8y^2 \\ 4x^2 + 3xy + y^2 \\ -3x^2 + 4xy - 3y^2 \\ 7x^2 - 6xy + 2y^2 \\ \underline{-x^2 + xy - y^2} \end{array}$$

20
$$\begin{array}{r} 1 + 2ab - 3bc + 4ac \\ -6 - ab + 8bc - 2ac \\ 12 + 10ab - 10bc + 7ac \\ -3 \qquad\quad + 2bc + ac \\ \underline{2 - 3ab - bc + 5ac} \end{array}$$

21
$$\begin{array}{r} x^3 - 3ax^2 + 3a^2x - a^3 \\ 4x^3 - 5ax^2 + 6a^2x - 15a^3 \\ 3x^3 + 4ax^2 + 2a^2x + 6a^3 \\ -17x^3 + 19ax^2 - 15a^2x + 8a^3 \\ \underline{-13ax^2 - 27a^2x + 18a^3} \end{array}$$

Add together

22 $a-b$, $b-c$, $c-d$ and $d-e$

23 $x-2y+3z$, $3y-x-4z$, $3x-5y-5z$, $x-z$ and $4y-x$

24 $2a+c+d$, $2b+a+e$, $c-d$, $e-3a-f$ and $d-2c-2e$

25 $2a^2+3b^2$, $4b^2-3a^2$, $4a^2-8b^2$, $5b^2-6a^2$ and $3a^2+7b^2$

26 $5x^3-2mn+3r^2$, $3mn+10r^2-2x^3$, $18r^2+6x^3-8mn$ and $mn+x^3-20r^2$.

27 $3x^2-xy+4x-3y$, $4xy+4y-x^2$, $7xy-5x^2+3x$, $2x^2+6y-5x$ and $2x-8y-8xy$

28 ax^3-2bx^2-3cx, $3bx^2-3ax^3+4d$, $2cx-d+bx^2$, $5d-5cx-4bx^2$, $4ax^3-2d$ and $6cx-2ax^3-1$

29 $xy-3x^2-x^3$, $4x^3-4x^2+3y^2$, $8x^2-2xy-y^2$, $6x^3-x^2+4y^2$, $-5xy-2x^3$ and $4xy-8y^2$

30 $2p^2-3pq-4q^2$, $3pr-2q^2-r^2$, $p^2-2qr+5r^2$ $3pq-6pr-3p^2$, $3pr-2p^2+5qr$ and $4q^2-3qr+2p^2$

Add together

31. $3x^3+2y^3+z^3+8yz^2$, $y^3+3x^2y+2xy^2+z^3-25x^2z$,
$x^3+2xyz+4x^2y+12x^2z-9y^2z+6yz^2$,
$2x^3-3y^3+4xyz-6xy^2$, $4y^3-x^3+5x^3z-15xyz+3y^2z-14yz^2$
and $6x^2z-15xyz+4xy^3-7x^2y+6y^2z$

56 Examination upon Chapter III

1. State the Rule for the addition of Like Terms.

2 What is a *literal* and what a *numerical* coefficient? Find the coefficients of x in $3x$, $-4ax$, $2ab^3x$, x, $-3a^2bcx$, m^8x, $-x$ and $(a+2)x$, and state which of these are *literal* and which *numerical*

3 Find the coefficients of x^2 in $x^2+3x^2-y^2+1-2x^2$

4. Find the coefficients of x and y in the sum of $5x-2y+1$ $8y-2x-3$ and $10-4x-6y$.

5. Find the coefficient of ax and of x, in the sum of $5ax$, $4by$, $-2ax$, x and $-3by$.

6. State the rule for the addition of Polynomials

7 Shew that the sum of $4ax+3bx-2c$, $2cx-2ax+b$ and $-4bx+cx+3a$ is equal to that of $2ax-5bx+c$, $3bx+4cx-9a$ and $-2cx+4ax+4b$, if $x=3$.

8 To what expression must $2x^2-x^3+5x-1$ be added, so as to give 0?

9 What expression must be added to $3x^2-5x+2y$ so as to give $x-y$ for the sum?

10 A sum of money was divided between A and B, if A received x rupees and B as much as A and 3 rupees more, what was the sum?

11 A has as much as B, and C has 2 rupees less than B If A has a rupees, find how much they have altogether

12. If a person owe m rupees to A, n rupees to B, and $2p$ rupees to C, what is the amount of his debt?

CHAPTER IV.

SUBTRACTION—BRACKETS.

SUBTRACTION

57 **Rule for Subtraction.** We have seen that to subtract a whole expression is the same as to subtract each of its terms in succession [§ 49] Hence to subtract an expression from another

is the same as to add the former to the latter when the sign of each of its terms is changed Hence the **Rule** —*Change the sign of every term of the subtrahend, and when thus changed, add it to the minuend*

Example From $5a-3b-2c$ take $3a-4b+c$

Remainder $=5a-3b-2c-3a+4b-c$

$=5a-3a-3b+4b-2c-c$ [§47]

$=2a+b-3c$ [§ 55]

The usual method however is to place the subtrahend under the minuend, taking care that *like* terms be placed under *like* terms, and then to combine the like terms after *mentally* changing the signs of those in the subtrahend Thus the above example is usually worked as shewn below—

$$\begin{array}{ll} \text{(Minuend)} & 5a-3b-2c \\ \text{(Subtrahend)} & \underline{3a-4b+c} \\ \text{(Remainder)} & 2a+b-3c \end{array}$$

Examples.

Ex 1 Min a / Sub $\underline{-b}$ / Rem $a+b$

Ex 2 Min $8x-3y$ / Sub $\underline{3x-2y}$ / Rem $5x-y$

Ex. 3 From $15a+16$ / take $\underline{3a+1}$ / Rem $12a+15$

Ex 4 From $6ax-3by+3$ / take $\underline{8ax-5by-2}$ / Rem $-2ax+2by+5$

Ex 5. From $8x^2+20ab-y^2$ / take $\underline{3x^2+25ab+2y^2}$ / Rem $5x^2-5ab-3y^2$

Ex 6 From $ax^2+2hxy+by^2+c$

take $\underline{gx^2-2fxy+hy^2-d}$

Rem $ax^2-gx^2+2hxy+2fxy+by^2-hy^2+c+d$

Ex 7 From $35x^2+20x-3y$ take $18y-3x+2c-1$.

Place like terms under like terms, thus

$$\begin{array}{l} 35x^2+20x-3y \\ \underline{-3x+18y+2c-1} \\ \text{Rem } 35x^2+23x-21y-2c+1 \end{array}$$

8 From $2a$ take $a-x$ 9 From $x+y$ take $x-y$

10 From $10a+20$ take $-5a+8$ 11 From $a+x-1$ take $-a+x-2$

12 From $x+\frac{1}{2}y-1$ take $\frac{1}{2}x-y+1$.

13 From $3mn-mx+n$ take $4mn-my-n$

14. From $2x+3y-4z$ take $x-2y+3z$.

15. Take $3x+2y-4z$ from $2x-z$.

16. Take $-a-3b-c$ from $4a-2b+1$.

17. Take $-4b+5d-9$ from $a-3b+2c$.

18. From $3p^2-4pq+q^2+1$ take $3p^2+q^2-5pq$.

19. From $\frac{2}{3}x-\frac{1}{2}xy+$ [illegible] take [illegible].

20. From $x^2-y^2+2z^2$ take $3x^2-2y^2-z^2$.

21. Subtract $2ab+ab-ac+1$ from [illegible] $-3ab-3ac$.

22. Subtract a^2-ax+x^2 from $2a^2-3ax-x^2$.

23. Subtract $2ax+x^2+3a^2$ from $4ax-5x^2+a^2$.

24. Subtract px^2-qx^2+rx from ax^2-bx^2+x.

25. From x^3+5x^2-10 subtract $3x^2-4x-8$.

26. Take x^3-2x^2-2x-4 from x^3-1.

27. Subtract [illegible] from [illegible].

28. From [illegible] take [illegible].

29. From $7a^3+2a^2b-3ab^2$ subtract $7b^3-2a^2b+2ab^2$.

30. Take $-x^4+2x^3+4x^2-3x-1$ from $x^4-2x^3+4x^2-x-1$.

31. Subtract [illegible]

from [illegible].

Brackets.

58. Brackets. We have from § 49, (2)

$$E+(a-b+c-d)=E+a-b+c-d \qquad \text{(i)};$$

hence also $\quad E+a-b+c-d=E+(a-b+c-d) \qquad \text{(ii)}.$

Again from § 49, (1) we have

$$E-(a-b+c-d)=E-a+b-c+d \qquad \text{(iii)};$$

$\therefore$ conversely $\quad E-a+b-c+d=E-(a-b+c-d) \qquad \text{(iv)}.$

Hence we get the rules for the removal and insertion of brackets [See §§ 59 and 61].

59. Rules for the removal of Brackets. From (i) and (iii) of § 58, we get the following rules for removing brackets.

Rule I. *If a + sign precedes a bracket, remove the bracket without changing the sign of the included terms.*

Rule II. *If a − sign precedes a bracket, remove the bracket and change the sign of each of the included terms.*

Examples

Ex. 1 Simplify $5a+(2a-b)-(a+b)$

The given expression $=5a+2a-b-(a+b)$, by Rule I,

$=5a+2a-b-a-b$, by Rule II,

$=6a-2b$ [§ 55]

Ex 2. Simplify $2x-(-x+y)+\overline{3x+2y}-(7x-y)$

The given expression

$=2x+x-y+\overline{3x+2y}-7x+y$ [Rule II]

$=2x+x-y+3x+2y-7x+y$ [Rule I]

$=-x+2y$

Ex 3 Simplify $-3x+(-y+2x)-\overline{8x-3y+z}-(-9x+y)$.

The given expression

$=-3x-y+2x-\overline{8x-3y+z}-(-9x+y)$ [Rule I]

$=-3x-y+2x-8x+3y-z+9x-y$ [Rule II]

$=y-z$

Remove the brackets from

4 $x+(y-z)-(x-z)$ 5 $(2a-3x)+(-a+2x)$

6 $(ax+by)-(by-1)$ 7 $(a+b+c+f)-(-a+b-c+f)$

8. $a-(b-c+d)-\overline{e-2a+b}+(d-c)$

9 $x+y+z-(x+y-z)-(-y-z+x)$

10 $2a-3b-(a+2b)-(8b-6a)$

11. $3a-(b-4c)+2d+(a-b)-\overline{2c-d}$

12 $4m-3\overline{m-4}+\overline{m-6}-(2m-8)$

13. $(ax+bx)+(by+cy)+(ax-bx)-(by-cy)$

14 $(x^2-2xy+3y^2)-(x^2+3xy-2y^2)-(4y^2-5xy)$

60 Two or more Brackets Sometimes expressions occur, involving more than one pair of brackets The usual method in such cases is to begin with either the *innermost* pair, or the *outermost* pair, and to proceed to the *next in order* till all are removed

Examples

Ex 1 Simplify $2a-[a-\{a-(x+\overline{a-x})\}]$

Given expression $=2a-[a-\{a-(x+a-x)\}]$ [Rule I]

$=2a-[a-\{a-(+a)\}]$

$=2a-[a-\{a-a\}]$ [Rule II]

$=2a-[a]=2a-a=a$

or thus :—Given expn $=2x-a+\{a-(x+\overline{a-x})\}$ [Rule II.]

$=a+a-(x+\overline{a-x})$ [Rule I]

$=2a-x-\overline{a-x}$ [Rule II]

$=2a-x-a+x$ [Rule II] $=a$

REMARK. It is usually advantageous to begin with the innermost pair.

Simplify

2. $x-\{x-(x-y)\}$.
3. $3a-\{2b-(a-4b)\}-(a+b)$.
4. $1-[1-\{1-x-(1-\overline{1-x})\}]$.
5. $4x-[y-\{x+(y-3a)\}]$.
6. $(a+x)-\{a-x-[a-2x-(2a-x)]\}$
7. $-[+\{+(-x)\}]-[-\{+(-\overline{-x})\}]$.
8. $a+x-[b+y-\{a-x-\overline{b-2y}\}]$.
9. $xy-\{xy-(2xy-\overline{2xy-y^2})\}$.
10. $16a-[3b+\{5a+2b-(3a-\overline{2a-4b})\}]$.
11. $17a-4b-[3b+2a-\{5b-6a-(2b-a)\}]$.
12. $11x-\{7x-[8x-(9x-\overline{12a-6x})]\}$.
13. $ab-\{(3bce-2ab)-(5bce-bef)+(3ab-3bef)\}$.
14. $2a-\{3b+(2b-c)-4c+[2a-(3b-\overline{c-2b})]\}$.
15. $16-x-[(-7x)-\{8-(+9x)-(-6x-\overline{-3})\}]$
16. $a+\{-3b-[-6c+(-3a-\overline{-2b-5c})]\}$.
17. $-[15x-\{14y-(15z+12y)-(10x-15z)\}]$.
18. $-\{a-[-2a-(-3a-\overline{-4a-5b})]+(-5a-3b)\}$.
19. $-5a-\{-4b-[-8c-(-4b+\overline{-6a-7c})]\}$
20. $-m-\{-3m+2n-(-5m-8n)+(-3m-2n-\overline{-3n+m})\}$.

61. Rules for the insertion of Brackets From (ii) and (iv) of § 58, we get the following rules for inserting brackets

RULE I *If a + sign is to precede a bracket, keep the signs of the included terms unchanged*

Thus $a-b+c-d=a+(-b+c-d)=a-b+(+c-d)=a-b+c+(-d)$.

RULE II. *If a − sign is to precede a bracket, change the sign of each of the included terms.*

Thus $a-b+c-d=a-(+b-c+d)=a-b-(-c+d)=a-b+c-(+d)$.

Examples

Ex 1 Enclose in a bracket with a *positive* sign, the last 3 terms of $x-y+z-1$

Given expression $=x+(-y+z-1)$ [Rule I] $=x+(z-y-1)$

Ex 2 Enclose the same terms in a bracket with a *negative* sign

Given expression $=x-(+y-z+1)$ [Rule II] $=x-(y-z+1)$

3. Enclose in a bracket, (1) with a positive, and (2) with a negative sign, the last 4 terms of $a-b+c-d-e+f$

4 Of the expression $a-b+c-d-e+f-g+h$, enclose in a bracket

(1) every 2 terms beginning with the first ,
(2) .. .. second ,
(3) . 3 ... . first ,
(4) 3 . .. .. second ,
(5) the last 5 terms ,
(6) the second, third, fourth, and fifth terms ,
(7) the fifth, sixth, seventh and eighth terms ,
(8) all the terms, except the first

62 Examination upon Chapter IV

1 What is left when $5a-3b$ is taken from the sum of $3-2a+x$, and $4a-5b-7$?

2 If the difference of $3x^2-2y^2$ and $3xy-5y^2$ be taken from unity, what is the remainder ?

3 The sum of two quantities is $3a-x-y+1$, and one of them is $3a-5x+2y$, find the other

4 What expression must be subtracted from 0 so as to leave a remainder x^2-2x+1 ?

5 Find the remainder when $3x^2-4x+1$ is subtracted from 0, and the value of the remainder when $x=1\frac{1}{3}$

6 By how much does x exceed $5-2x$? Find the value of th difference when $x=\frac{5}{3}$

7 I have a rupees, out of which I give x and y rupees to persons , what have I left ?

8 A is x years older than B , if A's age be a years, w B's age ?

9 A house and a garden together cost R1500 , if the va the house be x rupees, what is the value of the garden ?

10 If $x=3a-b$ and $y=a-3b$, find the value of $x-y$

11 State the rules for the insertion and removal of Bracke

Miscellaneous Examples I

1 the

1. Find the value of $\frac{7b+c}{3a+4b}$ when $a=5$, $b=4$, $c=3$

2 Find the sum of $4ar$, $-a^2x$, $8ar^2$, $-2ar$, $-(-3a^2x)$, $-2ax^2$ and $-3ax$, and the value of the result when $x=1$

3 Subtract $-5a+b$ from a^2, and simplify the result when $a=$ $b=-2$

4 Remove the brackets from $x-\{2y-(2y+3z)-x\}$

5 By how much does $x-y$ exceed $x+y$?

the sum

---

6 If $a=4$ and $b=6$, find the value of $\frac{3a+2b}{2b-a}+\frac{a^3-}{2}$ from 0, and

7 What must be added to $a-b$ that the result m

8 Subtract the sum of $3x^2$ and $1-2x$ from unity value of the difference when $x=1$?

$y)]\}$

9 Simplify $3p-\{q+(2p-q)-(p-q)\}$

10 Add together $4x^2-2xy+y^2$ and $5x^2-2y$ $Z=9a^2-5ab+3b^2$, sum by $8x^2-2xy+y^2$.

---

11. If $a=1$, $b=4$, $c=6$, $d=0$, find the val $-(1-F)]\}$

$$\frac{1+bd+a^2b^2}{a^2+b^2+d^2}-\frac{1-cd+a^2}{a^2+c^2+a}$$

12 Add together AND DIVISION.

$3-(2+x)$, $x-(3-2x)$, $4x+(8-x)$, 2

13 Subtract $4a-(3b-2c)+(2a-c)$ from 3

14 Remove the brackets from o be multiplied by

$2a-\{5b+[c-(a+b-2c)]\}-e$ as there are units in as many 8s as there

15 Find the quantity which when s plication is the *shortest* $3xy$ leaves a remainder $2x^2-3ax+3xy+1$ n a certain *number of* fficient as long as the

---

mber and not a *fraction.*

16 If $a=6$, $b=4$, $c=3$, $d=0$ and $e=$ -"*To multiply one number*

$5(a-b)(c^2+bd-e)$ — *one to unity to obtain the*

17. Add together $-b+2a+3c$, $6a-($

18 Take $2+\{x-(2y+3z)-5\}$ fro he following examples.

19. Simplify $4x-[3y+\{2-3z-(1$ ur times, that is,

[§ 30].

20. What is that quantity from w

the remainder is $\frac{3}{4}a^3+2ab-\frac{1}{4}b^2$? $+28$ ....(1),

$-7=-28$ .... (11).

.r $a=8$, $b=0$ and $c=3$, find the value of

$$2a-(3b-c)-(8c-a)-\{4c-(a-b)\}$$

22 Find the sum of $5x-(4y+z)$, $2x+(2z-y)$, and

$$9z-(6x-6y+5z)$$

23 Subtract

$a-(b-c)+\{2c-(3b-5a)\}$ from $a-2b-\{3a-(b-c)-5c\}$

Remove the brackets from

3

negative $4x^2-\{2x^2-(2x-3)\}-\{x^2-(5x-2x^2)\}-\{3-(2x^2-x)\}$

4 Of tl sum of two expressions is $2x^2+3xy$, and one of them is x^2, find the other

(1)

(2)

(3) e numerical value of

(4) $-3y)-\{4y+2x-(x-2y)\}$, when $x=3$ and $y=14$.

(5) $b-2c)-d$, $b-\{a-(2c-d)\}$, $c-\{2b-(3d-a)\}$, and

(6) th

(7) the

(8) all t ference when $a^2-2ab+b^2-3$ is taken from

62 Ex $\{2m-\overline{m-(2m+n)}\}-(3m-2n)]$

1 What is left wh of two expressions is $2a+3ab-5b$, and the and $4a-5b-7$? what is the other?

2 If the difference what is the remainder?

3 The sum of two $=4$, and $l=-5$, find the value of is $3a-5x+2y$, find th

4 What expressic $-(\sqrt{4bl+b^3}+b\sqrt{ab-l})$ remainder x^3-2x+1? $x^2-x^2)\}$, $-3ax+\{x^2-(a^2-2ax)\}$,

5 Find the remaind $+2x^3$ and the value of the rem

6 By how much doe $3z-\frac{2y}{3}+\frac{x}{4}-1$ exceed $\frac{x}{2}-\frac{y}{3}+\frac{z}{4}-2$? difference when $x=\frac{5}{3}$ $3b-[3c-(8a+b)+4a]+2a-(2c+b-a)$

7 I have a rupees, out expressions is $5xy-\frac{5}{2}x^2-\frac{1}{2}y^2$ and the persons, what have I left? the greater

8 A is x years older th re B's age?

9 A house and a garde the house be x rupees, wha $=8$, find the value of $\sqrt[3]{2a+9b(c-3d)^3}$

10 If $x=3a-b$ and $y=$ n

11 State the rules for $2c^2-\{(a^2-3b^2)+\frac{1}{2}c^2\}$

38 From $a-\{3b-(2a-c)\}$ take $3a-(2b-c)$, and add the remainder to $b+1$.

39. Simplify $5x-\{3y-(x-y)\}+8x-(3y+z)-\{z-(x-2y)\}$

40. If $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, shew that

$$x+y+z=0$$

---

41. If $a=9$, $b=1$, $c=25$, $d=8$, find the value of

$$\sqrt[3]{2a+9b}(c-3d)^3$$

42 Add together $3a-\frac{b}{2}+\frac{c}{4}$ and $4c-\frac{a}{2}$, and subtract the sum from a

43 Subtract $3x^3-2x^2+4$ from unity, and $3x^2-4x^3$ from 0, and add the two results together.

44 Add to the sum of

$$3x-\{2y-(3z-2x)\} \text{ and } x-\{3y-[5z-(x-y)]\}$$

the difference of $5z$ and $3y-\{x+(2y-3z)\}$

45 If $X=3a^2-2ab+5b^2$, $Y=7a^2-8ab+5b^2$, $Z=9a^2-5ab+3b^2$, $V=5a^2+ab+3b^2$, find the value of

(1) $Y-(X+Z-V)$, (2) $Y-\{Z-[X-(Y-V)]\}$.

CHAPTER V

FUNDAMENTAL LAWS MULTIPLICATION AND DIVISION.

MULTIPLICATION.

63 Definitions One number is said to be multiplied by another, when the first is taken as many times as there are units in the second Thus to multiply 8 by 5 is to take as many 8s as there are units in 5 In this sense therefore Multiplication is the *shortest method* of finding the *sum* of a number, taken a certain *number of times* But this definition is obviously sufficient as long as the number by which we multiply is a *whole number* and not a *fraction*. We therefore define Multiplication thus —"*To multiply one number by a second is to do to the first what is done to unity to obtain the second*".

We shall illustrate this definition by the following examples.

(1) To obtain 4, unity is repeated four times, that is,

$$4=1+1+1+1 \text{ [§ 30]},$$

therefore $7\times4=7+7+7+7=+28$ ....(i),

and $(-7)\times4=-7-7-7-7=-28$ ....(ii).

(2) To obtain -4, unity is subtracted four times, that is,

$$-4=-1-1-1-1 \text{ [§ 35]},$$

therefore $$7\times(-4)=-7-7-7-7^*=-28 \quad . \quad . \qquad \text{(iii)},$$

and $$(-7)\times(-4)=-(-7)-(-7)-(-7)-(-7)^*$$
$$=+7+7+7+7 \text{ [§ 43]}=+28. \quad . \qquad \text{(iv)}$$

Thus the definition holds when the multiplier is an *integer*, positive or negative It will likewise hold when the multiplier is a *fraction*, positive or negative

(3) To multiply $\frac{7}{8}$ by $\frac{3}{5}$, we must do to $\frac{7}{8}$ what is done to unity to obtain $\frac{3}{5}$ Now to obtain $\frac{3}{5}$, unity is divided into *five* equal parts and *three* of these parts are taken, hence to multiply $\frac{7}{8}$ by $\frac{3}{5}$, we divide $\frac{7}{8}$ into *five* equal parts and take *three* of there parts But when $\frac{7}{8}$ is divided by 5, each of the parts is $\frac{7}{8\times5}$, and when three of these parts are taken, we get $\frac{7\times3}{8\times5}$,

$$\frac{7}{8}\times\frac{3}{5}=\frac{7\times3}{8\times5}$$

(4) To multiply $\frac{7}{8}$ by $-\frac{3}{5}$ To obtain $-\frac{3}{5}$, we divide unity into 5 equal parts and *subtract* three of these parts in succession, therefore here we divide $\frac{7}{8}$ into 5 equal parts, and *subtract* three of these parts in succession, now $\frac{7}{8}\div5$ gives $\frac{7}{8\times5}$ as one of the parts,

$$\frac{7}{8}\times\left(-\frac{3}{5}\right)=-\frac{7}{8\times5}-\frac{7}{8\times5}-\frac{7}{8\times5}=-\frac{7\times3}{8\times5}$$

(5) Similarly it may be shewn that

$$-\frac{7}{8}\times\frac{3}{5}=-\frac{7\times3}{8\times5}, \text{ and } -\frac{7}{8}\times-\frac{3}{5}=+\frac{7\times3}{8\times5}$$

When one number is multiplied by another, the former is called the **Multiplicand**, and the latter is called the **Multiplier**, and the result is called the **Product** Thus in the *product* ab, a is the *multiplicand*, and b is the *multiplier*

Remark The student should carefully note that in a product ab, we shall always call as in Arithmetic, the *first* number the multiplicand, and the *second* number the *multiplier*

* Here we see that a negative multiplier makes a contrary repetition of the multiplicand, which idea we first meet with in the works of a Hindu algebraist "It has been before observed that negation is contrariety A negative multiplier therefore, is a contrary one, that is, it makes a contrary repetition of the multiplicand Such being the case, if the multiplicand be positive, (the multiplier being negative), the product will be negative, if the multiplicand be negative, the product will be affirmative"—Krishna Bhatta's *Commentary* on Bhaskara's *Vijaganita*

64 The Law of Signs From (i), (ii), (iii) and (iv) of § 63, we get the following rule that regulates the sign of the product :— *If the multiplicand and multiplier both have the same sign, the sign of the product is +, and if they have different signs, the sign of the product is −.*

Thus whatever a and b may be, *integral* or *fractional*, *positive* or *negative*, we have

$$(+a)\times(+b)=+ab \qquad \text{(i)},$$
$$(-a)\times(+b)=-ab \qquad \text{(ii)};$$
$$(+a)\times(-b)=-ab \qquad \text{(iii)},$$
$$(-a)\times(-b)=+ab \qquad \text{(iv)}.$$

The above rule is called the Law of Signs and is briefly enunciated thus *Like signs produce +, and unlike signs produce −.*

Corollary. Hence it is easy to see that *a product has the + sign if the number of* NEGATIVE *factors producing it be* EVEN* *and the − sign, if their number be* ODD*

For $a\times(-b)=-ab$, $a(-b)(-c)=-ab\times(-c)=+abc$;
$a(-b)(-c)=+abc\times(-d)=-abcd$,
$a(-b)(-c)(-d)(-f)(-d)=+abcd\times(-f)=+abcdf$, and so on.

Examples

If $a=-2$, $b=5$, $c=-3$, $x=4$ and $y=-1$, find the value of

Ex 1 $3a^3xy^2=3\times(-2)^3\times4\times(-1)^2=3\times(-8)\times4\times(+1)=-96.$

Ex 2. $-a^3x^2y^5=-(-2)^3\times4^2\times(-1)^5=-(-8)\times16\times(-1)=-128$

3 $4a^2y^2$ 4 $3abc$ 5 $-b^2y^5$ 6 $-x^3y^4$
7 $2a^2bc.$ 8. $-ac^2x.$ 9 $-x^2y^3$ 10. $3a^3c^3$
11. $5ab^2c$ 12. $-2b^2xy^2$ 13 $3bcy^4.$ 14. $-2abc^3$
15. $-a^2x^2y^5.$ 16. $4a^3bc^2.$ 17. $-6ax^2y^3$ 18. $-c^3a^2y^5.$

If $a=-5$, $x=5$, $y=-3$, $z=-2$, find the value of

19 $a^2-x^2.$ 20 x^3-y^3 21 $az^3-xy.$
22 a^4-xyz 23 $a^3-(y+z)^3.$ 24 $a^3+x^3-y^3$
25 $(3a-2x)(a+2x).$ 26 $(5x+4y)(2x-3y).$
27. $3x^2-5ay+2z^3.$ 28 $y^2-3xz-a^3$
29 $x^2-y^2-z^2-2yz.$ 30 $y^3+z^3+3ayz-a^3.$

* A number is said to be *even* when it is divisible by 2, thus 2, 4, 6, .. and *odd*, when it is not divisible by 2, thus 1, 3, 5, ...

65 The factors in a product may be bracketed in any manner We shall first prove that

$$abc=a(bc)=(ab)c,$$

where a, b and c are *any numbers*, positive or negative, integral or fractional

Put b stars in a row, and let there be c such rows, the stars being placed under one another, in columns, as shewn below

*	*	*	*	....	*	row (1)
*	*	*	*		*	row (2)
*	*	*	*	.....	*	row (3)
	. ... .	. .		. .		
*	*	*	*	... ..	*	row (c)
col (1)	col (2)	col (3)	col (4)		col (b)	

Since there are b columns, each row contains b stars, and as there are c rows, the number of stars altogether is bc Now put a units in the place of *each* star, thus there are altogether bc times a units, that is the total number of units is $a\times(bc)$ . . . .(i)

Also in the first row there are a units repeated b times, and as there are c rows, these are again repeated c times, therefore the total number of units is $a\times b\times c$ .. . .(ii)

Again, in the first row there are a units repeated b times, i e, there are ab units, and as there are c rows, ab is repeated c times, thus the total number of units is $(ab)\times c$ . .. . (iii).

Hence from (i), (ii), and (iii) $a\times b\times c=a\times(bc)=(ab)\times c$, that is,

$$abc=a(bc)=(ab)c$$

Thus the proposition is proved when a, b and c are *integers*

It is likewise true when a, b, and c are *fractions*, for we know that

$$\frac{7}{8}\times\frac{3}{5}\times\frac{9}{11}=\frac{7\times3\times9}{8\times5\times11},$$

and, by the above proof for integers, we have

$$\frac{7\times3\times9}{8\times5\times11}=\frac{7\times(3\times9)}{8\times(5\times11)}=\frac{(7\times3)\times9}{(8\times5)\times11};$$

$$\therefore\ \frac{7}{8}\times\frac{3}{5}\times\frac{9}{11}=\frac{7}{8}\times\left(\frac{3}{5}\times\frac{9}{11}\right)=\left(\frac{7}{8}\times\frac{3}{5}\right)\times\frac{9}{11}$$

Next let c be *negative*, then also

$$a\times b\times -c=a\times(b\times -c)=(a\times b)\times -c,$$

for each of these products has the *same absolute value*, viz, abc and the *same sign*, viz, the sign $-$ [§ 64]. In the same way, it

may be shewn that the proposition holds whether *all* or *any* of the numbers a, b, c be negative

Thus for *all values* of a, b, c, we have $abc = a(bc) = (ab)c$.

Similarly it may be proved that

$$abcd\ldots = (ab)cd\ldots = (ab)(cd) = a(bcd)\ldots = (abc)d\ldots = \&c$$

Hence we learn that *the factors of a product may be grouped in any manner.* This is termed the Law of Association

66 Theorem *To shew that $ab = ba$, whatever a and b may be, positive or negative, integral or fractional*

First, let a and b be positive *integers*

Write down a units in a row, and take b such rows, placing the units under one another, in columns, as shewn below.

1	1	1	1........	1	row (1)
1	1	1	1........	1	row (2)
1	1	1	1........	1	row (3)
......	......	......	......	......	
1	1	1	1........	1	row (b)
col (1)	col. (2)	col (3)	col (4)	col. (a)	

Now each row obviously contains a units and there are b such rows; therefore there are altogether a units repeated b times, i e, $a \times b$ units, thus the *total number* of units is $a \times b$ Again each column contains b units and there are a such columns, therefore there are altogether b units repeated a times, i e, $b \times a$ units, thus the *total number* of units is $b \times a$ Hence $a \times b = b \times a$, that is, $ab = ba$

Secondly, let $a = \frac{7}{5}$ and $b = \frac{3}{8}$. Now we have seen [§ 63] that $\frac{7}{5} \times \frac{3}{8} = \frac{7 \times 3}{5 \times 8}$ and by the above proof (since 7, 3, 8 and 5 are all integers) $\frac{7 \times 3}{5 \times 8} = \frac{3 \times 7}{8 \times 5}$ Thus $\frac{7}{5} \times \frac{3}{8} = \frac{3}{8} \times \frac{7}{5}$

Similarly if $a = \dfrac{m}{n}$ and $b = \dfrac{p}{q}$, we shall have $\dfrac{m}{n} \times \dfrac{p}{q} = \dfrac{p}{q} \times \dfrac{m}{n}$

Thus the proposition is true when a and b are *positive fractions*

Lastly, (i) let $a = -m$, $b = n$ Now $mn = nm$, and by the Law of Signs $(-m) \times n = -mn$ and $n \times (-m) = -nm$; therefore

$$(-m) \times n = n \times (-m)$$

(ii) Let $a = -m$ and $b = -n$ Here, as above, $mn = nm$; and $(-m) \times (-n) = +mn$ and $(-n) \times (-m) = +nm$, therefore

$$(-m) \times (-n) = (-n) \times (-m).$$

67 The order of factors in a Product is indifferent We shall first shew that

$$abc=acb=bac=bca=cab=cba$$

Now $\qquad abc=a(bc)=(ab)c$ [§ 65],

and by § 66, $a(bc)=(bc)a$, $(ab)c=c(ab)$, and $bc=cb$, $ab=ba$,

$$abc=acb=bca=bac=cab=cba$$

Similarly it may be shewn that

$$abcd=acbd=bcad=\&c,$$

and so on for any number of factors

Hence we have for Multiplication the same **Law of Commutation** as for Addition and Subtraction [§ 47] *viz —The order in which multiplications may be made is indifferent*, that is, the product will be the same *in whatever order the factors are taken*

REMARK Though it is perfectly indifferent in what order the factors appear, a numerical factor is always put first and then the factors in letters are placed in *alphabetical* order Thus the product of $2a$, $3b$, x, and $4y$ is usually written $24abxy$

Examples Multiply $2a$ by $3x$, $5x$ by $-3ax$, and $-4ax$ by $-6ab$

$$2a\times 3x=2\times 3\times a\times x=6ax$$

$$5x\times(-3ax)=-5x\times 3ax=-5\times 3\times a\times x\times x=-15ax^2$$

$$(-4ax)\times(-6ab)=+4ax\times 6ab=4\times 6\times a\times a\times b\times x=24a^2bx$$

68 Product of Powers Since $a^2=aa$ and $a^3=aaa$, we get

$$a^2\times a^3=aa\times aaa=aaaaa=a^5=a^{2+3},$$

similarly $\quad a^5\times a^4=aaaaa\times aaaa=aaaaaaaaa=a^9=a^{5+4}$,

and so on Thus, since *the index of a power indicates the number of factors in that power*, we have

$$a^m=aaaa\ \ldots\ \text{to } m \text{ factors},$$

and $\qquad a^n=aaaa\ldots\ \ldots\ \text{to } n \text{ factors},$

$$\therefore\ a^m\times a^n=(aaa\ldots\text{to } m \text{ factors})\times(aaa\ldots\text{to } n \text{ factors})$$

$$=aaaa\ldots\ldots\text{to } (m+n) \text{ factors}=a^{m+n},$$

thus $\quad a^m\times a^n=a^{m+n}$, where m and n are *positive integers*

Hence in *multiplying* powers of the same quantity, we *add the indices of the powers*. This law is called the **Index Law.**

Corrollary. Hence $a^m \times a^n \times a^p = a^{m+n+p}$, where m, n and p are positive integers.

For $a^m \times a^n \times a^p = (a^m \times a^n) \times a^p$ [§ 65] $= a^{m+n} \times a^p = a^{m+n+p}$.

And *generally* $a^m \times a^n \times a^p \times \ldots\ldots = a^{m+n+p+\cdots}$

Example. Multiply $2a^4$ by $3a$, $-4a^2$ by $-ab$; and $7a^3x$ by $-5ax^2$.

$$2a^4 \times 3a = 2 \times 3 \times a^4 \times a = 6a^{4+1} = 6a^5$$

$$(-4a^2) \times (-ab) = +4a^2 \times ab = 4a^2a \times b = 4a^{2+1}b = 4a^3b.$$

$$7a^3x \times (-5ax^2) = -7a^3x \times 5ax^2 = -35a^{3+1}x^{1+2} = -35a^4x^3.$$

Division

69. Definitions. Division is defined to be *the inverse operation to that of Multiplication*. Hence to divide a by b is the same as to find a quantity c which when multiplied by b gives a, that is, *the quantity c is such that* $c \times b = a$.

Thus the definition may be symbolically expressed in the following two forms. *If* $c \times b = a$, then

$$c = a \div b \qquad \text{(i)},$$

and therefore, $\quad (a \div b) \times b = a$, i.e., $a \div b \times b = a \qquad$ (ii)

The quantity divided is called the DIVIDEND, the quantity by which it is divided is called the DIVISOR and the result is called the QUOTIENT. Thus, in $a \div b = c$, a is the dividend, b is the divisor, and c is the quotient.

70. Theorem. *To prove that* $a \div b \div c = a \div bc$.

Now $\quad (a \div b \div c) \times bc = \{(a \div b) \div c\} \times c \times b$ [§ 67]

$= \{(a \div b) \div c \times c\} \times b$ [§ 65]

$= (a \div b) \times b$, by def. form (ii), § 69,

$= a$, by def., form (ii), § 69;

$\therefore \quad a \div b \div c = a \div bc$, by def., from (i), § 69.

Thus to divide a number a by two numbers b and c successively is the same as to divide a at once by their product bc.

Corollary. Hence *generally* to divide a by b, c, d, ... in succession is the same as to divide a at once by bcd..; and *vice versâ*. That is,

$$a \div b \div c \div d \div \ldots = a \div bcd\ldots;\ \text{and}\ a \div bcd\ldots = a \div b \div c \div d \div \ldots$$

71 Division otherwise expressed The operation of division is often expressed in the form of a *fraction* the dividend being the *numerator*, and the divisor, the *denominator*, thus $a \div b$ is the same as $\frac{a}{b}$

Hence if A and B be two expressions which have no *common factor*, then the fraction $\frac{A}{B}$ expresses the *quotient*, when A is divided by B Thus the quotient of 3 divided by 2 is $\frac{3}{2}$

72 Theorem *To prove that to divide a quantity by d is the same as to multiply it by $\frac{1}{d}$, where d is an integer*

Since $\frac{1}{d} = 1 \div d$ [§ 71], we have $\frac{1}{d} \times d = 1 \div d \times d = 1$, by def, form (ii)

$\therefore\ a \times \frac{1}{d} \times d = a \times 1 = a$, that is, $\left(a \times \frac{1}{d}\right) \times d = a$ [§ 65],

thus $$a \times \frac{1}{d} = a \div d, \text{ by def., form (i)}$$

Hence to divide a quantity by another is the same as to multiply the former by the *reciprocal* of the latter

Definition Quantities are said to be *reciprocal* to one another when their product is *unity* Thus if $ab = 1$, a is said to be reciprocal to b, and *vice versâ* Hence 2 and $\frac{1}{2}$, $\frac{5}{3}$ and $\frac{3}{5}$, a and $\frac{1}{a}$, &c are other examples of reciprocal quantities

73 Divisions may be performed in any Order For

$$a \div b \div c = a \times \frac{1}{b} \times \frac{1}{c} \text{ [§ 72]} = a \times \frac{1}{c} \times \frac{1}{b} \text{ [§ 67]} = \left(a \times \frac{1}{c}\right) \times \frac{1}{b} \text{ [§ 65]}$$
$$= a \div c \div b \text{ [§ 72]}$$

Thus $$a \div b \div c = a \div c \div b \qquad (i)$$

Similarly it may be shewn that

$$a \div b \div c \div d \div \&c = a \div d \div c \div b \div \&c, = \&c$$

Thus we have the same **Law of Commutation** for Division as for Multiplication [§ 67] *viz* —*The order in which a series of divisions may be performed is indifferent*

This law may be extended to the case where divisions and multiplications are combined, as shewn below

(1) $a\times b\div c=a\times b\times\frac{1}{c}$ [§ 72] $=a\times\frac{1}{c}\times b$ [§ 67] $=\left(a\times\frac{1}{c}\right)\times b$ [§ 65]

$=a\div c\times b$ [§ 72] (ii).

(2) $a\div b\times c=a\times\frac{1}{b}\times c$ [§ 72] $=a\times c\times\frac{1}{b}$ [§ 67] $=(a\times c)\times\frac{1}{b}$ [§ 65]

$=a\times c\div b$ [§ 72] (iii).

(3) $a\times b\div c\div d=a\times b\times\frac{1}{c}\times\frac{1}{d}$ [§ 72] $=a\times\frac{1}{d}\times b\times\frac{1}{c}$ [§ 67]

$=\left(a\times\frac{1}{d}\right)\times\left(b\times\frac{1}{c}\right)$ [§ 65] $=a\div d\times b\div c$ [§ 72] (iv).

Similarly other proposed cases may be proved. Hence *multiplications and divisions when combined may be performed in any order.*

The results (i), (ii), (iii) and (iv) may also be deduced directly from the definition of Division. Thus to prove (i)

$(a\div b\div c)\times b\times c=\{(a\div b)\div c\}\times c\times b$ [§ 67]

$=\{(a\div b)\div c\times c\}\times b$ [§ 65]

$=(a\div b)\times b$ [Def. form (ii)] $=a$.

Again $(a\div c\div b)\times b\times c=\{(a\div c)\div b\times b\}\times c$ [§ 65]

$=(a\div c)\times c$ [Def. form (ii)] $=a$.

Thus $(a\div b\div c)\times b\times c=(a\div c\div b)\times b\times c$,

i e, $a\div b\div c=a\div c\div b$.

The other cases are left as exercises to the student.

74. The Law of Signs. Since [§ 64]

$(+a)\times(+b)=+ab$, we have $+ab\div(+a)=+b$ [Def., § 69],

$(-a)\times(+b)=-ab$, we have $-ab\div(-a)=+b$;

$(+a)\times(-b)=-ab$, we have $-ab\div(+a)=-b$;

$(-a)\times(-b)=+ab$, we have $+ab\div(-a)=-b$.

Hence *when the dividend and the divisor both have the same sign, the quotient has the sign +, and when they have different signs, the quotient has the sign −.* This is the Law of Signs for Division and may be briefly stated thus:—*Like signs produce +, and unlike signs produce −.*

75. Division of powers of the same quantity. Since $a^3\times a^2=a^6$ [§ 68], we have $a^3=a^6\div a^2$, by the definition of Division, form (i); and $a^3=a^{6-2}$ identically; therefore $a^{6-2}=a^6\div a^2$.

Similarly, since $a^5\times a^4=a^9$, we have $a^5=a^9\div a^4$, that is,

$$a^{9-4}=a^9\div a^4.$$

Hence generally, since $a^{m-n}\times a^n=a^m$ [§ 65] we have as above

$$a^{m-n}=a^m\div a^n \quad i\ e, \quad a^m\div a^n=a^{m-n},$$

where m and n are both *positive integers* and $m>n$

Thus in *dividing* one power of a quantity by another, we *subtract the index of the divisor from that of the dividend*

Example Divide a^4x^5 by a^3x and $-25a^6b^2$ by $5a^4b$

$$a^4x^5\div a^3x=a^4\times x^5\div a^3\div x\ [\S\ 70]=a^4\div a^3\times x^5\div x\ [\S\ 73]$$
$$=a^{4-3}\times x^{5-1}=a\times x^4=ax^4$$
$$-25a^6b^2\div 5a^4b=-25\times a^6\times b^2\div 5\div a^4\div b\ [\S\ 70]$$
$$=-25\div 5\times a^6\div a^4\times b^2\div b\ [\S\ 73]$$
$$=-5\times a^{6-4}\times b^{2-1}=-5\times a^2\times b^1=-5a^2b.$$

Cor. Meaning of a^0 When $m=n$, we have

$$a^{m-n}=a^m\div a^n=a^m\div a^m=1\ ,$$

and then $m-n=0$, and therefore $a^{m-n}=a^0$ Thus $a^0=1$.

Multiplication and Division of Polynomials

76 Theorem *To prove that $(a+b)n=an+bn$, whatever a, b and n may be, positive or negative, integral or fractional*

(i) Let n be a positive *integer*, then whatever a and b may be,

$$(a+b)n=(a+b)+(a+b)+(a+b)+\ \ldots \text{ repeated } n \text{ times}$$
$$=a+a+a+a+\ \ldots \text{ repeated } n \text{ times}$$
$$+b+b+b+b+\ \ldots \text{ repeated } n \text{ times}$$
$$=an+bn$$

(ii) Let n be a positive *fraction*, and let $n=\frac{p}{q}$, where p and q are integers, then since $\frac{p}{q}=p\div q$, we have

$$(a+b)\times\frac{p}{q}=(a+b)\times p\div q=\{(a+b)\times p\}\div q\ [\S\ 65]=(ap+bp)\div q$$

Now since division is the inverse of multiplication, we get $(x+y)\div z=x\div z+y\div z$, where z is a positive integer, hence

$$(a+b)\times\frac{p}{q}=ap\div q+bp\div q=a\times p\div q+b\times p\div q=a\times\frac{p}{q}+b\times\frac{p}{q}.$$

Therefore replacing $\frac{p}{q}$ by n, we have $(a+b)n=an+bn$

(iii) Let n be *any negative quantity*, and let $n=-p$, where p is any *positive* quantity.

Thus from (i) and (ii) above we have

$$(a+b)n=(a+b)\times(-p)=-(a+b)p=-(ap+bp)$$
$$=-ap-bp=a(-p)+b(-p)=an+bn.$$

Hence for *any value whatever of* a, b and n, we have

$$(a+b)n=an+bn$$

Thus *if the sum of any two quantities be multiplied by a third the product shall be equal to the sum of the partial products obtained by multiplying each of the quantities separately by the third*

This is termed the **Law of Distribution** for multiplication

Explanation. Since a and b may be positive or negative or fractional, we see that $(a-b)n=an-bn$, $(-a-b)n=-an-bn$, and $(-a+b)n=-an+bn$

77. To prove that $(a+b+c+...)n=an+bn+cn+...$

From § 76, we have $(p+q)n=pn+qn$, for all values of p, q and n Let $p=a+b$: thus

$$\{(a+b)+q\}n=(a+b)n+qn=an+bn+qn,$$
$$\therefore \quad (a+b+q)n=an+bn+qn$$

Similarly by putting $q=c+d$, we may shew that

$$(a+b+c+d)n=an+bn+cn+dn$$

Hence *generally* $(a+b+c+\ldots\ldots)n=an+bn+cn+\ldots\ldots$

Thus *if a polynomial be multiplied by a monomial, the product shall be equal to the sum of the partial products of each of its terms multiplied separately by the monomial*

Explanation Any one or more of the quantities a, b, c, &c. may be negative or fractional

Cor Hence conversely $an+bn+cn+\ldots=(a+b+c+\ldots\ldots)n.$

78 To prove that $(a+b)(m+n)=am+bm+an+bn.$

From § 76, we have $\quad (m+n)f=mf+nf,$

that is, $\quad f(m+n)=fm+fn$ [§ 66]

Put $f=a+b$; thus $(a+b)(m+n)=(a+b)m+(a+b)n$

$$=am+bm+an+bn \text{ [§ 76]}.$$

Similarly we may shew that $(a+b+c)(m+n+p)$

$$=(a+b+c)m+(a+b+c)n+(a+b+c)p$$
$$=am+bm+cm+an+bn+cn+ap+bp+cp,$$

and that $(a+b+c+\ldots\ldots)(m+n+p+\ldots\ldots)$

$$=(a+b+c+\ldots\ldots)m+(a+b+c+\ldots\ldots)n+(a+b+c+\ldots)p+\ldots$$
$$=am+bm+cm+\ldots\ldots+an+bn+cn+\ldots\ldots+ap+bp+cp+\ldots$$

Thus *if one polynomial be multiplied by another, the product is equal to the sum of the partial products obtained by multiplying each term of the multiplicand by each term of the multiplier*

79 Theorem *To prove that* $(a+b)\div n=a\div n+b\div n$ *whatever, a, b and n may be, positive or negative, integral or fractional*

By the Theorem of § 72,

$$(a+b)\div n=(a+b)\times\frac{1}{n}=a\times\frac{1}{n}+b\times\frac{1}{n}\ [\S\ 76]=a\div n+b\div n\ [\S\ 72]$$

Thus *if the sum of any two quantities be divided by a third, the quotient shall be equal to the sum of the partial quotients obtained by dividing each of the quantities separately by the third*

This is called the **Law of Distribution** for Division

80. To prove that $(a+b+c+\ \ldots)\div n$

$$=a\div n+b\div n+c\div n+\ \ldots$$

We have $(p+q)\div n=p\div n+q\div n$,

whatever p, q and n may be Let $p=a+b$, and $q=c+d$, thus

$$(a+b+c+d)\div n=(a+b)\div n+(c+d)\div n=a\div n+b\div n+c\div n+d\div n$$

Hence *generally* $(a+b+c+\ \ )\div n=a\div n+b\div n+c\div n+\ .$

Thus *if a polynomial be divided by a monomial, the quotient shall be equal to the sum of the partial quotients obtained by dividing each of its terms separately by the monomial*

REMARK. It is to be noted that the Law of Distribution has *full* application to Multiplication, but only a *partial* application to Division In Multiplication *both* the Multiplicand and Multiplier may be distributed [see § 78], whereas in Division, the *dividend only* can be distributed but not the divisor Thus it is true that $(a+b)\div c=a\div c+b\div c$, but it is not true that $a\div(b+c)=a\div b+a\div c$

81 Examination upon Chapter V

1. Define *Multiplication*, and illustrate the definition by multiplying 4 by 6, $\frac{5}{8}$ by $\frac{3}{4}$ and -4 by -6

2 State the Law of Signs for Multiplication and Division. Find the values of $(-1)(-\frac{1}{2})$, $\frac{1}{2}\times(-\frac{3}{4})$ and $(-\frac{1}{2})(-1)(-\frac{2}{5})$

3, Shew that the factors of a product may be grouped in any manner.

4 Prove that $ab=ba$, whatever a and b may be.

5 Prove that the factors of a product may be arranged in any manner.

6 Shew that $a^m\times a^n=a^{m+n}$. What is the restriction as to the values of m and n?

7. Define *Division*, and from the definition, prove that

$(a\div b\times c\div d)\times(bd)=ac$, and that $a\div b\times c\div d=a\div d\times c\div b$.

8 Distinguish between $a\div b\times c$ and $a\div bc$, and shew from the definition of division that $a\div bc=a\div b\div c$.

9. Shew that $a\div d=a\times\dfrac{1}{d}$, and that $a^0=1$.

10 Prove that $a^m\div a^n=a^{m-n}$ What is the limitation as to the values of m and n?

11. Prove that $(a+b)n=an+bn$, and that $(a+b)\div n=a\div n+b\div n$.

12. Prove that $(a+b)(c+d)=ac+bc+ad+bd$.

CHAPTER VI.

MULTIPLICATION

82 Multiplication of Monomials We have seen how Monomials are multiplied and the sign of the product determined [§§ 67, 68 and 64] We shall now give a few examples

Examples.

Ex. 1. Multiply $-12x^8$ by $-5x^4$

Product $=(-12x^8)(-5x^4)=+(12x^8\times 5x^4)$ [§ 64]

$=+(12\times 5\times x^8\times x^4)$ [§ 67, Remark]

$=+(60\times x^{8+4})$ [§ 68] $=60x^{12}$.

Ex 2 Multiply $4a^3x^2y$ by $-3ax^3y^2$

Product $=(4a^3x^2y)\times(-(3ax^3y^2))=-(4a^3x^2y\times 3ax^3y^2)$ [§ 64]

$=-(4\times 3\times a^3\times a\times x^2\times x^3\times y\times y^2)$ [§ 67]

$=-(4\times 3\times a^{3+1}\times x^{2+3}\times y^{1+2})$ [§ 68] $=-12a^4x^5y^3$.

Ex. 3 Multiply together ab, $-2bc$, $-3ac$ and $-4bc^2$.

Here the number of negative factors is 3, which is an *odd* number, therefore the product must be *negative* [§ 64, Cor] Hence

Product $=-(ab\times 2bc\times 3ac\times 4bc^2)$

$=-(2\times 3\times 4\times ab\times bc\times ac\times bc^2)$ [§ 67]

$=-(24\times a\,a\times b\,b\,b\times c\,c\,c^2)$ [§ 67]

$=-24a^{1+1}b^{1+1+1}c^{1+1+2}$ [§ 68] $=-24a^2b^3c^4$.

Ex 4 Simplify $(-ax^3)\times 2f^2g\times(-4fy^2)\times(-15cxy)$

Product $= -(ax^3\times 2f^2g\times 4fy^2\times 15cxy)$ [§ 64, *Cor.*]

$= -(2\times4\times15\times a\times c\times f^2 f\times g\times x^3 x\times y^2 y)$ [§ 67]

$= -(120\times a\times c\times f^{2+1}\times g\times x^{3+1}\times y^{2+1})$ [§ 68]

$= -120acf^3gx^4y^3$

Ex 5 Find the value of $(-ax)^3$

Reqd value $=(-ax)(-ax)(-ax) = -(ax\times ax\times ax)$

$= -(a\,a\,a\times x\,x\,x) = -(a^{1+1+1}\times x^{1+1+1}) = -a^3x^3.$

REMARK. We have shewn the work in the above examples at some length, but it is always advisable to do the work *mentally* and put down the result at once, thus

Ex 1 $(-12x^8)\times(-5x^4) = +60x^{12}$

Ex 4 $(-ax^3)\times 2f^2g\times(-4fy^2)\times(-15cxy) = -120acf^3gx^4y^3$.

Ex 5 $(-ax)^3 = -a^3x^3$

Similarly $(-ax)^2 = +a^2x^2$, $(-ax)^5 = -a^5x^5$, &c

Multiply

6 $3x$ by ax 7. $5xy$ by $-2y$ 8 $-8ax$ by $13px$

9 $-2abx$ by ac 10 $-7abc$ by $-8bc$ 11 $-pqr$ by $-qrx$

12 $25axy$ by $-31bcz$ 13 $-312mn$ by $-56pq$

14 $54abz$ by $12cdy$ 15 $-12xyz$ by $3ax$

16 $-16abpq$ by $-5abxy$ 17 $20ars$ by $-17bruz$

18 $3a^5$ by $2a$ 19 $10x^6$ by $5x^4$ 20 $15y^5$ by $-4y$.

21. $2a^3$ by $-5a^2$ 22. $-12m^9$ by $-6m^9$. 23 $-7z^{10}$ by $-11z^3$

24 $3ax^2$ by $-a^2x$ 25 $2x^3y$ by $-3xy^3$

26 $-a^2bc^3$ by $-2abc$. 27 $-m^3n^2x$ by $-4mx^3$

28. $3xyz^2$ by $-2x^3z$. 29. $-4ab^3$ by $-6a^2b^3$

30 $2a^3b^3c^3d^4$ by $-abcd$ 31 $-4a^2x^3by^4$ by $-3a^2xb^3y$.

32 $-7a^2b^8m$ by $3a^3b^4m^2$ 33 $-8a^2xy^3z$ by $-20ay^2z^3$.

Simplify

34 $(-1)^2$, $(-1)^3$, $(-1)^4$, $(-1)^5$

35 $(-4a)^2$, $(-2ab)^3$, $(-abc)^5$; $(-3x^3)^4$.

Multiply together

36 $-ax$, bx and $-acx$ 37 $5ab$, $-2cd$ and $-3ef$.

38 $-a$, $2b$, $-4c$ and $-15d$

39 $3ab$, $-5a$, $-b$, $2c$, $-4bc$ and $-8ac$

Multiply together

40. $6a, -5bc$ and $12px$ 41. $-16ab, -2cx$ and $-20yz$

42 $-ab, 2c, -15r, -3z, -4s, 6x$ and $-y$.

43 $20ab, -3abc, -4cx, 5xy, -2by$ and $-8c$.

Find the continued product of

44 $3x^3$ $-2x$ and $-x^4$ 45 $4c, -8c^2, -7c$ and $-mc^3$.

46 $-a^2, 3a^3, -4a^6$ and $-5a^4$. 47 $-ax^3, byz$ and $-3a^2xy$

48. $-6ab^2x^3, 12a^3xy^3, -c^2xy$ and $-3a^3cy^2$

49 $a^3x^2, -b^3y^3, mc^2z, -axy^2$ and $5xyz$

83. Multiplication of a Polynomial by a Monomial

The result of § 77 enables us to multiply a polynomial by a monomial.

Examples (i)

Ex. 1. Multiply $a-b+c$ by m

Required product$=(a-b+c)m=am-bm+cm$ [§ 64].

The operation is, however, conveniently performed thus

$$\begin{array}{l} a-b+c \\ \underline{m} \\ am-bm+cm \end{array}$$

Ex. 2 Multiply $-3a^2b^2+2ab^2c-8abc^2+a^4$ by $-18abcx$.

$$\begin{array}{l} -3a^2b^2+2ab^2c-8abc^2+a^4 \\ \underline{-18abcx} \\ +54a^3b^3cx-36a^2b^4c^2x+144a^2b^2c^3x-18a^5bcx \end{array}$$

Ex. 3. Multiply $-\frac{1}{2}a^2x-\frac{3}{4}ab^2-\frac{1}{4}a^3x^2+3ab^3$ by $-\frac{1}{3}abx^2$.

$$\begin{array}{l} -\frac{1}{2}a^2x-\frac{3}{4}ab^2-\frac{1}{4}a^3x^2+3ab^3 \\ \underline{-\frac{1}{3}abx^2\phantom{-\frac{1}{4}a^3x^2+3ab^3+\frac{1}{12}a^4bx^4-a^2b^4x^2}} \\ +\frac{1}{6}a^3bx^3+\frac{1}{4}a^2b^3x^2+\frac{1}{12}a^4bx^4-a^2b^4x^2 \end{array}$$

Here the *fractional* coefficients are multiplied as in Arithmetic.

Multiply

4 $a-x$ by $3ax$ 5. $x-y$ by $-a$ 6 $-a-b$ by $-b$.

7. $2ax-by$ by ay. 8 $4ab-c^2$ by $3abc^2$.

9 $-ab-cd$ by $-5bc$. 10. $-x+y-2$ by $2a$

11 $x-3b-ab^3$ by $-bx$ 12 $bc+ca-ab$ by $-abc$

13. $2x-y-4z$ by $-3xyz$ 14 $-a^2-b^2-c^2$ by $-2a^2b^2c^2$

15 $3a+2b-4c+d$ by ad 16. $2a^2bc-3ab^2c-4abc^2$ by $-abc$.

Multiply

17 $5x-2y+4z$ by $\frac{3}{20}x^2$

18 $4a-3b-2$ by $-\frac{5}{12}a^2bc$

19 $-\frac{1}{3}a-\frac{1}{2}b-1$ by $-\frac{6}{7}ab$

20 $\frac{2}{3}a^2x^3-\frac{3}{4}a^3x^2$ by $\frac{5}{8}ax^2$.

21 $\frac{1}{3}a^3b-\frac{3}{4}a^2b^2-\frac{2}{3}b^3$ by $-\frac{6}{5}a^3b^3$

22 $-\frac{3}{5}a^2x-\frac{2}{7}by^2+\frac{1}{5}c^3z^3$ by $-\frac{10}{3}b^2cx^3$

23 $-2a^3-3b^3-4c^3$ by $-5a^2b^2c$

24 $2a^2b-8b^2c-5ac^2$ by $3ab^3c^3$

25. $a^3x^3-2a^2x^5-3a^4x$ by $-4ax^5$

26. $-8x^3y^2z^4+7xyz^3-3x^2y^2z$ by $-2xy^2z^3$

27 $x^3-3x^2y+3xy^2-y^3$ by $8xy$

28 $3a^3b^2c-b^3c^2d-2c^3d^2a+4d^3a^2b$ by $-3abcd$

29 $5a^2b^3c-3ax^2y-abx+cx^2y^4$ by $-4abxy$

Examples (ii)

Ex 1 Simplify $8x^2(2x-3)-4x(4x^2-3)+12x(2x-1)$

Given expression

$$=(8x^2\times 2x-8x^2\times 3)-(4x\times 4x^2-4x\times 3)+(12x\times 2x-12x\times 1)$$

[here we have inverted the order of multiplicand and multiplier (§66)],

$$=(16x^3-24x^2)-(16x^3-12x)+(24x^2-12x)$$

$$=16x^3-24x^2-16x^3+12x+24x^2-12x=0$$

Ex 2 Simplify $2x^2a+2x^2(x-4a)-2x^2(x-7a)$

Given expression $=2x^2a+(2x^3-8x^2a)-(2x^3-14x^2a)$

$$=2ax^2+2x^3-8ax^2-2x^3+14ax^2=8ax^2$$

Otherwise Since $2x^2$ is common to every term, we see as in § 77 that the given expression

$$=2x^2\{a+(x-4a)-(x-7a)\}$$

$$=2x^2(a+x-4a-x+7a)=2x^2\times 4a=8ax^2$$

Simplify

3 $3a^2(1-a)-4a^3(a-2)-a(3a+5a^3)$

4 $4x^2(x-4)-3x(x^2-5x)+2(x^2-1)$.

5 $2ax(a-1)-2ax(x-1)+2ax(a-x)$

6 $3x^3(13a-12b)+3x^3(14b-4a)-3x^3(2a-5b)$

7 $6a(3a-4b+1)-8b(3b-3a+1)-2(3a-4b)$

8 $8x^2(2x-3y+4)-6y^2(4x-2y)-8x(2x^2-3xy-3y^2)$.

9 $5x^3y(3x^2y^2-2xy^3-2y^3)-3xy^3(5x^4-2x^3y-4x^2)+2x^3y^2(2x^2y^2-xy+1)$.

Simplify

10. $a^2(b-c)+a^2(3c+a)-a^2(b-2a)+a^2(a-b)$

11. $ab(a-b)+bc(b-c)-b^2(c-a)+b(c^2-a^2)$

12. $a^3(b-c)+b^3(c-a)+c(a^3-b^3)-ab(a^2-b^2)$.

13. $x^3(y^2z-yz^2)+z^2(x^2y-xy^2)-x^2z^2(yz-xy)-xz(x^2y^2-y^2z^2)$

84 Degree of a Term. Each of the letters in a product is called a DIMENSION of the product, and the *number* of the letters in it, denotes its DEGREE

Thus abc has *three* letters, it has therefore *three* dimensions, or is of the *third* degree; a^2xy^3 has six letters, for it $=aaxyyy$, it has therefore *six* dimensions or is of the *sixth* degree, $5xy^2z$ has *four* letters, it has therefore *four* dimensions, or is of the *fourth* degree, and so on The numerical co-efficient is not counted

Hence *the degree of a term is determined by adding the indices of the letters*: thus $2ax^2y^3z^4$ is of $1+2+3+4$, or 10 dimenisons

It is sometimes important to consider the degree of a term *with respect to a particular symbol*, involved in it, which is then called the *symbol of reference* Thus the term $3a^3bx^4$ is of the *fourth* degree with reference to x, of the *third* degree with reference to a, and of the *first* degree with reference to b

85 Degree of an Expression The degree of a polynomial is the degree of the term of *highest* dimensions in it Thus the polynomial x^2+3x-1 is of the *second* degree, because the first term is of the *second* degree and no other term in it has *higher* dimensions; so also, $2x+3x^3+4$ is of the *third* degree, for the second term is of the *third* degree, and no other term in it is of a *higher* degree, and so on

It is often useful to consider the degree of a given polynomial *with respect to a particular symbol*, say x. The polynomial is then said to be an **expression in** x and the symbol x is said to be the **symbol of reference** Thus $x^3+2ax^2+3a^2x+2a^4$ is considered as an expression of the *third* degree when we take x to be the symbol of reference, and an expression of the *fourth* degree when a is taken as the symbol of reference

Hence the polynomial

$$5a^3x^4+3a^2by-ab^2xy^2$$

is of the *fourth* degree with respect to x, of the *second* degree with respect to y, of the *third* degree with respect to a, of the *second* degree with respect to b, and of the *seventh* degree when *all* the letters are considered.

86 Homogeneous Expression When all the terms of a polynomial are of the *same* degree, the polynomial is said to be **Homogeneous** Thus

$$2a+b+c,$$
$$a^3-5a^2b+2abc,$$
$$ax^4+bx^3y-8c^3x^2+3y^5, \&c,$$

are homogeneous expressions, and are respectively of the *first*, *third* and *fifth* degrees

87 Arrangement of an expression according to powers of a letter An expression is said to be arranged according to the *ascending* or *descending* powers of some letter, when the power of that letter *increases* or *decreases* in each successive term beginning with the first Thus the expression

$$1+x+x^2+x^3+x^4+x^5$$

is arranged according to the *ascending* powers of x, for the index of the first term is 0 [§ 75, *Cor*], of the second term is 1, of the third term is 2, &c, *i e*, the index of the power, and therefore the power itself, increases in every *successive* term Also, the same expression when written

$$x^5+x^4+x^3+x^3+x+1$$

is arranged according to the *descending* powers of x, for the index diminishes *successively*, that of the first term being 5, of the second term being 4, &c Thus the *same expression* if arranged according to the *descending* powers of *some* letter, will be arranged according to the *ascending* powers of *that* letter, if the expression be written in a *reversed order*, and *vice versa*

As another example, we see that

$$x^5+3x^4y+4x^3y^2+5x^2y^3+6xy^4+y^5$$

is arranged according to the *descending* powers of x and *ascending* powers of y, and, when written in a reversed order, it will be arranged according to the *ascending* powers of x and *descending* powers of y

Again an expression, if it contains two or more letters, may be arranged *differently* Thus for example, the expression

$$a^3(b-c)+b^3(c-a)+c^3(a-b),$$

when arranged according to the descending powers of a, b and c, will be written respectively

$$(b-c)a^3-(b^3-c^3)a+(b^3c-bc^3),$$
$$(c-a)b^3-(c^3-a^3)b+(c^3a-ca^3),$$

and
$$(a-b)c^3-(a^3-b^3)c+(a^3b-ab^3)$$

Corollary. Hence it is easy to see that if an expression be arranged in *either* order, there can be but *one* *highest* term and *one* *lowest* term in it; and that when it is arranged in *descending* powers, the *first* term is the *highest* term and the *last* term is the *lowest* term in it, and *vice versa*

88 Multiplication of two Polynomials The Rule established in § 78, enables us to multiply one polynomial by another

Two Binomials Examples (1).

Ex 1 Multiply $a+b$ by $c+d$

Proceed according to the Rule, thus the partial products are $(+a)(+c)=+ac$, $(+b)(+c)=+bc$, $(+a)(+d)=+ad$, $(+b)(+d)=+bd$, hence the product required $=ac+bc+ad+bd$.

The work is usually shewn thus —

Required product $=(a+b)(c+d)=(a+b)c+(a+b)d$

$$=(ac+bc)+(ad+bd)=ac+bc+ad+bd$$

The common practice however is to place the multiplicand over the multiplier, and then to proceed according to the Rule, thus .—

$$\begin{array}{l} a+b \\ \underline{c+d} \\ ac+bc \\ \underline{\quad\;\; +ad+bd} \\ ac+bc+ad+bd \end{array}$$

Ex. 2 Multiply $x+2$ by $x+3$.

$$\begin{array}{l} x+2 \\ \underline{x+3} \\ x^2+2x \\ \underline{\quad\;\; +3x+6} \\ x^2+5x+6 \end{array}$$

REMARK. The second product is shifted *one place* to the right to bring *like* terms under *like* terms, which facilitates addition Whenever multiplicand and multiplier are arranged according to powers of some common letter [§ 87], this artifice is often useful

Ex. 3 Multiply $\frac{1}{2}x+3y$

by $2x-\frac{1}{3}y$

$$\begin{array}{l} x^2+6xy \\ \underline{\quad\;\; -\frac{1}{6}xy-y^2} \end{array}$$

Product $x^2+\frac{35}{6}xy+y^2$.

REMARK The fractional co efficients are multiplied as in Arithmetic It is to be noted here that $\frac{1}{2}x$ is the same as $\frac{x}{2}$, $\frac{1}{3}y$ is the same as $\frac{y}{3}$, and so on

Ex 4 Multiply $a+b$ by $a+b$

$$\begin{array}{rr} & a+b \\ \text{by} & a+b \\ \hline & a^2+ab \\ & +ab+b^2 \\ \hline \text{Product} & a^2+2ab+b^2 \end{array}$$

Remark $(a+b)(a+b)=(a+b)^2$

Ex. 5 Multiply $a-b$ by $a-b$

$$\begin{array}{rr} & a-b \\ \text{by} & a-b \\ \hline & a^2-ab \\ & -ab+b^2 \\ \hline \text{Product} & a^2-2ab+b^2 \end{array}$$

Remark $(a-b)(a-b)=(a-b)^2$

Ex 6 Multiply $a+b$ by $a-b$

$$\begin{array}{rr} & a+b \\ \text{by} & a-b \\ \hline & a^2+ab \\ & -ab-b^2 \\ \hline \text{Product} & a^2 \qquad -b^2 \end{array}$$

Ex. 7 Multiply $x+a$ by $x+b$

$$\begin{array}{rr} & x+a \\ \text{by} & x+b \\ \hline & x^2+ax \\ & +bx+ab \\ \hline \text{Product} & x^2+(a+b)x+ab \end{array}$$

Ex 8 Simplify $(5x-6y)(3x+8y)$.

$$\begin{aligned}(5x-6y)(3x+8y) &= (5x-6y)3x+(5x-6y)8y \\ &= (15x^2-18xy)+(40xy-48y^2) \\ &= 15x^2-18xy+40xy-48y^2 \\ &= 15x^2+22xy-48y^2\end{aligned}$$

Ex. 9. Simplify $(2a-b)(5x-3y)-(2a-b)(8x-3y)$

Put $2a-b=m$, thus

given expn $=m(5x-3y)-m(8x-3y)=m\{(5x-3y)-(8x-3y)\}$
$=m(5x-3y-8x+3y)=m(-3x)=(2a-b)(-3x)$
$=-6ax+3bx=3bx-6ax$

Multiply

10. $x+5$ by $x-3$ 11 $x-7$ by $x+8$ 12 $x-11$ by $x-9$
13 $-a+6$ by $a-2$. 14 $x+a$ by $x-b$. 15. $x-a$ by $x+b$
16 $x-a$ by $x-b$ 17 $-x-7$ by $-x-9$

Simplify

18 $(a+b)(c-d)$ 19 $(a-b)(c-d)$ 20 $(a-b)(c+d)$
21 $(2m-9)(m+13)$ 22. $(2x-3y)(3x-2y)$
23 $(ax-by)(ax+by)$ 24 $(\frac{3}{5}x-a)(\frac{3}{4}x-\frac{1}{2}a)$
25 $(\frac{5}{6}ax-1)(\frac{3}{10}bx+1)$ 26 $(2x-1)(3x-2)-(2x-1)(5x-2)$.
27 $(5q-8r)(3q-14r)-(3q-7r)(5q-16r)$
28 $2x(x-1)+(x-1)(x+2)-2(x-1)(x+3)$
29 $a^3(a-1)+a(a+1)(a-1)-2a(a+2)(a-2)$

Polynomial by a Binomial Examples (ii).

Ex 1. Multiply a^2-ab+b^2

by $a+b$

$$\begin{array}{l} a^3-a^2b+ab^2 \\ \quad\;\; +a^2b-ab^2+b^3 \\ \hline a^3 \qquad\qquad\quad +b^3 \end{array}$$

Product a^3+b^3

2. Multiply a^2+ab+b^2

by $a-b$

$$\begin{array}{l} a^3+a^2b+ab^2 \\ \quad\;\; -a^2b-ab^2-b^3 \\ \hline a^3 \qquad\qquad\quad -b^3 \end{array}$$

Product a^3-b^3.

Ex 3. Multiply $x^4+x^3y+x^2y^2+xy^3+y^4$

by $x-y$

$$\begin{array}{l} x^5+x^4y+x^3y^2+x^2y^3+xy^4 \\ \quad\;\; -x^4y-x^3y^2-x^2y^3-xy^4-y^5 \\ \hline x^5 \qquad\qquad\qquad\qquad\qquad -y^5 \end{array}$$

Product x^5-y^5

Ex. 4. Multiply $ax^2+2hxy+by^2$

by $gx+fy$

$$\begin{array}{l} agx^3+2ghx^2y+bgxy^2 \\ \qquad\quad +afx^2y+2fhxy^2+bfy^3 \\ \hline agx^3+(2gh+af)x^2y+(bg+2fh)xy^2+bfy^3 \end{array}$$

Product $agx^3+(2gh+af)x^2y+(bg+2fh)xy^2+bfy^3$

REMARK. Here multiplicand and multiplier are both *homogeneous* and of the *third* and *second* degree respectively also the product is *homogeneous* and of the (3+2)th, i.e., *fifth* degree [See § 90, *post*.]

Multiply

5. a^2+2a+7 by $a+5$

6. x^2-2x+4 by $x+2$

7. x^2+3x+9 by $x-3$

8. x^3+x^2+x+1 by $x-1$.

9. $p^2+4pq+4q^2$ by $p+2q$.

10. $3x^2-5xy+7y^2$ by $2x-3y$.

11. $25x^2-20xy+4y^2$ by $5x-2y$.

12. a^3-2a+3 by a^2+2.

13. $x^5+2x^3y-y^3$ by x^2-2y.

14. $5a^3-7a^2-3a+8$ by $7a^2+8$

15. $25a^2+10ab+4b^2$ by $5a-2b$

16. $25x^2-40xy+64y^2$ by $5x+8y$

17. $a^4-a^3b+a^2b^2-ab^3+b^4$ by $a+b$.

18. x^2+ax-b by $ax-b$.

Two Polynomials Examples (iii).

Ex. 1 Multiply $4x^5-8x^4+9$ by x^2+2x+4

Here the given expressions are arranged according to the descending powers of x

$$\begin{array}{rl}
 & 4x^5 - 8x^4 + 9 \\
 & \underline{x^2 + 2x + 4} \\
 & 4x^7 - 8x^6 + 9x^2 \\
 & \qquad + 8x^6 - 16x^5 + 18x \\
 & \qquad\qquad\quad \underline{+ 16x^5 - 32x^4 + 36} \\
\text{Product} & 4x^7 \qquad\quad + 9x^2 + 18x - 32x^4 + 36,
\end{array}$$

or arranged according to the descending powers of x,

$$\text{Product} = 4x^7 - 32x^4 + 9x^2 + 18x + 36.$$

A little experience will enable the student to at once put down the product properly *arranged*, at the time of adding up the partial products, instead of re-arranging it as we have done here.

REMARK Here the multiplicand is of the *fifth* degree, the multiplier is of the *second* degree, and the product is of the $(5+2)$th, *i e*, *seventh* degree [See § 90, *post*] Also the *highest* term in the product, *viz*, $4x^7$ is the product of $4x^5$ and x^2, which are respectively the *highest* terms in the multiplicand and multiplier, and the *lowest* term in the product, *viz*, 36 is the product of 9 and 4, which are the *lowest* terms in the multiplicand and multiplier respectively [See § 91, *post*]

Ex 2 Multiply $x^2 - 2xy - y^2$ by $y^2 + 2xy - x^2$

Here the multiplicand is arranged according to the *descending* powers of x, but not the multiplier Arrange the multiplier therefore according to the *descending* powers of x.

$$\begin{array}{rl}
 & x^2 - 2xy - y^2 \\
 & \underline{-x^2 + 2xy + y^2} \\
 & -x^4 + 2x^3y + x^2y^2 \\
 & \qquad + 2x^3y - 4x^2y^2 - 2xy^3 \\
 & \qquad\qquad\quad \underline{+ x^2y^2 - 2xy^3 - y^4} \\
\text{Product} & -x^4 + 4x^3y - 2x^2y^2 - 4xy^3 - y^4
\end{array}$$

The same result will of course follow if we multiply out the expressions as they are given, *without arranging them*

$$\begin{array}{rl}
 & x^2 - 2xy - y^2 \\
 & \underline{y^2 + 2xy - x^2} \\
 & x^2y^2 - 2xy^3 - y^4 \\
 & \qquad + 2x^3y - 4x^2y^2 - 2xy^3 \\
 & \qquad\qquad\quad \underline{- x^4 \quad + 2x^3y + x^2y^2} \\
\text{Product} & -2x^2y^2 + 4x^3y - 4xy^3 - y^4 - x^4
\end{array}$$

REMARK Here the multiplicand and multiplier are each *homogeneous* and of the *second* degree, also the product is *homogeneous* and of the $(2+2)$th, *i e*, *fourth* degree [See § 90, *post*]

Multiply

3. x^2-ax+a^2 by x^2+ax+a^2 [see § 102, Ex 5]
4. $a^2+2ab+b^2$ by $a^2-2ab+b^2$
5. $5a^2-7ab-3b^2$ by $2a^2+3ab+9b^2$.
6. $x^2+xy-3y^2$ by $x^2-xy-3y^2$. 7. a^2+ab-b^2 by a^2+ab+b^2
8. $x^2-xy+y^2+x+y+1$ by $x+y-1$.
9. $a^2+b^2+c^2-ab-bc-ca$ by $a+b+c$.
10 $ax^2-2bx+c$ by $px+2q$. 11. x^2-2x+1 by x^3-3x+2.
12. $3x^2+xy-y^2$ by $x^2-2xy+3y^2$.
13 $2x^3-5x^2y+y^3$ by $y^3+5xy^2+2x^3$
14 $3a^3b-2a^2b^2+ab^3$ by $2a^2-ab-5b^2$.
15 $a^2-2ab+b^2+c^2$ by $a^2+2ab+b^2-c^2$.
16. $7a-2a^2+8a^3-4a^4$ by $5a+3a^2-7a^3$.
17. ax^2-px+q by bx^2+cx-r. 18 $bc+ca+ab$ by $a+b+c$.
19. $x^4-(p-1)x^3+(q-p+1)x^2-(p-1)x+1$ by $x-1$
20 $1+2x+3x^2+4x^3$ by $1-3x+2x^2-x^3$
21. $1-\frac{x}{2}+\frac{x^2}{3}$ by $1+\frac{x}{3}-\frac{x^2}{3}$ [see Ex. (i), 3].
22 $1-\frac{x}{2}+\frac{2x^2}{3}$ by $1+\frac{2x}{3}-\frac{x^2}{2}$ [see Ex. (i), 3]
23. x^3-2x^2+4x-8 by $x^3-\frac{x^2}{2}+\frac{x}{4}-\frac{1}{8}$ [see Ex (i), 3]
24. $ax^4+bx^3+cx^2+dx+f$ by $ax^4-bx^3+cx^2-dx+f$.
25. $x^2+y^2+2gx+2fy+1$ by $lx+my-1$

89 Multiplication of Several Polynomials The method will be seen from the following examples

Examples.

Ex. 1. Multiply together $a+b$, $a+b$, and $a+b$.

Now $(a+b)(a+b)=a^2+2ab+b^2$; [§ 88, Ex. (i), 4]

$$\begin{array}{rl}
\text{Multiply} & a^2+2ab+b^2 \\
\text{by} & \underline{a+b\qquad\qquad} \\
 & a^3+2a^2b+ab^2 \\
 & \underline{\qquad +a^2b+2ab^2+b^3} \\
\text{Required product} = & a^3+3a^2b+3ab^2+b^3 \\
 & =a^3+3ab(a+b)+b^3.
\end{array}$$

Remark. $(a+b)(a+b)(a+b)=(a+b)^3$.

Ex 2 Multiply together $a-b$, $a-b$, and $a-b$

Now $(a-b)(a-b)=a^2-2ab+b^2$, [§ 88, Ex (1), 5]

$$\begin{array}{rl}
\text{Multiply} & a^2-2ab+b^2 \\
\text{by} & a-b \\
\hline
& a^3-2a^2b+ab^2 \\
& \quad\;\; -a^2b+2ab^2-b^3 \\
\hline
\end{array}$$

Required product $=a^3-3a^2b+3ab^2-b^3$
$\qquad\qquad\qquad\;\; =a^3-3ab(a-b)-b^3$

Remark $(a-b)(a-b)(a-b)=(a-b)^3$

Ex 3 Multiply together $x+1$, $x+2$, and $x+3$

$$\begin{array}{l}
x+1 \\
x+2 \\
\hline
x^2+x \\
\quad\; +2x+2 \\
\hline
x^2+3x+2 \\
x+3 \\
\hline
x^3+3x^2+2x \\
\quad\;\; +3x^2+9x+6 \\
\hline
\end{array}$$

Required product $=x^3+6x^2+11x+6$

Ex 4 Multiply together $x+a$, $x+b$ and $x+c$

Now $(x+a)(x+b)=x^2+(a+b)x+ab$, [§ 88 Ex (1) 7]

$$\begin{array}{rl}
\text{Multiply} & x^2+(a+b)x+ab \\
\text{by} & x+c \\
\hline
& x^3+(a+b)x^2+abx \\
& \qquad +cx^2+(ac+bc)x+abc \\
\hline
\end{array}$$

Required product $=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$

Ex 5 Multiply together $x-a$, $x-b$ and $x-c$.

Now $(x-a)(x-b)=x^2-(a+b)x+ab$, [§ 88, Ex. (1), 16]

$$\begin{array}{rl}
\text{Multiply} & x^2-(a+b)x+ab \\
\text{by} & x-c \\
\hline
& x^3-(a+b)x^2+abx \\
& \qquad -cx^2+(ac+bc)x-abc \\
\hline
\end{array}$$

Required product $=x^3-(a+b+c)x^2+(ab+ac+bc)x-abc.$

Ex. 6. Multiply together $1-x$, $1+x$, $1+x^2$ and $1+x^4$.

[See § 102, Ex 6]

$1-x$	$1-x^2$	$1-x^4$
$1+x$	$1+x^2$	$1+x^4$
$1-x$	$1-x^2$	$1-x^4$
$+x-x^2$	$+x^2-x^4$	$+x^4-x^8$
$1-x^2$	$1-x^4$	$1-x^8$ Req. prod

Multiply together

7 $a+x$, $b+y$ and $c+z$

8 $1-ax$, $1-bx$ and $1-cx$

9 $a+b$, $b+c$ and $c+a$

10 $a-b$, $b-c$ and $c-a$

11 $x+1$, $x+2$, $3-x$ and $4-x$

12 $x+a$, $x+b$, $x+c$ and $x+d$

13 $x-a$, $x-b$, $x-c$ and $x-d$

14 $1-ax$, $1-by$ and $1-cz$

15 $ax-by$, $ax+cy$ and $ax-dy$

16 $a+b-c$, $b+c-a$ and $c+a-b$

17 $3x-4y$, $x-2y$, $x+2y$ and $3x+4y$

18 $a+x$, $a-x$, a^2-ax+x^2 and a^2+ax+x^2

19. $x+a$, $x-a$, x^2+a^2, x^4+a^4 and x^8+a^8

20 a^2-ax+x^2, a^2+ax+x^2 and $a^4-a^2x^2+x^4$.

21 $a+b+c$, $b+c-a$, $c+a-b$ and $a+b-c$

89a Multiplication of Quantities with any exponents The Index Law, $a^m \times a^n = a^{m+n}$, has been proved for *positive integers* [§ 68] We shall now *assume*, what will be proved later on, that the Law is true for all values of m and n Hence, quantities having any exponents, *positive* or *negative*, *integral* or *fractional*, are multiplied *by adding their indices*

Examples

Ex 1. Multiply $a^{\frac{3}{4}}$ by $a^{\frac{1}{4}}$, $2x^{\frac{1}{3}}y^{\frac{1}{2}}$ by $-4x^{\frac{1}{2}}y^{\frac{1}{3}}$, $a^{\frac{5}{2}}b^{-1}$ by $3a^{-\frac{1}{2}}b$

$$a^{\frac{3}{4}} \times a^{\frac{1}{4}} = a^{\frac{3}{4}+\frac{1}{4}} = a^1 = a$$

$$2x^{\frac{1}{3}}y^{\frac{1}{2}} \times -4x^{\frac{1}{2}}y^{\frac{1}{3}} = -8x^{\frac{1}{3}+\frac{1}{2}}y^{\frac{1}{2}+\frac{1}{3}} = -8x^{\frac{5}{6}}y^{\frac{5}{6}}.$$

$$a^{\frac{5}{2}}b^{-1} \times 3a^{-\frac{1}{2}}b = 3a^{\frac{5}{2}-\frac{1}{2}}b^{-1+1} = 3a^{\frac{4}{2}}b^0 = 3a^2 \; [\because b^0 = 1]$$

Ex. 2 Multiply x^{-3} by x^2; $-a^x$ by a^{-2x}; $2a^{m+2}$ by $-3a^{2m-1}$

$$x^{-3} \times x^2 = x^{-3+2} = x^{-1}$$

$$-a^x \times a^{-2x} = -a^{x-2x} = -a^{-x}.$$

$$2a^{m+2} \times -3a^{2m-1} = -6a^{m+2+2m-1} = -6a^{3m+1}$$

Ex. 3. Multiply $x^{-2}y^n$ by $ax^{n+1}y$, $-2x^{m+1}y^3$ by $-5x^{2m-1}y^a$

$$x^{-2}y^n \times ax^{n+1}y = ax^{-2+n+1}y^{n+1} = ax^{n-1}y^{n+1}$$

$$-2x^{m+1}y^3 \times -5x^{2m-1}y^a = +10x^{m+1+2m-1}y^{3+a} = 10x^{3m}y^{a+3}$$

Ex. 4 Multiply together $-x^{-\frac{2}{3}}, x^{-1}, 2x^{\frac{5}{3}}, -3x^{\frac{7}{3}}$ and $4x^{-\frac{1}{3}}$.

Required product $= +24x^{-\frac{2}{3}-1+\frac{5}{3}+\frac{7}{3}-\frac{1}{3}} = 24x^2$

Ex 5 Multiply together $-x^{2a+1}, x^{b-2}, -x^{b-a}$, and $-x^{c+3}$

Required product $= -x^{2a+1+b-2+b-a+c+3} = -x^{a+2b+c+2}$.

Ex 6. Multiply $2a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{-3} + 3ab^{-\frac{1}{3}}$ by $a^{\frac{1}{3}}b^{-\frac{2}{3}}$.

$$\begin{aligned}\text{Reqd product} &= 2a^{\frac{2}{3}} \times a^{\frac{1}{3}}b^{-\frac{2}{3}} - a^{\frac{1}{3}}b^{-3} \times a^{\frac{1}{3}}b^{-\frac{2}{3}} + 3ab^{-\frac{1}{3}} \times a^{\frac{1}{3}}b^{-\frac{2}{3}}\\ &= 2a^{\frac{2}{3}+\frac{1}{3}}b^{-\frac{2}{3}} - a^{\frac{1}{3}+\frac{1}{3}}b^{-3-\frac{2}{3}} + 3a^{1+\frac{1}{3}}b^{-\frac{1}{3}-\frac{2}{3}}\\ &= 2ab^{-\frac{2}{3}} - a^{\frac{2}{3}}b^{-\frac{11}{3}} + 3a^{\frac{4}{3}}b^{-1}\end{aligned}$$

Ex 7 Multiply

$$\begin{array}{rl} & a^mb^n + 2a^{-2}b^3 - 3a^{2m+1} + 5b^{n+1}.\\ \text{by} & \underline{-15ab^{-1}\qquad\qquad\qquad\qquad\qquad\qquad}\\ \text{Product} & -15a^{m+1}b^{n-1} - 30a^{-1}b^2 + 45a^{2m+2}b^{-1} - 75ab^n\end{array}$$

Ex 8. Multiply

$$\begin{array}{rl} & a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\\ \text{by} & \underline{a^{\frac{1}{3}} + b^{\frac{1}{3}}\qquad}\\ & a - a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}}\\ & \underline{\qquad + a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b}\\ \text{Product} & a \qquad\qquad\qquad\qquad + b\end{array}$$

Ex. 9 Multiply

$$\begin{array}{rl} & 3x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + 2y^{\frac{2}{3}}\\ \text{by} & \underline{3x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + 2y^{\frac{2}{3}}}\\ & 9x^{\frac{4}{3}} - 3xy^{\frac{1}{3}} + 6x^{\frac{2}{3}}y^{\frac{2}{3}}\\ & \qquad + 3xy^{\frac{1}{3}} - x^{\frac{2}{3}}y^{\frac{2}{3}} + 2x^{\frac{1}{3}}y\\ & \underline{\qquad\qquad + 6x^{\frac{2}{3}}y^{\frac{2}{3}} - 2x^{\frac{1}{3}}y + 4y^{\frac{4}{3}}}\\ \text{Product} & 9x^{\frac{4}{3}} \qquad + 11x^{\frac{2}{3}}y^{\frac{2}{3}} \qquad + 4y^{\frac{4}{3}}.\end{array}$$

Ex 10. Multiply mx^m+nx^n
by x^m+x^n

$$\begin{array}{l} mx^{2m}+nx^{m+n} \\ \qquad +mx^{m+n}+nx^{2n} \\ \hline \end{array}$$

Product $mx^{2m}+(m+n)x^{m+n}+nx^{2n}$

Multiply

11. $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$; $2x^{\frac{2}{3}}$ by $-x^{\frac{1}{3}}$; $-a^{-2}x^{\frac{1}{2}}$ by $3a^{\frac{1}{2}}x^2$.
12. a^2 by a^{-5}; $3a^{-2}b^{\frac{4}{3}}$ by $-ab^{-1}$; a^{n-1} by $5a^{n+2}$.
13. a^mb^{-2} by $6a^{-1}b^2$; $4a^2b^{\frac{1}{2}}c^{-2}$ by $-a^{-1}b^3c^{\frac{1}{2}}$.
14. $3a^m$ by $-8a$; $-18x^{m-1}$ by $-3x^2$; $4y^{4+n}$ by $8y^{n-3}$.
15. $-3ax^m$ by $2a^2x$; $5a^mb^{n+1}$ by $-4a^2b^n$; $-7ax^{m-1}y^2$ by $-6xy^{n-2}$

Multiply together

16. $-a^2$, $2a^{-1}$, a^2 and a^{-4} 17. x^3, $-3x^{-\frac{1}{2}}$, $-x^{\frac{4}{3}}$ and $5x^{-1}$
18. $8a^{-1}b^{\frac{1}{2}}$, $a^{\frac{1}{2}}b^{-1}$, $-2ab^{\frac{3}{2}}$ and $a^{\frac{1}{2}}b^{-\frac{1}{2}}$
19. x^{a-1}, $-x^{2a}$, $-x^{2a-1}$ and x^3.
20. $5x^2y^n$, $-2x^{n-1}z$ and $-yz^2$
21. $-ax^my^2$, $-3x^2y^n$ and $4a^mx^{1-m}y^{n-1}$
22. $a^{m+2}b^{m-1}c$, $-b^2c^{m+1}$, $-c^2a$ and abc

Multiply

23. $ab^{m-n}+c$ by ab^{n-m}. 24. $a^{n-1}b^{r-1}-1$ by $a^{1-n}b$
25. $x^{a-b}y^{c+1}+z^{a-1}$ by $x^{b-a}z^{1-a}$.
26. $3a^xm^y-a^2m^x+2a^{x-1}m^{y-1}$ by $8am$.
27. $ax^{n+1}-2a^2x^{n-2}+3a^3x^{n-1}$ by $-5a^2x^2$
28. $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}+y^{\frac{1}{2}}$. 29. $2x^{\frac{1}{2}}+3y^{\frac{1}{2}}$ by $2x^{\frac{1}{2}}-3y^{\frac{1}{2}}$.
30. $a^{\frac{3}{2}}+b^{\frac{3}{2}}$ by $a^{\frac{3}{2}}-b^{\frac{3}{2}}$. 31. $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.
32. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$. 33. $a+a^{\frac{1}{2}}b^{-\frac{1}{3}}+b^{-\frac{2}{3}}$ by $a^{\frac{1}{2}}-b^{-\frac{1}{3}}$
34. $a^{-\frac{2}{3}}-a^{-\frac{1}{3}}+1$ by $a^{-\frac{1}{3}}+1$.
35. $2x^m-3x^{m-1}+1$ by x^2-1
36. $x^{\frac{3}{2}}-x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}$ by $x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}$.
37. $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ by $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$
38. $x^{\frac{1}{2}}+x^{-\frac{1}{2}}-1$ by $x^{\frac{1}{2}}+x^{-\frac{1}{2}}+1$.
39. $ab^{-1}+1+ba^{-1}$ by $ab^{-1}-1+ba^{-1}$.

Multiply

40 $2a^{\frac{1}{2}}-a^{\frac{1}{4}}x^{\frac{1}{4}}-3x^{\frac{1}{2}}$ by $2a^{\frac{1}{2}}+a^{\frac{1}{4}}x^{\frac{1}{4}}-3x^{\frac{1}{2}}$.

41 $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}-c^{\frac{1}{3}}a^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$

42 $4x^{\frac{4}{3}}+y^{\frac{4}{3}}+9z^{\frac{4}{3}}+2x^{\frac{2}{3}}y^{\frac{2}{3}}-6x^{\frac{2}{3}}z^{\frac{2}{3}}+3y^{\frac{2}{3}}z^{\frac{2}{3}}$ by $2x^{\frac{2}{3}}-y^{\frac{2}{3}}+3z^{\frac{2}{3}}$.

43 $x^{2a}+x^{a}y^{a}+y^{2a}$ by $x^{2a}-x^{a}y^{a}+y^{2a}$

44 $x^{2m}-x^{m}a^{n}+a^{2n}$ by $x^{2m}+x^{m}a^{n}-a^{2n}$.

90 Degree of Product We have remarked in Ex 1, [§ 88 Examples (iii)], that the multiplicand and multiplier being of the *fifth* and *second*, degree respectively, the product is of the (5+2)th, *i e*, *seventh* degree Similarly if other examples be examined, we shall see that the degree of the product will always be the sum of the degrees of the multiplicand and multiplier Hence we generally conclude that *if an expression of any degree be multiplied by another expression of a different degree, the product will be of a degree which denotes the sum of the degrees of the given expressions*

As a consequence of the above conclusion, it is clear that *if two homogeneous expressions of any degrees, be multiplied together, the product will also be homogeneous and of a degree denoted by the number which is the sum of the degrees of the given expressions* [See *Remark*, Ex 4, § 88, Examples (ii), and *Remark*, Ex 2, § 88 Examples, (iii)]

Remark. It is very important to remember this principle, for it will enable the student to test the accuracy of his work For instance, in Ex 4 § 88, Examples (ii), multiplicand and multiplier are both *homogeneous* and of 3 and 2 dimensions respectively, therefore the product will also be *homogeneous* and of 3+2 or 5 dimensions Hence, if any term of the product be found to be of a *higher* or *lower degree than the fifth*, we should at once conclude that it is wrong Thus, for instance, if the last term were bfy^2 instead of bfy^3, we might at once see that it was wrong, for it is of *four* dimensions, whereas it should have been of *five*, and on reference to the work, the error could have been easily detected

91 Highest and Lowest terms of an expression From § 87, Corollary, we learn that when an expression is arranged, in either ascending or descending powers of some letter, there is in it only *one highest* term and *one lowest* term in that letter

This being the case, we learn from *Remark*, Ex 1, [§ 88 Examples (iii)], that when the factors of a product are arranged in *descending* powers of a letter as also the product itself, the *first* term of the product is of the *highest* degree in that letter, and is the product of the *first* terms of the factors, and the *last* term of the product is of the *lowest* degree in that letter, and is the product of the *last* terms of the factors If, on the other hand, the factors, as well as the product, be arranged in *ascending* powers, the reverse will be the case

92 Examination upon Chapter VI.

1 Explain the terms *dimension* and *degree* How is the degree of a term determined? State the degree of each of the terms — $3a^4b^2c$, $5x^6$, $-a^3bcx^2$ and $-2abc$.

2 How is the degree of an expression ascertained? State the degree of the following expressions —

$$(1)\ ax^3-bx^2+cx-d\,,$$
$$(2)\ 3x^5-2a^2x^4+4ax^3+1\,;$$
$$(3)\ a^4x^3+3a^3b^3x^2-2ab^2x+a^5\,,$$

(1) when the symbol of reference is x, and (2) when it is a.

3 What is a *homogeneous* expression? State which of the following expressions are homogeneous and which not, and also their degrees —(1) $2bc+ca-4ab$, (2) $3x^4-2x^2y+y^4$, (3) $ax^2+2hxy+by^2$.

4 Arrange $x^3-2x^2+3x^4-1+4x$ (1) according to the ascending, and (2) according to the descending, powers of x.

5 One of the terms in the product of $3x^2+15xy+4y^2$ by $2x+3y$ is seen to be $39xy$, is the result right? Give your reason.

6. Find the number, which is 5 times the number that exceeds x by 10

7. A's age is x years and B's age is 3 times of what A's age was 8 years ago, what is the age of B?

8 There are $(a+b)$ mangoes in one heap, twice as many in another, and 5 times as many in a third, what is the total number of mangoes? If $(7a-b)$ mangoes be sold out of them, what number remains?

9 My monthly income is $2x$ rupees, and expenditure is $(x-1)$ rupees, what are my annual savings?

CHAPTER VII.

Division.

93 Division of Monomials. Rule:—*Indicate the operation in the form of a fraction* [§71]; *strike out the common factors as in Arithmetic, the remaining factors will be the quotient required.*

Examples (1).

Ex 1 Divide $20abcd$ by $5ac$.

$$\text{Required quotient} = \frac{4\,5ac\,bd}{5ac} = 4bd$$

Ex 2 Divide $156abxyz$ by $-13axz$.

$$\text{Required quotient} = \frac{12 \cdot 13axz \cdot by}{-13axz} = -12by$$

Ex 3 Divide $-2macxy$ by $-mxy$

$$\text{Required quotient} = \frac{-2\,mxy\,ac}{-mxy} = 2ac$$

Ex 4 Divide $13abc$ by $2ad$

$$\text{Required quotient} = \frac{a\,13bc}{a\,2d} = \frac{13bc}{2d} \quad [\S 71]$$

Ex. 5 Divide $24xyz$ by $-28axyz$

$$\text{Required quotient} = \frac{6\,4xyz}{-7a\,4xyz} = -\frac{6}{7a} \quad [\S\, 71]$$

Divide

6 $abxyz$ by ax

7 $-120acdxy$ by $15cxy$

8 $391mnpxy$ by $-23npy$

9 $-15abcx$ by $6aby$

10. $-420axyz$ by $-70axz$

11 $-10abqr$ by $15axpr$

12 $13bcrs$ by $-13abrs$

13 $-2mnxy$ by $-3amyz$

When the Monomials involve powers of the same quantity the method is as follows

Examples (ii)

Ex. 1 Divide $20a^5$ by $5a^2$

We proceed as in § 75, thus

$$\text{Required quotient} = 20a^5 \div 5a^2 = 20 \times a^5 \div 5 \div a^2 \quad [\S\, 70]$$
$$= 20 \div 5 \times a^5 \div a^2 \; [\S\, 73] = 4 \times a^{5-2} = 4a^3$$

Or more conveniently thus —

$$\text{Required quotient} = \frac{20a^5}{5a^2} = \tfrac{20}{5}a^{5-2} = 4a^3.$$

Ex 2. Divide $150a^5x^4$ by $30a^2x^3$

$$\text{Required quotient} = \frac{150a^5x^4}{30a^2x^3} = \frac{150}{30} \times a^{5-2}x^{4-3} = 5a^3x$$

Ex 3. Divide $20a^2x^3y$ by $-4ax^2y$

$$\text{Required quotient} = \frac{20a^2x^3y}{-4ax^2y} = \frac{20}{-4} \times a^{2-1}x^{3-2}y^0 = -5ax$$

Ex 4. Divide $-144x^3y^{10}$ by $-30xy^3$.

Required quotient $=\frac{-144x^3y^{10}}{-30xy^3}=\frac{-144}{-30}\times x^{3-1}y^{10-3}=\frac{24}{5}x^2y^7$.

Divide

5. a^4x^3 by $-a^2x$
6. $3a^2bc$ by ac.
7. $16a^3x^2y^2$ by $-4a^2xy$
8. $-180a^5b^4c^2d^2$ by $45b^3c^2d$
9. $-129x^4y^3z^3$ by $-3x^3y^2z$
10. $-3x^3yz^4$ by $2ax^2yz^3$
11. $-16a^4x^8y^2$ by $-24a^3x^5y$.
12. $-20x^3y^6z^9$ by $5x^2y^5$.
13. $-50a^8b^3m^7$ by $-75a^6bm^4$.
14. $4a^2x^3y^5$ by $5xy^4$.

94 Division of a Polynomial by a Monomial We follow the rule established in § 80

Ex 1. Divide $ax+bx$ by x

Required quotient $=(ax+bx)\div x=ax\div x+bx\div x=a+b$

Otherwise, more conveniently thus :—

Required quotient $=\frac{ax+bx}{x}=\frac{ax}{x}+\frac{bx}{x}=a+b$ [§ 93]

Ex. 2 Divide $3ax^2-2abx+a^2cx$ by $-ax$

Required quotient $=\frac{3ax^2-2abx+a^2cx}{-ax}=\frac{3ax^2}{-ax}-\frac{2abx}{-ax}+\frac{a^2cx}{-ax}$
$=-3x+2b-ac$

Ex 3. Divide $-12a^2x^3+5abx^2-10bx$ by $-2x$

Required quotient $=\frac{12a^2x^3}{2x}-\frac{5abx^2}{2x}+\frac{10bx}{2x}=6a^2x^2-\frac{5}{2}abx+5b$

Divide

4. $ab+bc$ by b
5. $2ax-bx$ by $-x$
6. $3a^2x-2abx$ by ax
7. $-15x^3+5x^2$ by $-5x$
8. $5ab^2-abc+abd$ by ab
9. $-ax^3+a^2x^2-a^3x$ by $-ax$
10. $2x^5-x^4+3x^3$ by $-x^2$
11. $2x^3+6x^2y-9xy^2$ by $3x$
12. $-24a^2x^2y-3axy+6x^2y^2$ by $-3xy$
13. $-144a^3-108a^2b+96ab^2$ by $-12a$
14. $5a^3bx^3y^2-3a^2x^2y^2+2a^4xy^3z^2$ by $-a^2xy^2$.
15. $18a^4b^2c^2-24a^3b^3c^2+30a^2b^4c^2$ by $6a^2b^2c^2$
16. $16a^3xy-14a^2x^2+4a^2x^3$ by $4a^2x$.
17. $30a^3b^2c-24a^4c^3+12a^3c^3x-6a^3c$ by $6a^3c$.
18. $3p^4q-9p^3q^2+3p^2q^3-2pq^4$ by $-3pq$
19. $-12x^6y^8+4x^5y^7-6x^4y^6-3x^3y^5$ by $-3x^3y^5$

95 Division of one Polynomial by another It is easy to see that

$$(x+5)(x^2+2x+3)=x^3+7x^2+13x+15\ ,$$

therefore $(x^3+7x^2+13x+15)\div(x+5)=x^2+2x+3$, by the definition of division, and $x^3+7x^2+13x+15$ is the dividend, $x+5$ is the divisor, and x^2+2x+3 is the quotient [§ 69] Thus if the dividend and the divisor are arranged in *descending* powers of some letter, as x here, the quotient will be arranged in *descending* powers of that letter, and consequently the first term of the dividend will be the *highest* term in it, and will be the product of the *first* terms of the divisor and the quotient [§ 91] Thus *the first term of the quotient is obtained by dividing the first term of the dividend by the first term of the divisor*, that is, the first term of the quotient here is $x^3\div x=x^2$

Again

$$x^3+7x^2+13x+15=(x+5)x^2+(x+5)2x+(x+5)3$$

Subtract from the dividend $(x+5)x^2$, *i e*, x^3+5x^2, which is the product of the divisor into the *first* term of the quotient, thus the remainder is

$$2x^2+13x+15, \text{ which is obviously } =(x+5)2x+(x+5)3$$

Thus if the divisor be multiplied by the first term of the quotient and the result be subtracted from the dividend, we get a *remainder* arranged also in *descending* powers of x and which is equal to the product of the divisor into the remaining terms of the quotient Hence *the second term of the quotient will be obtained by dividing the first term of this remainder by the first term of the divisor* [§ 91], that is, the second term of the quotient here is $2x^2\div x=2x$

Now from $2x^2+13x+15$, subtract $(x+5)2x$, *i e*, $2x^2+10x$, which is the product of the divisor into the *second* term of the quotient, we have thus, the remainder

$$3x+15 \text{ which } =(x+5)3 \text{ evidently}$$

Thus if the product of the divisor into the *second* term of the quotient be subtracted from the *first* remainder we get a *second* remainder, arranged as before in *descending* powers of x, and which is equal to the product of the divisor into the remaining terms of the quotient Hence *if we divide the first term of this remainder by the first term of the divisor, we get the third term of the quotient*, that is, here this term $=3x\div x=3$

If we now subtract $(x+5)3$, *i e*, $3x+15$, which is the product of the divisor into the *third* term of the quotient from the *second* remainder $3x+15$, there is *no more remainder* and consequently the division *terminates*

The process explained above is concisely shewn thus—

$$
\begin{array}{r|l}
x+5 & x^3+7x^2+13x+15\ (\,x^2+2x+3 \\
 & \underline{x^3+5x^2} \\
 & \quad 2x^2+13x+15, \text{ the } \textit{first} \text{ remainder} \\
 & \quad \underline{2x^2+10x} \\
 & \qquad 3x+15, \text{ the } \textit{second} \text{ remainder} \\
 & \qquad \underline{3x+15}
\end{array}
$$

Similarly, the process might have been explained, if the dividend and divisor were arranged in *ascending* powers of x

Hence to divide one polynomial by another, we have the following

RULE —*Arrange both dividend and divisor according to the* DESCENDING *or* ASCENDING *powers of some common letter* [§ 87]

Divide the first term of the dividend by the first term of the divisor to obtain the FIRST TERM *of the quotient; multiply the divisor by this term and subtract the product from the dividend and put down the remainder*

Consider the remainder as a new dividend and repeat the above process; thus the SECOND TERM *of the quotient is obtained*

Continue the same operation with the successive remainders to obtain the OTHER TERMS *of the quotient till there is no remainder left.*

Examples.

Ex. 1 Divide $a^2-2ab+b^2$ by $a-b$

Here the dividend and the divisor are arranged according to the *descending* powers of a

$$
\begin{array}{r|l}
a-b & a^2-2ab+b^2\ (\,a-b \\
 & \underline{a^2-\ ab} \\
 & \quad -\ ab+b^2 \\
 & \quad \underline{-\ ab+b^2}
\end{array}
$$

REMARK We may also arrange the dividend and the divisor according to the *ascending* powers of a and obtain the same result: thus

$$
\begin{array}{r|l}
-b+a & b^2-2ab+a^2\ (\,-b+a, \text{ the same result as before.} \\
 & \underline{b^2-\ ab} \\
 & \quad -\ ab+a^2 \\
 & \quad \underline{-\ ab+a^2}
\end{array}
$$

Ex 2. Divide $11x^2+2x^3+5+17x$ by $2x+5$

Here the divisor is arranged according to the *descending* powers of x, but not the dividend, arrange it then according to the *descending* powers of x

$$
\begin{array}{r|l}
2x+5 & 2x^3+11x^2+17x+5 \ (x^2+3x+1 \\
 & \underline{2x^3+5x^2} \\
 & \qquad 6x^2+17x \\
 & \qquad \underline{6x^2+15x} \\
 & \qquad\qquad 2x+5 \\
 & \qquad\qquad \underline{2x+5}
\end{array}
$$

[Work this example by arranging it according to the *ascending* powers of x]

Ex 3 Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$

$$
\begin{array}{r|l}
ax-b & ax^3-a^2x^2-bx^2+b^2 \ (x^2-ax-b \\
 & \underline{ax^3-bx^2} \\
 & \qquad -a^2x^2 \\
 & \qquad \underline{-a^2x^2+abx} \\
 & \qquad\qquad -abx+b^2 \\
 & \qquad\qquad \underline{-abx+b^2}
\end{array}
$$

Ex 4 Divide $81x^4-1$ by $3x+1$.

$$
\begin{array}{r|l}
3x+1 & 81x^4-1 \ (27x^3-9x^2+3x-1 \\
 & \underline{81x^4+27x^3} \\
 & \qquad -27x^3-1 \\
 & \qquad \underline{-27x^3-9x^2} \\
 & \qquad\qquad 9x^2-1 \\
 & \qquad\qquad \underline{9x^2+3x} \\
 & \qquad\qquad\quad -3x-1 \\
 & \qquad\qquad\quad \underline{-3x-1}
\end{array}
$$

Divide, by arranging (1) according to the *descending* and (2) according to the *ascending*, powers of x—

5 x^2+4x+3 by $x+1$

6 $x^2+7x+12$ by $x+4$

7 $2x^2+5x+2$ by $x+2$

8 $4x^2+23x+15$ by $4x+3$

9 $14x^2+5x-1$ by $7x-1$

10 $x^2-5xy+6y^2$ by $x-2y$

Divide, by arranging (1) according to the *descending* and (2) according to the *ascending*, powers of x—

11. $4x^2-9$ by $2x-3$.

12. $27x^3-1$ by $3x-1$.

13. $4a^3-3a-1$ by $a-1$.

14. $4x^3-2x^2+1$ by $2x+1$.

15. $6x^3-5x^2+6x+8$ by $3x+2$.

16. $m^3-6m^2+11m-6$ by $m-2$.

17. $24x^3-22x^2+17x-5$ by $12x-5$.

18. x^3+a^3 by $x+a$.

19. $6a^2b^2-12b^4-ab^3$ by $3ab+4b^2$.

20. $81-16x^4$ by $2x+3$.

21. $ax^3+(2-a^2)x^2-ax+2$ by $ax+2$.

22. $x^3-a(a+1)x+a^2$ by $x-a$.

23. $a^2x^3-b(a^2+b)x+ab^2$ by $ax-b$.

Ex. 24. Divide $x^4-10x^2+24x-27$ by x^2-2x+3.

$$
\begin{array}{r|l}
x^2-2x+3 & x^4-10x^2+24x-27 \;(\,x^2+2x-9 \\
 & \underline{x^4-2x^3+3x^2} \\
 & \quad 2x^3-13x^2+24x \\
 & \quad \underline{2x^3-\;\;4x^2+\;\;6x} \\
 & \qquad -9x^2+18x-27 \\
 & \qquad \underline{-9x^2+18x-27.}
\end{array}
$$

Ex. 25. Divide $x^3+y^3+3xy-1$ by $x+y-1$.

Arrange *both* dividend and divisor according to the *descending* powers of x [§ 87].

$$
\begin{array}{r|l}
x-1+y & x^3+3xy-1+y^3 \;(\,x^2+x-xy+1+y+y^2 \\
 & \underline{x^3-x^2+x^2y} \\
 & \quad x^2-x^2y+3xy \\
 & \quad \underline{x^2-x\;\;+\;\;xy} \\
 & \quad -x^2y+\;\;x\;\;+2xy \\
 & \quad \underline{-x^2y+xy-xy^2} \\
 & \qquad x+xy+xy^2-1 \\
 & \qquad \underline{x-1\;\;+\;\;y} \\
 & \qquad\quad xy+xy^2-y \\
 & \qquad\quad \underline{xy-\;\;y\;\;+y^2} \\
 & \qquad\qquad xy^2-y^2+y^3 \\
 & \qquad\qquad \underline{xy^2-y^2+y^3.}
\end{array}
$$

Ex 26 Divide $4x^5+5x^3y^2-11x^2y^3-16y^5$ by $2x^2+3xy+4y^2$

Here both the expressions are *homogeneous* [§ 86] and are arranged according to the *descending* powers of x

$$
\begin{array}{r|l|l}
2x^2+3xy+4y^2 & 4x^5+5x^3y^2-11x^2y^3-16y^5 & 2x^3-3x^2y+3xy^2-4y^3 \\
 & \underline{4x^5+6x^4y+8x^3y^2} & \\
 & -6x^4y-3x^3y^2-11x^2y^3 & \\
 & \underline{-6x^4y-9x^3y^2-12x^2y^3} & \\
 & 6x^3y^2+x^2y^3-16y^5 & \\
 & \underline{6x^3y^2+9x^2y^3+12xy^4} & \\
 & -8x^2y^3-12xy^4-16y^5 & \\
 & \underline{-8x^2y^3-12xy^4-16y^5} &
\end{array}
$$

REMARK Here the dividend, the divisor and the quotient are all *homogeneous*

Divide

27 $x^3+8x^2+11x-6$ by x^2+2x-1

28 $14x^3-15x^2-2x^4-35x-4$ by x^2-3x-4

29 $3x^5+11x^2-8x^3-4x^4-2$ by $1-2x+x^3$

30 $30a^2x^2+8x^4-2a^4-29ax^3-7a^3x$ by $3ax-x^2-2a^2$

31 $1-a^2-b^2+2ab$ by $1+a-b$

32 $x^4-y^4+a^4+2a^2x^2$ by $x^2-y^2+a^2$

33 $x^4+x^2y^2+y^4$ by x^2+xy+y^2 34 a^4+4b^4 by $a^2-2ab+2b^2$

35 $4x^4y^4+1$ by $2x^2y^2-2xy+1$

36 $256x^4+16x^2y^2+y^4$ by $16x^2+4xy+y^2$

37. $2-x+16x^3-8x^4$ by $2+3x-2x^2$

38 $a^4-9ab^3+18b^4$ by a^2+3b^2-3ab

39 $4x^5-x^3+4x$ by $3x+2+2x^2$

40. $6x^4-19x^3+17x^2-5x$ by $1-3x+3x^2$

41 $a^5-4a^3b^2-8a^2b^3-17ab^4-12b^5$ by $a^2-2ab-3b^2$

42 $14a^4+15a^3b+33a^2b^2+36ab^3+28b^4$ by $7a^2-3ab+14b^2$

43 $9x^4-12x^3y+13x^2y^2-4xy^3+y^4$ by $9x^2-3xy+y^2$

44. $2y^2-5xy+2x^2-ay-ax-a^2$ by $2y-x+a$

45 $x^6+2x^3y^3+y^6$ by $x^2+2xy+y^2$

46 $5x^6+6x^5+1$ by x^2+2x+1

47 $m^6-6mn^5+5n^6$ by $m^2-2mn+n^2$

48. $x^3-3ax^2+3a^2x-a^3+b^3$ by $x-a+b$

Divide

49 $a^3+b^3-3ab+1$ by $a+b+1$. 50 $1+x^3-8y^3+6xy$ by $1+x-2y$

51 $8x^3-y^3+z^3+6xyz$ by $y-z-2x$.

52 $a^3-b^3-27c^3-9abc$ by $3c-a+b$

53 $21x^5-2x^4-70x^3-23x^2+33x+27$ by $3x^3-2x^2-5x-3$

54. $15a^5-17a^4x+20a^3x^2+39a^2x^3-7ax^4-13x^5$ by $5a^2+ax-2x^2$.

55 $8a^6+21a^3x^3-x^6-24a^5x$ by $3ax-x^2-a^2$.

56. $50x^3y^2-6y^5+25xy^4-45x^2y^3-41x^4y+20x^5$ by $5xy^2-3y^3-4x^2y+5x^3$

57 $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$

58 $a^4+b^4+2a^2b^2-2c^2d^2-c^4-d^4$ by $a^2+b^2-c^2-d^2$

59 $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$ by $a^2+b^2-c^2+2ab$

60. $x^7-x^4y^3-x^3y^4+y^7$ by $x^4-x^3y-xy^3+y^4$

61 $x^8+x^6y^2+x^4y^4+x^2y^6+y^8$ by $x^4+x^3y+x^2y^2+xy^3+y^4$

[In the following examples, remove the brackets, arrange each example, according to powers of some letter, and then proceed as usual]

62 $(a+b)^2-c^2$ by $a+b+c$. 63 $8x^3+y^6$ by $2x+y^2$.

64 $x^3-(y-z)^3$ by $x-y+z$ 65 $27a^6-8x^9$ by $3a^2-2x^3$

66 x^3+mx^2-mx-1 by $x-1$ 67 a^5+b^5 by $a+b$.

68 $a^6-3a^4x-a^3x^2+4a^2x^3-x^5$ by a^2-x^2.

69 $y^6-my^4+ny^3-ny^2+my-1$ by $y-1$

70. $x^3-apx^2+a^2px-a^3$ by $x-a$ 71 $\frac{1}{3}-6a^2+27a^4$ by $1-6a+9a^2$

72 $\frac{1}{4}-\frac{19}{24}x+\frac{5}{8}x^2+\frac{1}{8}x^3-x^4+\frac{3}{4}x^5$ by $\frac{1}{4}-\frac{2}{3}x+\frac{1}{2}x^2$

73 $x^3+(4ab-b^2)x-(a-2b)(a^2+3b^2)$ by $x-a+2b$

74 $x^4-a^4+2ax^3+2na^3x-(n^2-1)a^2x^2$ by $x^2+a^2-(n-1)ax$

75 $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $c^2-(a+b)c+ab$

76 $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $c^2-(a+b)c+ab$.

77. $xy^3+2xyz^2+y^3z+yz^3-2xy^2z-xz^3-y^2z^2$ by $xy-xz+yz$.

78. $x^2+y^2-xy\{(x-y)^2+2\}+1$ by x^2-xy+1

79 $(a+b)(x+y)(ax+by)-bx-ay-1$ by $ax+by-1$

95a Division of Quantities with any exponents. It has been proved [§ 75] that $a^m \div a^n = a^{m-n}$ where m and n are *positive integers* and m *greater than* n We shall here *assume* that this relation always holds for *all values* of m and n

Examples

Ex. 1 Divide $a^{\frac34}$ by $a^{\frac14}$, x^2 by x^{-3}, $-a^x$ by a, $6x^5$ by $3x^{-2}$

$a^{\frac34}\div a^{\frac14}=a^{\frac34-\frac14}=a^{\frac24}=a^{\frac12}$

$x^2\div x^{-3}=x^{2+3}=x^5.$

$-a^x\div a=-a^{x-1}$

$6x^5\div 3x^{-2}=2x^{5+2}=2x^7.$

Ex. 2 Divide $3x^{\frac25}y^{\frac45}$ by $-x^{\frac15}y^{-\frac35}$, $25x^4y^{m+1}$ by $-10x^5y^m$

$3x^{\frac25}y^{\frac45}\div(-x^{\frac15}y^{-\frac35})=-3x^{\frac25-\frac15}y^{\frac45+\frac35}=-3x^{\frac15}y^{\frac75}$

$25x^4y^{m+1}\div(-10x^5y^m)=-\frac{25}{10}x^{4-5}y^{m+1-m}=-\frac52x^{-1}y$

Ex. 3. Divide $a+b$ by $a^{\frac13}+b^{\frac13}$

$$
\begin{array}{r|l}
a^{\frac13}+b^{\frac13}\,)\;a+b & (a^{\frac23}-a^{\frac13}b^{\frac13}+b^{\frac23} \\
a+a^{\frac23}b^{\frac13} & \\
\hline
-a^{\frac23}b^{\frac13}+b & \\
-a^{\frac23}b^{\frac13}-a^{\frac13}b^{\frac23} & \\
\hline
a^{\frac13}b^{\frac23}+b & \\
a^{\frac13}b^{\frac23}+b. & \\
\hline
\end{array}
$$

Simplify

4 $a^{\frac83}\div a^{\frac54}$, $x^{\frac32}\div x^{-\frac12}$, $8a^{-\frac38}\div 4a^{-\frac58}$; $15ay^{-2}\div 5a^{-\frac12}y^2$.

5. $4a^{\frac23}x^{\frac34}\div -2a^{\frac13}x^{\frac12}$, $a^2b^{-\frac23}\div 2a^{-1}b^{\frac12}$, $-3x^{-\frac34}y^{-1}z^{\frac12}\div 4xy^{-2}z^{-2}$.

Divide

6 $x-y$ by $x^{\frac13}-y^{\frac13}$.

7 $x^{\frac43}-xy^{\frac13}-x^{\frac13}y+y^{\frac43}$ by $x-y$

8. $x^{-3}-x^{-2}-2x^{-1}+2$ by $x^{-1}-1$

9. x^2-y^2 by $x^{\frac12}+y^{\frac12}$

10 $a^{\frac34}+b^{\frac34}$ by $a^{\frac14}+b^{\frac14}$

11 $x^3-x^{\frac32}y^{\frac32}+2x^{\frac34}y^{\frac94}-y^3$ by $x^{\frac32}+x^{\frac34}y^{\frac34}-y^{\frac32}$

12. $a+b+c-3a^{\frac13}b^{\frac13}c^{\frac13}$ by $a^{\frac13}+b^{\frac13}+c^{\frac13}$

13. $a^{2m}-3a^mc^n+2c^{2n}$ by a^m-2c^n

14. $2x^{m+n}-x^my^{-n}+2x^ny^n-1$ by x^m+y^n

15 $x^{-3}+y$ by $x^{-2}-x^{-1}y^{\frac13}+y^{\frac23}$.

16 $x^{-6}+2x^{-5}y-x^{-4}y^{\frac43}+y^2$ by $x^{-3}-x^{-2}y^{\frac23}+y$.

17 $x^{2n}-4x^2y^{-1}+4xy^{\frac12}-y^{\frac32}$ by $x^n-2xy^{-\frac12}+y^{\frac34}$

96 Interminable Division Let us divide $2x^3+5x^2+8x+5$ by x^2+x+2

$$\begin{array}{r|l} x^2+x+2 & 2x^3+5x^2+8x+5 \;(\,2x+3 \\ & \underline{2x^3+2x^2+4x} \\ & \qquad 3x^2+4x+5 \\ & \qquad \underline{3x^2+3x+6} \\ & \qquad\qquad x-1 \end{array}$$

If we continue the process, we get a *fractional* term *viz.*, $\frac{1}{x}$ in the Quotient and by further continuing the operation, a *series* of fractional terms will be obtained Hence as soon as a *fractional* term enters a Quotient, the division ceases to be terminable, and the Quotient becomes *infinite* In such a case the Quotient may be represented as in Arithmetic, by putting, after the *integral part*, the *last remainder* over the Divisor, in the form of a fraction

Thus in the above example, the quotient $=2x+3+\dfrac{x-1}{x^2+x+2}$

Definition From the above it is easy to see that when Dividend and Divisor are both arranged according to the *descending* powers of the symbol of reference, *that residue which is of a lower degree than the divisor is called the Remainder*

Corollary Since $27=4\times6+3$, where 4 is the divisor, 6 the quotient and 3 the remainder, we get

$$2x^3+5x^2+8x+5=(x^2+x+2)(2x+3)+(x-1)$$

Hence generally if D represents the Dividend, d the Divisor, Q the Quotient and R the Remainder, we have

$$D=dQ+R.$$

Examples

Ex 1 Divide x^2-5x+7 by $x-2$, and find the remainder

$$\begin{array}{r|l} x-2 & x^2-5x+7 \;(\,x-3 \\ & \underline{x^2-2x} \\ & \quad -3x+7 \\ & \quad \underline{-3x+6} \\ & \qquad\quad 1 \end{array}$$

$\therefore$ remainder $=1$

Ex 2 Divide x^2-px+q by $x-a$, and find the remainder

$$\begin{array}{r|l} x-a & x^2-px+q \;(\,x+(a-p) \\ & \underline{x^2-ax} \\ & (a-p)x+q \\ & \underline{(a-p)x-a(a-p)} \end{array}$$

$\therefore$ remainder $=a(a-p)+q=a^2-pa+q$

Ex 3 Divide px^3-qx^2+rx-s by $x-a$, and find the remainder

$$
\begin{array}{l}
x-a\,)\,px^3-qx^2+rx-s\,(px^2+(ap-q)x+(a^2p-aq+r) \\
\quad\underline{px^3-apx^2} \\
\qquad (ap-q)x^2+rx \\
\qquad \underline{(ap-q)x^2-(a^2p-aq)x} \\
\qquad\qquad (a^2p-aq+r)x-s \\
\qquad\qquad \underline{(a^2p-aq+r)x-(a^3p-a^2q+ar)} \\
\qquad\qquad \text{remainder } =pa^3-qa^2+ra-s
\end{array}
$$

REMARK. From Ex 2 and Ex. 3, it is evident that, when the divisor is $x-a$, the remainder is of the form of the Dividend with a written in the place of x We shall call attention to this fact in a subsequent Chapter [See § 275]

As the quotient, in an Interminable Division, is not finite, we can only find a limited number of terms

Ex 4. Divide a by $1-x$, keeping 4 terms of the quotient

$$
\begin{array}{l}
1-x)\,a\quad(a+ax+ax^2+ax^3+\&c \\
\quad\underline{a-ax} \\
\qquad ax \\
\qquad\underline{ax-ax^2} \\
\qquad\qquad ax^2 \\
\qquad\qquad\underline{ax^2-ax^3} \\
\qquad\qquad\qquad ax^3 \\
\qquad\qquad\qquad\underline{ax^3-ax^4} \\
\qquad\qquad\qquad\qquad ax^4
\end{array}
$$

$\therefore$ required quotient $=a+ax+ax^2+ax^3$.

Ex. 5 Divide $1+x$ by $1-x$, and keep 5 terms of the quotient

$$
\begin{array}{l}
1-x)\,1+x\,(1+2x+2x^2+2x^3+2x^4+\&c \\
\quad\underline{1-x} \\
\qquad 2x \\
\qquad\underline{2x-2x^2} \\
\qquad\qquad 2x^2 \\
\qquad\qquad\underline{2x^2-2x^3} \\
\qquad\qquad\qquad 2x^3 \\
\qquad\qquad\qquad\underline{2x^3-2x^4} \\
\qquad\qquad\qquad\qquad 2x^4 \\
\qquad\qquad\qquad\qquad\underline{2x^4-2x^5} \\
\qquad\qquad\qquad\qquad\qquad 2x^5
\end{array}
$$

$\therefore$ required quotient $=1+2x+2x^2+2x^3+2x^4$

Ex. 6. Divide $x^{pq}-1$ by x^q-1, and write down the last 3 terms of the quotient.

$$
\begin{array}{r|l}
x^q-1 & x^{pq}-1 \quad (x^{pq-q}+x^{pq-2q}+\&c \\
 & \underline{x^{pq}-x^{pq-q}} \\
 & \quad x^{pq-q}-1 \\
 & \quad \underline{x^{pq-q}-x^{pq-2q}} \\
 & \qquad x^{pq-2q}-1,
\end{array}
$$

To find the last three terms, invert the order of the terms in the Dividend and the Divisor, thus

$$
\begin{array}{r|l}
-1+x^q & -1+x^{pq} \quad (1+x^q+x^{2q}+\&c \\
 & \underline{-1+x^q} \\
 & \quad -x^q \\
 & \quad \underline{-x^q+x^{2q}} \\
 & \qquad -x^{2q} \\
 & \qquad \underline{-x^{2q}+x^{3q}} \\
 & \qquad\quad -x^{3q}
\end{array}
$$

$\therefore$ the required quotient $=x^{pq-q}+x^{pq-2q}+\ldots\ldots+x^{2q}+x^q+1$.

Ex 7. Divide x^n-y^n by $x-y$.

$$
\begin{array}{r|l}
x-y & x^n-y^n \quad (x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3+\&c. \\
 & \underline{x^n-x^{n-1}y} \\
 & \quad x^{n-1}y-y^n \\
 & \quad \underline{x^{n-1}y-x^{n-2}y^2} \\
 & \qquad x^{n-2}y^2-y^n \\
 & \qquad \underline{x^{n-2}y^2-x^{n-3}y^3} \\
 & \qquad\quad x^{n-3}y^3-y^n \\
 & \qquad\quad \underline{x^{n-3}y^3-x^{n-4}y^4} \\
 & \qquad\qquad x^{n-4}y^4-y^n
\end{array}
$$

If x be the symbol of reference, find the remainder in the division of

8. $10x^2+14x-9$ by $2x+4$. 9 $8x^2-26xy+14y^2$ by $4x-3y$

10 $18x^4-45x^3+82x^2-62x+37$ by $6x^2-7x+8$

11. $1+x^3-6y^3+6xy$ by $1+x-2y$

12. $(a^2+a)x^3+(2a-1)x^2+ax+3$ by $x-a$

13. $x^2+(m-n)x+1-m^2$ by $x-m-n$

14. $x^4-(p+q)x^3+(p+pq+q)x^2-(p^2+q^2-1)x+1$ by x^2-px+q.

15 Divide $a-x$ by $a+2x$, and keep four terms of the quotient

16 Divide 1 by $1-ax$

17 Divide $1-x+x^2$ by $1-x$, and keep the n^{th} term and the last 3 terms of the quotient

18 Divide $a-bx$ by $a+cx$, and write down the first 4 terms of the quotient

19 What number must be added to $x^3+x^2-4(x+3)$ that it may be divisible by $x-6$?

20 If the dividend be x^3+5x^2+6x-4, the divisor x^2+2x-1 and the remainder $x-1$, what is the quotient?

97 Degree of quotient Since *divisor × quotient = dividend*, the degree of the dividend is the *sum* of the degrees of the divisor and the quotient [§ 90], and therefore *the degree of the quotient is the difference of the degrees of the dividend and the divisor* Thus for example if the dividend be of the *fourth* degree and divisor of the *first* degree, the quotient will be of the $(4-1)$th, *i e*, *third* degree, and so on [See § 95, Exs 1—4]

Hence *if the dividend and the divisor be both homogeneous, the quotient will also be homogeneous and of a degree denoted by the number which is the difference of the degrees of the dividend and the divisor* [See § 95, Ex 26]

REMARK. This principle is useful to remember as it enables us to test the accuracy of the work For example if the *third* term of the quotient in Ex 26, [§ 95] were $3xy$ instead of $3xy^2$, we could at once say that it was wrong, for *each of the terms of the quotient must be of* $5-2$ or 3 *dimensions*

97a Examination upon Chapter VII

1 State the rule for the division of one polynomial by another and shew clearly by an example how you obtain this rule

2 Find the quotient when $5x^2+4x-10$ is divided by 5

3 Find the quotient when $ax-2bx+c$ is divided by x

4 Express symbolically the quotient when the difference of x and y is divided by their product. Find also the quotient in a simple form

5. If one of the terms in the quotient of $x^4-2a^2x^2+16a^3x-15a^4$ by $x^2+2ax-3a^2$ be $-2a^2x$, would the result be right? If not, why?

6 If D represent the dividend, d the divisor and Q the quotient, express the relation between D, d and Q.

7 If the divisor be $x(x-1)+a(x+1)$ and the quotient be $x(x+1)+a(x-1)$, what is the dividend?

8 If the dividend be $4a^2b^2+2(3a^4-2b^4)-ab(5a^2-11b^2)$ and the quotient be $2(a+b)a+a^2-b^2$, what is the divisor?

9 The product of two expressions is $ax^3-(2a-3)x^2-7x+2$ and one of them is $x-2$, find the other

10. A person leaves x rupees and a property of the same value, to be divided among his four sons, find the value of each share.

11 Find the number that is a sixth of the number which is equal to the sum of x and y diminished by unity

Miscellaneous Examples II

1. If $a=4$, $x=5$, $y=3$, find the numerical value of

$$\sqrt[3]{[5(x^2-y^2)-a^2]}+\sqrt[4]{3[a(a^2-y^2)-1]}.$$

2. Add together $3(a^2-xy+c^2)$, $(a-x)(c-y)$ and $cx+ay-ac$

3 Subtract $ax-by$ from $(a+b)(x-y)$.

4 Multiply x^2+ax-a by $x-a$.

5 Divide $p^4-2p^3q+p^2q^2-1$ by p^2-pq-1

6 If $x=a+2b+3c$, $y=b+2c+3a$, $z=c+2a+3b$, shew that

$$x+y+z=6(a+b+c).$$

---

7 If $a=4$, $x=5$ and $y=3$, find the value of

$$2a^2-x\sqrt[4]{(x+y)^3(4y-a)}+\sqrt{y(2a-x)(x^2-a^2)}$$

8 Add together $(2x-1)(x-2y)$, $2(1-x)(x-y)$ and $x(2y-1)$

9 Subtract $(a-b)x-(c-d)y$ from $(a+b)x+(c-d)y$.

10 Multiply $3y^2-2x(y-2x)$ by $4x^2-y(y+2x)$ and arrange the result according to the ascending powers of x

11. Divide $1-x-3x^2-x^5$ by $(x+1)^2$

12 If $A=b^2+c^2-a^2$, $B=c^2+a^2-b^2$, $C=a^2+b^2-c^2$, $D=a^2+b^2+c^2$, shew that $$A+B+C+D=2(a^2+b^2+c^2)$$

---

13 If $a=4$, $x=5$ and $y=3$, find the value of

$$\frac{\sqrt[3]{(a^3+x^3+y^3)}+2\sqrt{(y^2+a^2)}}{\sqrt[3]{4(x^2-y^2)}}.$$

14 Add $5(x^2+y^2)$ to the difference of $(x-3y)(2y-x)$ and $5x(y-x)$

15. Take $2(a+x)+10-14(x+y)$ from $5(a+x)-10-8(x+y)$

16 Multiply $x^3(x^2-1)+2$ by $x(x^2-1)+2$

17. Divide $a^2x^3-(a^2+b^2)x-ab$ by $ax+b$.

18. Shew that $a(a^2-bc)+b(b^2-ca)+c(c^2-ab)+3abc=a^3+b^3+c^3$.

---

19 If $a=2$, $b=4$, $c=1$, $x=3$, find the value of $ax^a-ca^b+xb^{ca}$.

20 Add together $3\{x-4(y-3z)\}$ and $5y-2\{z-4(3x-y)\}$

21. Take $3x-8$ from the sum of $(x+1)(x-3)$ and $(4-x)(x-2)$

22. Multiply $4(x^2-y^2)+2x(x-y)$ by $3x(2x-y)+y(x+4y)$.

23 Divide $6a^4+4a^3x-9a^2x^2-3ax^3+2x^4$ by $2a(x+a)-x^2$.

24. If $a=x(1+y)$, $b=y(1+z)$, $c=xy+x(1+y)$,

shew that $a+b-c=x+y-z$

---

25. If $a=2$, $b=4$, $c=1$, $x=3$, find the value of

$$\sqrt{(3abx+bc^x-x^3)}+3\sqrt{(b^a+x^2)}$$

26 Add $x+y$, $3x-y-3z$ and $2y-2x+z$, and multiply the result by $x-y-z$

27. Subtract $(x-1)(x-7)$ from $(x-2)(x-3)$, and add the result to $(x-1)(x-2)$

28 Multiply $2x^2-5xy+3y^2$ by $x-7y$, and divide the product by $x^2-8xy+7y^2$

29 Divide $a^3-3a^2+3a+b^3-1$ by $a+b-1$

30 If $A=ax+by$ and $B=bx-ay$, find the value of $(Aa+Bb)-(a^2+b^2)$

---

31 If $a=2$, $b=4$, $c=1$, $x=3$, find the value of

$$\sqrt[3]{(4a^2b^a-10b^c)}+\sqrt[3]{(3bx-67)}$$

32 What must be added to $2a^3-6a^2x+3ax^2-1$ in order that the sum may be a^3-ax^2-1?

33 Simplify

$3x-4(2y-x)-2\{x-(4y-3x)\}$ and $2x-3(x-2y)-4\{y-2(3x-2y)\}$, and multiply the results together

34. Divide $8m^4-5mn^3-3n^4$ by $2m^2-mn-n^2$.

35 Shew that $(a+bx)(b-ax)+(ax+b)(bx-a)+2x(a+b)(a-b)=0$

---

36 Find the value of $\dfrac{a-5\sqrt[3]{a}}{\sqrt{(a-48)}}-(\sqrt{a}-7)(\sqrt[3]{a}-2)$, when $a=64$.

37 Simplify $a^2-3\{b^2-2c(a-3b)-2b(b+3c)\}$ and find its numerical value when $a=-2$, $b=3$, $c=-4$

38 Simplify the product $(x+a)(x+b)$ and from the result deduce the value of $(x+1)^2$

39 Divide $x(x^2-yz)+y(y^2-zx)+z(z^2-xy)$ by $x+y+z$

40 If $x=b-c$, $y=c-a$, $z=a-b$, find the values of

$$ax+by+cz \text{ and } (b+c)x+(c+a)y+(a+b)z.$$

41. Shew that $(ax+b)(cx+d)-(a+bx)(c+dx)+(ac-bd)(1-x^2)=0$

---

42. Find the value of $\frac{x^2}{y^2}-\sqrt{\frac{1+x}{1-y}}+\frac{1+x}{1-y}$, when $x=\frac{1}{4}$; $y=\frac{1}{5}$.

43 Subtract $2x^4-3x^3y+4x^2y^2$, from $2y^4-3y^3x+4y^2x^2$, and arrange the result according to the descending powers of x

44. Multiply a^3-x^3 by $a+x$, and divide the product by $a-x$.

45. Find the value of $2x^2-3x+4$, (1) when x is changed into $x+1$ and (2) when x is changed into $x-1$.

46 Divide the product of $2x^2-5x-12$ and x^2+x-30 by

$$2x^2-7x-15.$$

47. Shew that $(ax+b)(cx+d)-(cx+b)(ax+d)+(a-c)(b-d)x=0$.

---

48. Find the value of $x^{2x-1}-2x^{x+1}+(2x-1)^x$, when $x=2$.

49 If $a=-1$, $b=-2$, $c=-3$, find the value of

$$\{a-(b-c)\}^3+\{b-(c-a)\}^3+\{c-(a-b)\}^3.$$

50 Simplify $(x+1)(x+2)(x+3)-(x-1)(x-2)(x-3)$

51 The product of two expressions is $4a(a+b+c)+10bc-3(b^2+c^2)$ and one of them is $2a+3b-c$, find the other

52. Divide x^3+y^3 by $x+y$ and from the result deduce the quotient when $(a+b)^3+c^3$ is divided by $a+b+c$

53 If $A=a^2-bc$, $B=b^2-ca$, $C=c^2-ab$, shew that

$$(Aa+Bb+Cc)\div(A+B+C)=a+b+c$$

CHAPTER VIII.

ALGEBRAICAL FORMULÆ AND THEIR APPLICATION.

98 Formulæ In the present Chapter we propose to give certain important results in multiplication and to shew their application in reducing algebraical expressions These results, which are usually called *formulæ*, should be carefully committed to memory, as they are of great use in shortening the process of multiplication

Definition. Any general result expressed by means of symbols is called a FORMULÆ

99 Formula I $(a+b)^2=a^2+2ab+b^2$ [§ 88 Ex, (1), 4]

This formula corresponds to Euc II 4, and is enunciated thus — *The square of the* **sum** *of two quantities = the sum of their squares* **plus** *twice their product*

Cor Hence *conversely* $a^2+2ab+b^2=(a+b)^2$ [See § 123]

Examples

Ex. 1. Find the square of $x+3$

$$(x+3)^2=x^2+2\cdot 3\cdot x+3^2=x^2+6x+9$$

Ex. 2. Find the square of $3x+8y$

$$(3x+8y)^2=(3x)^2+2(3x)(8y)+(8y)^2=9x^2+48xy+64y^2$$

Ex 3 Find the square of $ax+by$

$$(ax+by)^2=(ax)^2+2(ax)(by)+(by)^2=a^2x^2+2abxy+b^2y^2$$

Ex 4 Find the square of $a+b+c$

Here are 3 terms, but enclosing $b+c$ in a bracket, we may consider these two terms as one Hence

$$\begin{aligned}(a+b+c)^2&=\{a+(b+c)\}^2\\ &=a^2+2\,a\,(b+c)+(b+c)^2\\ &=a^2+(2ab+2ac)+(b^2+2bc+c^2)\\ &=a^2+b^2+c^2+2ab+2ac+2bc\end{aligned}$$

Ex 5 Find the square of $a+b+c+d$.

Here are 4 terms, but enclosing $a+b$ and $c+d$ in brackets, we may consider them as single terms Hence

$$\begin{aligned}(a+b+c+d)^2&=\{(a+b)+(c+d)\}^2\\ &=(a+b)^2+2(a+b)(c+d)+(c+d)^2\\ &=(a^2+2ab+b^2)+2(a+b)c+2(a+b)d+(c^2+2cd+d^2)\\ &=(a^2+2ab+b^2)+(2ac+2bc)+(2ad+2bd)+(c^2+2cd+d^2)\\ &=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd\end{aligned}$$

Note Hence generally *the square of a polynomial = the sum of the squares of each term + twice the product of every two of them*

Ex 6 Multiply x^3+y^3+3xy by x^3+3xy

$$\begin{aligned}\text{Required product }&=\{(x^3+3xy)+y^3\}(x^3+3xy)\\ &=(x^3+3xy)^2+y^3(x^3+3xy)\\ &=(x^3)^2+2x^3(3xy)+(3xy)^2+x^3y^3+3xy^4\\ &=x^6+6x^4y+9x^2y^2+x^3y^3+3xy^4\\ &=x^6+6x^4y+x^3y^3+9x^2y^2+3xy^4\end{aligned}$$

Ex. 7. Find the value of $25a^2+10ab+b^2$, when $a=-2$, $b=7$.

Given expression $=(5a)^2+2(5a)b+b^2$

$=\{(5a)+b\}^2$ [Cor.] $=\{5(-2)+7\}^2$

$=(-10+7)^2=(-3)^2=(-3)(-3)=9$.

Ex. 8. Simplify $(x-2y+z)^2+2(x-2y+z)(x+2y-z)+(x+2y-z)^2$.

Let $a=x-2y+z$ and $b=x+2y-z$, thus given expression

$=a^2+2ab+b^2=(a+b)^2$ [Cor.]

$=\{(x-2y+z)+(x+2y-z)\}^2=(2x)^2=4x^2$.

Find the value of

9. $(2x+7)^2$ 10. $(a+2b)^2$ 11. $(2x+3a)^2$ 12. $(ax+1)^2$

13. $(3+2xy)^2$ 14. $(l^2+mn)^2$ 15. $(2pq+r^2)^2$ 16. $(ab+2c)^2$

17. $(2ax+3by)^2$ 18. $(a^2+x^2)^2$ 19. $(a^2x+ax^2)^2$. 20. $(2a^3+b^3)^2$

21. $(x^3+3yz)^2$. 22. $(x^4+y^4)^2$ 23. $(a^2+ab+b^2)^2$

24. $(3a+5b+4c)^2$ 25. $(2x+6y+z)^2$ 26. $(a^2+b^2+c^2)^2$

27. $(3mx+2ny+z^2)^2$. 28. $(x+2y+3z+4u)^2$. 29. $(4a+5b+c+2d)^2$.

30. $(2x+3y+4z)(3y+4z)$ 31. $(4a+7b+c)(4a+7b)$.

32. $(x^2+xy+y^2)y(x+y)$ 33. x^2+y^2, when $x=a+3$ and $y=b+2$

34. $4a^2+9b^2$, when $2a=p+1$, $3b=q+1$

35. $a^2+4ab+4b^2$, when $a=8$ and $b=-6$

36. $9x^2+24xy+16y^2$, when $x=-4$, $y=3$.

37. $36p^2+60pq+25q^2$, when $p=5$ and $q=-6$

38. $4m^2+28mn+49n^2$, when $m=7$ and $n=-1$

39. $9x^2+12xy+4y^2$, when $x=a-2$ and $y=a+3$

Simplify

40. $(x-y)^2+2(x-y)(x+y)+(x+y)^2$

41. $(2a+b-1)^2+2(2a+b-1)(2a-b+1)+(2a-b+1)^2$

42. $(x+y+z)^2+2(x+y+z)(x-y+z)+(x-y+z)^2$.

100. Formula II $(a-b)^2=a^2-2ab+b^2$ [§ 88, Ex. (1), 5]

This formula corresponds to Euc. II. 7, and is enunciated thus:— *The square of the* **difference** *of two quantities = the sum of their squares* **minus** *twice their product.*

Cor. Hence *conversely* $a^2-2ab+b^2=(a-b)^2=(b-a)^2$ [see § 123]

REMARK. It is to be observed that Formula I virtually includes Formula II. For if we put $-b$ for $+b$ in Formula I, it becomes $\{a+(-b)\}^2=a^2+2a(-b)+(-b)^2$ or $(a-b)^2=a^2-2ab+b^2$ by the Law of Signs. Thus

Formula II, which was obtained independently before, is now deduced from Formula I *by the substitution of* $-b$ for $+b$

Examples

Ex 1 Find the square of $2x-3y$

$$(2x-3y)^2=(2x)^2-2(2x)(3y)+(3y)^2=4x^2-12xy+9y^2.$$

Ex. 2 Find the square of $ax-by$

$$(ax-by)^2=(ax)^2-2(ax)(by)+(by)^2=a^2x^2-2abxy+b^2y^2$$

Ex. 3 Find the square of $a-b+c$

Here enclose the last two terms in a bracket, thus

$$\begin{aligned}(a-b+c)^2&=\{a-(b-c)\}^2\\&=a^2-2a(b-c)+(b-c)^2\\&=a^2-2a(b-c)+b^2-2bc+c^2\\&=a^2+b^2+c^2-2ab+2ac-2bc\end{aligned}$$

Note This result may be obtained by putting $-b$ for b in Ex 4 of § 99 [See *Remark*]

Ex. 4 Find the square of x^2-ax-b

$$\begin{aligned}(x^2-ax-b)^2&=\{(x^2-ax)-b\}^2\\&=(x^2-ax)^2-2(x^2-ax)b+b^2\\&=x^4-2x^2(ax)+(ax)^2-2bx^2+2abx+b^2\\&=x^4-2ax^3+(a^2-2b)x^2+2abx+b^2.\end{aligned}$$

Ex. 5 Find the square of $2a-b-3c+d$.

$$\begin{aligned}(2a-b-3c+d)^2&=\{(2a-b)-(3c-d)\}^2\\&=(2a-b)^2-2(2a-b)(3c-d)+(3c-d)^2\\&=(4a^2-4ab+b^2)-2(6ac-3bc-2ad+bd)\\&\qquad+(9c^2-6cd+d^2)\\&=4a^2+b^2+9c^2+d^2-4ab-12ac+4ad+6bc-2bd-6cd\end{aligned}$$

Ex 6 Multipl x^2+ax-b by $ax-b$ [§ 88, Ex. (ii), 18]

$$\begin{aligned}\text{Required product}&=(x^2+ax-b)(ax-b)\\&=\{x^2+(ax-b)\}(ax-b)\\&=x^2(ax-b)+(ax-b)^2\\&=ax^3-bx^2+a^2x^2-2abx+b^2\\&=ax^3+(a^2-b)x^2-2abx+b^2.\end{aligned}$$

Ex 7 Find the value of $16x^2-56xy+49y^2$ when $x=14$, $y=8$.
Given expression $=(4x)^2-2(4x)(7y)+(7y)^2=(4x-7y)^2$ [*Cor*]
$$=(4\times 14-7\times 8)^2=(56-56)^2=0.$$

Ex. 8 Simplify $(x-y+2z)^2-2(x+y-2z)(x-y+2z)+(x+y-2z)^2$.
Let $a=x-y+2z$ and $b=x+y-2z$, thus given expression
$$=a^2-2ab+b^2=(a-b)^2$$
$$=\{(x-y+2z)-(x+y-2z)\}^2=(-2y+4z)^2=(2y-4z)^2$$
$$=4y^2-16yz+16z^2$$

Find the value of

9. $(3x-4)^2$ 10. $(x-5y)^2$ 11. $(8-3ax)^2$. 12. $(3a-4b)^2$.
13. $(ab-xy)^2$ 14 $(1-abc)^2$ 15 $(c^2-2ab)^2$ 16. $(3pq-4r^3)^2$.
17 $(x^3-a^3)^2$. 18 $(a^4-b^4)^2$ 19 $(a^3-3a^2b)^2$. 20 $(m^2n^2-l^4)^2$.
21 $(2mx^2-3m^2x)^2$ 22. $(ax-by+cz)^2$ 23. $(ab-xy-c^2)^2$.
24. $(3x-2y-3z)^2$ 25 $(1-2x+3x^2)^2$ 26. $(a-b-c+d)^2$.
27. $(x^2-2xy-y^2+1)^2$ 28 $(3x^3-4ax^2-a^3)(3x^3-a^3)$.
29 $(4x^2-5ax+3a^2)a(5x-3a)$
30 $2x^2-y^2$, when $x=a-3$ and $y=2b-5$
31 $16x^2-25y^2$, when $4x=2p-1$ and $5y=3q-1$.
32 $25x^2-40xy+16y^2$, when $x=8$ and $y=10$.
33. $9a^2-42ab+49b^2$, when $a=3$ and $b=-2$.
34. $16m^2-48mn+36n^2$, when $m=-18$, $n=-12$
35. $4x^2y^2-12xy+9$, when $x=-3$ and $y=-4$.

Simplify

36 $(2x-1)^2-2(2x-1)(x-1)+(x-1)^2$
37. $(3x-2y+1)^2-2(3x+2y+1)(3x-2y+1)+(3x+2y+1)^2$.
38 $(2ax-by+3z)^2-2(2ax+by-3z)(2ax-by+3z)+(2ax+by-3z)^2$

***101.** From §§ 99 and 100, we have
$$(a+b)^2=(a-b)^2+4ab \qquad\text{(i)}$$
and
$$(a-b)^2=(a+b)^2-4ab. \qquad\text{(ii)}.$$
[For $(a+b)^2=a^2+b^2+2ab=a^2+b^2-2ab+4ab=(a-b)^2+4ab$;
and $(a-b)^2=a^2+b^2-2ab=a^2+b^2+2ab-4ab=(a+b)^2-4ab$]
Again from the same articles, we have
$$a^2+b^2=(a+b)^2-2ab=(a-b)^2+2ab \qquad\text{(iii)}.$$
[For $a^2+b^2=a^2+b^2+2ab-2ab=(a+b)^2-2ab$, or $=(a-b)^2+2ab$]

Also from (iii), by addition, we get

$$2(a^2+b^2)=(a+b)^2+(a-b)^2$$

or $$a^2+b^2=\tfrac{1}{2}(a+b)^2+\tfrac{1}{2}(a-b)^2 \quad \ldots\ \ldots\ldots \quad \text{(iv)}$$

Lastly, since $(a+b)^2-(a-b)^2=(a^2+2ab+b^2)-(a^2-2ab+b^2)=4ab$,

we have $$4ab=(a+b)^2-(a-b)^2,$$

or dividing by 4, $$ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2 \quad \ldots\ \ldots\ \ldots\ldots \quad \text{(v)}$$

The result (v) enables us to express the product of two or more factors *as the difference of two squares* [see § 127]

Examples

Ex 1 Find the values of $(x+y)^2$ and $x+y$, when $x-y=2$, $xy=15$, and of $(x-y)^2$ and $x-y$, when $x+y=8$, $xy=12$

$$(x+y)^2=(x-y)^2+4xy=2^2+4\times 15=64, \qquad x+y=\sqrt{64}=8$$
$$(x-y)^2=(x+y)^2-4xy=8^2-4\times 12=16, \qquad x-y=\sqrt{16}=4$$

Ex. 2 Find the value of $x^2+2xy+y^2$, when $x=625$ and $y=624$

$$(x+y)^2=(x-y)^2+4xy=(625-624)^2+4\times 625\times 624$$
$$=1^2+2500\times 4\times 156=1+10000\times 156=1560001$$

Ex 3 Find the value of x^2+y^2, (1) when $x+y=10$ and $xy=24$; (2) when $x-y=3$ and $xy=28$

(1) $x^2+y^2=(x+y)^2-2xy=(10)^2-2\times 24=52$ [from (iii)]

(2) $x^2+y^2=(x-y)^2+2xy=3^2+2\times 28=65$ [from (iii)]

Ex 4 Find the value of x^2+y^2, when $x+y=17$ and $x-y=9$

$$x^2+y^2=\tfrac{1}{2}(x+y)^2+\tfrac{1}{2}(x-y)^2=\tfrac{1}{2}\times(17)^2+\tfrac{1}{2}\times 9^2=\tfrac{289}{2}+\tfrac{81}{2}=185$$

Ex 5 Find the value of xy, when $x+y=17$ and $x-y=3$

$$xy=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2=\left(\frac{17}{2}\right)^2-\left(\frac{3}{2}\right)^2=\frac{289-9}{4}=70$$

Ex 6 Express $(a+b-2c)^2+2(b-c)(c-a)$ as the sum of two squares

Since $a+b-2c=(b-c)-(c-a)$, we have

Given expn $=\{(b-c)-(c-a)\}^2+2(b-c)(c-a)$

$=(b-c)^2+(c-a)^2$ [by (iii)]

Ex 7 Express $x(x-4)$, x^2+12x, and $(x-a)(x-3a)$ as the difference of two squares

(1) $x(x-4)=\left\{\dfrac{x+(x-4)}{2}\right\}^2-\left\{\dfrac{x-(x-4)}{2}\right\}^2$ [by v]

[here $a=x$, $b=x-4$]

$$=\left(\frac{2x-4}{2}\right)^2-\left(\frac{4}{2}\right)^2=(x-2)^2-2^2$$

(2) $x^2+12x=x(x+12)=\left\{\dfrac{x+(x+12)}{2}\right\}^2-\left\{\dfrac{x-(x+12)}{2}\right\}^2$ [by v]

$$=\left(\frac{2x+12}{2}\right)^2-\left(\frac{-12}{2}\right)^2=(x+6)^2-(-6)^2=(x+6)^2-6^2.$$

(3) $(x-a)(x-3a)=\left\{\dfrac{(x-a)+(x-3a)}{2}\right\}^2-\left\{\dfrac{(x-a)-(x-3a)}{2}\right\}^2$

[here $a=x-a$, $b=x-3a$]

$$=\left(\frac{2x-4a}{2}\right)^2-\left(\frac{2a}{2}\right)^2=(x-2a)^2-a^2$$

Ex. 8. Express as the difference of two squares x^4+4a^4

$$x^4+4a^4=(x^2)^2+(2a^2)^2=(x^2+2a^2)^2-2x^2(2a^2) \text{ [by (iii)]}$$
$$=(x^2+2a^2)^2-4a^2x^2=(x^2+2a^2)^2-(2ax)^2$$

[For other examples, see § 124]

Ex 9 Shew that $8xy(x^2+y^2)=(x+y)^4-(x-y)^4$

$$8xy(x^2+y^2)=(4xy)\times 2(x^2+y^2)$$
$$=\{(x+y)^2-(x-y)^2\}\{(x+y)^2+(x-y)^2\} \text{ [v \& iv]}$$
$$=(x+y)^4-(x-y)^4 \text{ [§ 103]}$$

Ex. 10 Shew that

$$(a+b)^2(x-y)^2+4xy(a-b)^2=(a-b)^2(x+y)^2+4ab(x-y)^2.$$

Left side $=\{(a-b)^2+4ab\}(x-y)^2+\{(x+y)^2-(x-y)^2\}(a-b)^2$ [i & v]

$$=(a-b)^2(x-y)^2+4ab(x-y)^2+(a-b)^2(x+y)^2-(a-b)^2(x-y)^2$$
$$=(a-b)^2(x+y)^2+4ab(x-y)^2.$$

Ex 11 If $x+y=a$, $xy=b$, shew that $(1+x^2)(1+y^2)=a^2+(1-b)^2$.

$$(1+x^2)(1+y^2)=1+x^2+y^2+x^2y^2=1+(x+y)^2-2xy+x^2y^2 \text{ [iii]}$$
$$=1+a^2-2b+b^2=a^2+(1-2b+b^2)=a^2+(1-b)^2$$

Find the value of

12 $x^2+2xy+y^2$, when $x-y=4$, $xy=117$.

13 $x-y$, when $x+y=5$, $xy=6$

14. $2x+a$, when $2x-a=13$, $ax=7$.

15. x^2+y^2, (1) when $x+y=13$, $xy=42$, (2) when $x-y=7$, $xy=24$; (3) when $x+y=18$, $x-y=4$

16 xy, when $x+y=15$ and $x-y=11$.

Find the value of

17. $m^2+2mn+n^2$, when $m=126$ and $n=125$

18. x^2+y^2, and of xy, when $x=621$ and $y=379$

19. x^2+y^2, when $x=a^2+ab+b^2$, $y=a^2-ab+b^2$

20. x^2+xy+y^2, when $x+y=5$ and $xy=6$

21. x^2-xy+y^2, when $x-y=4$ and $xy=21$

22. If $x=ab+cd$, $y=ab-cd$, find x^2+xy+y^2 and x^2-xy+y^2, in terms of a, b, c and d

23. Shew that

(1). $x^2-2(x-y)y=(x-y)^2+y^2$

(2) $(x+y-z)^2+2(x+y)z=(x+y)^2+z^2$

(3) $(x+y)^2+z^2=(x+y+z)^2-2(yz+zx)$

(4) $(a+b)^2+c^2=(a+b-c)^2+2(bc+ac)$

(5) $2(a+b)^2+2(a-b)^2=(2a)^2+(2b)^2$

(6). $2(a^2+ab)^2+2(ab+b^2)^2=(a+b)^4+(a^2-b^2)^2$

(7) $(a-2b+c)^2+4(a-b)(b-c)=(a-c)^2$

(8) $(x+2y)^2-4(x+y)y=x^2$

(9). $(a-b)^2-2(b-c)(c-a)=(b-c)^2+(c-a)^2$

(10). $(a-b)^2+2(b+c)(c+a)=(b+c)^2+(c+a)^2$

24. Shew that $(a^2-b^2)^2=\{(a+b)^2-4ab\}\{(a-b)^2+4ab\}$

25. Shew that $(a+b)^2(x-y)^2+4xy(a-b)^2=(a+b)^2(x+y)^2-16abxy$

26. If $x^2+y^2=2xy=2a^2$, shew that $(1+x)(1+y)=(1+a)^2$

27. If $x+y=2a$, $x-y=2b$, $xy=c^2$, shew that $c^2=a^2-b^2$

28. Express as the difference of two squares —

(1) x^2-10x, (2) $x(x+18)$, (3) $(x+a)(x+5a)$,
(4) $(x-a)(x+3a)$, (5) $(x+1)(x+2)(x+3)(x+4)$

29. Shew that $x^4-7x^2y^2+9y^4=(x^2-3y^2)^2-x^2y^2$

30. Prove that $(a^2+ab+b^2)^2+(a^2-ab+b^2)^2=2(a^4+3a^2b^2+b^4)$

31. Express $(a^2+b^2+c^2+2ab)^2-2(a+b)^2c^2$ as the sum of two perfect squares

102. Formula III $(a+b)(a-b)=a^2-b^2$ [§ 88, Ex. (1), 6]

This formula corresponds to Euc II 5, and is enunciated thus —*The product of the* **sum** *and* **difference** *of two quantities* $=$*the* **difference** *of their squares*

Cor Hence *conversely* $a^2-b^2=(a+b)(a-b)$ Thus the factors of an expression of the from a^2-b^2 are $a+b$ and $a-b$. [See § 124]

Examples

Ex. 1 Find the product of $x+2$ and $x-2$.
Required product $=x^2-2^2=x^2-4$.

Ex. 2 Find the product of $ax+by$ and $ax-by$.
Required product $=(ax)^2-(by)^2=a^2x^2-b^2y^2$.

Ex. 3 Find the product of $2x^2+5y^2$ and $2x^2-5y^2$.
Required product $=(2x^2)^2-(5y^2)^2=4x^4-25y^4$.

Ex. 4 Find the product of $a+b+c$ and $a+b-c$.
Required product $=\{(a+b)+c\}\{(a+b)-c\}$
$=(a+b)^2-c^2=a^2+2ab+b^2-c^2$ [1].

Ex. 5. Multiply x^2-ax+a^2 by x^2+ax+a^2 [§ 88, Ex. (iii), 3]
Required product $=(\overline{x^2+a^2}-ax)(\overline{x^2+a^2}+ax)$
$=(x^2+a^2)^2-(ax)^2$
$=x^4+2a^2x^2+a^4-a^2x^2=x^4+a^2x^2+a^4$

Ex. 6 Find the product of $1-x$, $1+x$, $1+x^2$ and $1+x^4$ [Ex. 6, § 89].
Required product $=(1-x)(1+x)(1+x^2)(1+x^4)$
$=(1-x^2)(1+x^2)(1+x^4)$
$=(1-x^4)(1+x^4)=1-x^8$

Find the product of

7. $x+5$ and $x-5$.
8. $2x+7$ and $2x-7$.
9. $3x+4y$ and $3x-4y$
10. $ax-3$ and $ax+3$.
11. $2ax+3by$ and $2ax-3by$
12. x^2+2y and x^2-2y.
13. a^2+x^2 and a^2-x^2.
14. $2ab+c^2$ and $2ab-c^2$.
15. $2a^3-3b^3$ and $2a^3+3b^3$.

Multiply together

16. $a+b-c$ and $a-b+c$
17. $a-b-c$ and $a+b-c$.
18. $a+b+c$ and $a-b+c$
19. $x+2y+3z$ and $x-2y-3z$
20. $x-2y+3z$ and $x-2y-3z$
21. $x+2y-3z$ and $x-2y+3z$
22. $a^2+2ab+b^2$ and $a^2-2ab+b^2$ [Ex. (iii), 4, § 88]
23. x^2+ab-b^2 and x^2+ab+b^2 [Ex. (iii), 7, § 88].
24. $a+x$, $a-x$, and a^2+x^2
25. x^4+y^4, x^2+y^2, $x+y$ and $x-y$.
26. x^2-x+1, x^2+x+1 and x^4-x^2+1.
27. $a-b-c+d$ and $a-b+c-d$
28. $x+y-z+1$ and $x-y+z+1$

Ex 29 Resolve $4x^2-9y^2$ into factors

$4x^2-9y^2=(2x)^2-(3y)^2=(2x+3y)(2x-3y)$,

the factors of $4x^2-9y^2$ are $2x+3y$ and $2x-3y$

Resolve into factors

(1) $1-x^2$	(2) m^2-16	(3) $64-q^2$
(4) $1-81z^2$	(5) $25y^2-1$	(6) $16a^2-9b^2$
(7) $25a^2x^2-49b^2$	(8) $144x^2-169a^2$	(9). $81q^2-64r^2$
(10) $625a^2x^4-121$	(11) $81p^2-100q^2$	(12) $144x^2-121y^2$
(13) $49a^2c^2-81d^4$	(14) $l^2m^2-n^2q^2$	(15) $9z^4-16x^2y^2$
(16) $25a^2z^2-4c^2y^2$	(17) a^4-x^4	(18) $16x^4-25a^6$

Ex 30 Shew that $(a-b+c)^2-(a+b-c)^2=4a(c-b)$

Given expression $=\{(a-b+c)+(a+b-c)\}\{(a-b+c)-(a+b-c)\}$

$=(a-b+c+a+b-c)(a-b+c-a-b+c)$

$=2a\times(2c-2b)=4a(c-b)$

31 Simplify $(2a-b+3c)^2-(2a+b-3c)^2$

32 Simplify $(x^2+xy-2y^2)^2-(x^2-xy+2y^2)^2$

33 Simplify $(2x-3y+4z)^2-(2x+3y-4z)^2$

34 Simplify $(a-2b)(a+2b)+(2b+3c)(2b-3c)+(3c-d)(3c+d)$

35 Simplify $(a-b+c)^2+2\{a^2-(b-c)^2\}+(a+b-c)^2$

36 Simplify $(2a+b+c)^2-2\{4a^2-(b+c)^2\}+(2a-b-c)^2$

37 Shew that $(x^2+2xy+3y^2)^2-(x^2-2xy+3y^2)^2=8xy(x^2+3y^2)$

38 Shew that $(a^2+ab-b^2)^2-(a^2-ab+b^2)^2=4a^2(ab-b^2)$.

39 Shew that $(a^2+ab)^2-(ab+b^2)^2=(a+b)^3(a-b)$

40 Find the value of $a^2x^2-b^2y^2$, when $x=2a-b$ and $y=a-2b$

41. Find the value of x^2-xy+y^2, when $x=ab+cd$ and $y=ab-cd$

103 Formula IV $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$=a^3+b^3+3ab(a+b)$ [§ 89 Ex 1]

That is, *the cube of the* **sum** *of two quantities = the* **sum** *of their cubes*, **plus** *three times their product into their* **sum**

Examples

Ex 1 Multiply $p^2+4pq+4q^2$ by $p+2q$ [§ 88, Ex (ii), 9].

$(p^2+4pq+4q^2)(p+2q)=(p+2q)^2(p+2q)$ [§ 99] $=(p+2q)^3$

$=p^3+3p^2(2q)+3p(2q)^2+(2q)^3$

$=p^3+6p^2q+12pq^2+8q^3$.

Ex. 2. Find the value of $(ax+by)^3$.

Required value $=(ax)^3+3(ax)^2(by)+3(ax)(by)^2+(by)^3$.

$=a^3x^3+3a^2bx^2y+3ab^2xy^2+b^3y^3$

Ex. 3. Find the value of $(x+y+z)^3$

Required value $=\{x+(y+z)\}^3$.

$=x^3+3x^2(y+z)+3x(y+z)^2+(y+z)^3$.

$=x^3+3x^2y+3x^2z+3x(y^2+z^2+2yz)+y^3+3y^2z+3yz^2+z^3$

$=x^3+y^3+z^3+3x^2y+3xy^2+3x^2z+3xz^2+3y^2z+3yz^2+6xyz$.

Multiply

4. $4x^2+12xy+9y^2$ by $2x+3y$. 5 $25m^2+20mn+4n^2$ by $5m+2n$.

Find the value of

6. $(3a+2b)^3$ 7. $(1+2x)^3$ 8. $(1+x^2)^3$

9 $(ax+by+1)^3$. 10. $(1+x+x^2)^3$. 11. $(1+2x+3x^2)^3$

12. $(2x+y+3z)^3$. 13 $(bc+ca+ab)^3$. 14 $(a^2+ab+b^2)^3$.

Cor. I Hence *conversely* $a^3+3a^2b+3ab^2+b^3=(a+b)^3$

Ex. 15 Resolve $x^3+6x^2y+12xy^2+8y^3$ into factors

Given expression $=x^3+3x^2(2y)+3x(2y)^2+(2y)^3=(x+2y)^3$

Thus the factors are $x+2y$, $x+2y$ and $x+2y$

Resolve into factors

16 a^3+3a^2+3a+1. 17 $1+3y+3y^2+y^3$

18. $8x^3+12x^2y+6xy^2+y^3$. 19 $27x^3+54ax^2+36a^2x+8a^3$.

Ex 20 Simplify $(a+b)^3+3(a+b)^2(a-b)+3(a-b)^2(a+b)+(a-b)^3$

Let $m=a+b$ and $n=a-b$, thus the given expression

$=m^3+3m^2n+3mn^2+n^3=(m+n)^3$

$=\{(a+b)+(a-b)\}^3=(2a)^3=8a^3$

Simplify

21 $(x+2)^3+3(x+2)^2(x-2)+3(x+2)(x-2)^2+(x-2)^3$.

22 $(2a-b)^3+3(2a-b)^2(a+b)+3(2a-b)(a+b)^2+(a+b)^3$

23 $(5x-3y)^3+12x(3y-x)(5x-3y)+(3y-x)^3$

24. $(3a-2b)^3+3(3a-2b)(3b-2a)(a+b)+(3b-2a)^3$

25 $(x-4)^3+(2x+7)^3+9(x-4)(2x+7)(x+1)$.

26 $(a+b+c)^3+6a\{a^2-(b+c)^2\}+(a-b-c)^3$.

27 $(2x+3y-1)^3+(2x-3y+1)^3+12x\{4x^2-(3y-1)^2\}$

Ex 28 Find the value of $8a^3+36a^2b+54ab^2+27b^3$, when $a=-$ and $b=3$

Given expn $=(2a)^3+3(2a)^2(3b)+3(2a)(3b)^2+(3b)^3$

$=(2a+3b)^3=\{2(-5)+3\times 3\}^3=(-10+9)^3=(-1)^3=-1.$

Find the value of

29 $x^3+12x^2y+48xy^2+64y^3$, when $x=3$ and $y=-2.$

30 $27k^3+108k^2+144k+72$, when $k=-2$

31 $8a^3+36a^2+54a+8$, when $a=-4$

32 $x^3+6x^2y+12xy^2-8y^3$, when $x=5$ and $y=-1$

Ex 33 If $x+y=3$, shew that $x^3+9xy+y^3=27$

Since $x+y=3$, we have $(x+y)^3=3^3$,

thus $x^3+3xy(x+y)+y^3=27$,

or $x^3+3xy\times 3+y^3=27$, i e $x^3+9xy+y^3=27$

34 If $x+y=1$, shew that $x^3+3xy+y^3=1$

35 If $x+y=a$, shew that $x^3+3axy+y^3=a^3$

36 If $x^3+y^3=1$, find the value of $x^9+3x^3y^3+y^9$

37 If $2x+3y=4$, find the value of $8x^3+72xy+27y^3$

Cor 2 Hence also $a^3+b^3=(a+b)^3-3ab(a+b)$

[For $a^3+b^3=a^3+b^3+3ab(a+b)-3ab(a+b)=(a+b)^3-3ab(a+b)$]

Another form of this result is $3ab(a+b)=(a+b)^3-a^3-b^3$

Ex 38 If $x+y=2$ and $xy=3$, find the value of x^3+y^3

$x^3+y^3=(x+y)^3-3xy(x+y)=2^3-3\times 3\times 2=8-18=-10$

Ex 39 If $a+b=2$ and $a^3+b^3=3$, find the value of ab

We have $3ab(a+b)=(a+b)^3-a^3-b^3$,

thus $3ab\times 2=2^3-3$, or $6ab=5$, i e $ab=\frac{5}{6}$

40 Find the value of x^3+y^3, (1) when $x+y=2$ and $xy=1$, (2) when $x+y=4$ and $xy=2$

41 If $2p+3q=4$ and $pq=5$, find the value of $8p^3+27q^3$

42 Given $2x+y=2$ and $8x^3+y^3=-28$, find the value of xy

104 Formula V $(a-b)^3=a^3-3a^2b+3ab^2-b^3$

$=a^3-b^3-3ab(a-b)$ [§ 89, Ex 2]

That is, *the cube of the* difference *of two quantities = the* difference *of their cubes,* minus *three times their product into their* difference

Remark It is easy to see that Formula IV virtually includes this formula

Examples

Ex. 1 Multiply $4x^2-4xy+y^2$ by $2x-y$

Multiplicand $=(2x-y)^2$ [II];

$$(4x^2-4xy+y^2)(2x-y)=(2x-y)^2(2x-y)=(2x-y)^3$$
$$=(2x)^3-3(2x)^2y+3(2x)y^2-y^3$$
$$=8x^3-12x^2y+6xy^2-y^3$$

Ex. 2 Find the value of $(3x-4y)^3$

Required value $=(3x)^3-3(3x)^2(4y)+3(3x)(4y)^2-(4y)^3$
$$=27x^3-108x^2y+144xy^2-64y^3.$$

Ex 3, Find the value of $(ax-by+cz)^3$

Reqd value $=\{ax-(by-cz)\}^3$
$$=(ax)^3-3(ax)^2(by-cz)+3ax(by-cz)^2-(by-cz)^3$$
$$=a^3x^3-3a^2bx^2y+3a^2cx^2z+3ax(b^2y^2-2bcyz+c^2z^2)$$
$$-\{(by)^3-3(by)^2(cz)+3by(cz)^2-(cz)^3\}$$
$$=a^3x^3-b^3y^3+c^3z^3-3a^2bx^2y+3ab^2xy^2+3a^2cx^2z$$
$+3ac^2xz^2+3b^2cy^2z-3bc^2yz^2-6abcxyz$ (clearing and arranging).

Multiply

4 x^2-4x+4 by $x-2$ 5 $25x^2-20xy+4y^2$ by $5x-2y$

Find the value of

6 $(x-1)^3$. 7 $(2x-3)^3$ 8 $(1-4x)^3$

9 $(2x-3y)^3$. 10 $(2-x^2)^3$ 11. $(ax-y^2)^3$

12. $(x-y+z)^3$. 13 $(2x-3y+1)^3$ 14 $(1-x-x^2)^3$.

Cor 1 We have *conversely* $a^3-3a^2b+3ab^2-b^3=(a-b)^3$ Thus an expression of the given form can be resolved into 3 factors of the form $a-b$, $a-b$ and $a-b$

Resolve into factors

15. $1-3x+3x^2-x^3$ 16 $8x^3-12ax^2+6a^2x-a^3$.

Simplify

17 $(2x-1)^3-3(2x-1)^2(2x+3)+3(2x-1)(2x+3)^2-(2x+3)^3$

18 $(a+b)^3-3(a+b)^2(a-b)+3(a+b)(a-b)^2-(a-b)^3$.

19 $(a+b+c)^3-(b+c-a)^3-6a\{(b+c)^2-a^2\}$

20 $(2a+3b)^3-(3a+2b)^3+3(a-b)(3a+2b)(2a+3b)$

Ex 21. Find the value of $m^3-6m^2n+12mn^2+8n^3$, when $m=7$ and $n=2$

Given expn $=\{m^3-3m^2(2n)+3m(2n)^2-8n^3\}+16n^3$
$$=(m-2n)^3+16n^3=(7-4)^3+16(2^3)=3^3+128=155$$

Find the value of

22. $x^3-9x^2y+27xy^2-27y^3$ when $x=-3$ and $y=-1$

23 $8x^3-60x^2+150x-135$, when $x=-2$

24 $2a^3-3ab(a-b)+b^3$, when $a=3$ and $b=5$

25 If $x-y=2$, shew that $x^3-6xy-y^3=8$

26 If $x-y=a$, shew that $x^3-3axy-y^3=a^3$

27 If $x^2-y^2=3$, shew that $x^6-9x^2y^2-y^6=27$

28 If $ax-by=c$, find the value of $a^3x^3-3abcxy-b^3y^3$

Cor 2 From this formula, we get $a^3-b^3=(a-b)^3+3ab(a-b)$

[For $a^3-b^3=a^3-b^3-3ab(a-b)+3ab(a-b)=(a-b)^3+3ab(a-b)$]

Another form of this result is $(a^3-b^3)-(a-b)^3=3ab(a-b)$

Ex 29 If $x-y=3$ and $xy=1$, find the value of x^3-y^3

$$x^3-y^3=(x-y)^3+3xy(x-y)=3^3+3\times3=27+9=36$$

Ex 30 If $x-y=2$ and $x^3-y^3=20$, find the value of xy

We have $3xy(x-y)=x^3-y^3-(x-y)^3$,

$$\therefore\ 3xy\times2=20-2^3, \text{ or } 6xy=12, \text{ i e } xy=2$$

31 Find the value of x^3-y^3, when $x-y=2$ and $xy=3$

32 Find the value of m^3-27n^3 when $m-3n=4$ and $mn=1$

33 Given $x-y=-5$ and $x^3-y^3=10$, find the value of xy

105 Formula VI $(a+b)(a^2-ab+b^2)=a^3+b^3$

[§ 88, Ex (ii), 1]

That is, *the* sum *of two quantities into the sum of their squares diminished by their product=the* sum *of their cubes*

Examples

Ex 1 Multiply x^2-2x+4 by $x+2$ [§ 88, Ex (ii), 6]

Required product$=(x^2-2x+2^2)(x+2)=x^3+2^3=x^3+8$,

[since here $a=x$ and $b=2$ evidently]

Ex 2 Multiply $4a^2-6ab+9b^2$ by $2a+3b$

$$\text{Required product}=\{(2a)^2-(2a)(3b)+(3b)^2\}(2a+3b)$$
$$=(2a)^3+(3b)^3=8a^3+27b^3$$

Ex 3 Write down the value of $(4x^2-6xy+9y^2)(2x+3y)$.

Since $4x^2=(2x)^2$, $6xy=(2x)(3y)$ and $9y^2=(3y)^2$, the given expression is of the form $(a^2-ab+b^2)(a+b)$ Hence

value required$=(2x)^3+(3y)^3=8x^3+27y^3$

Multiply

4 x^2-xy+y^2 by $x+y$ 5 $a^2-2ax+4x^2$ by $a+2x$.

6. $9x^2-3x+1$ by $3x+1$. 7. $1-4x+16x^2$ by $4x+1$.

8 x^4-x^2+1 by $1+x^2$. 9 $16a^4-20a^2x+25x^2$ by $4a^2+5x$.

10 $25x^2-40xy+64y^2$ by $5x+8y$ [§ 88, Ex. (ii), 16]

Write down the value of

11 $(x^2-3x+9)(x+3)$ 12 $(4a^2-2a+1)(2a+1)$

13 $(9m^2-6m+4)(3m+2)$ 14 $(4a^2-2ab+b^2)(2a+b)$

15 Simplify $(x^2-xy+y^2+1)(x+y)-(x^3+y^3)$

Cor We have *conversely* $a^3+b^3=(a+b)(a^2-ab+b^2)$. Hence an expression of the form a^3+b^3 can be resolved into factors.

Ex 16 Resolve $8x^3+1$ into factors

$$8x^3+1=(2x)^3+1^3=(2x+1)\{(2x)^2-2x\cdot1+1^2\}$$
$$=(2x+1)(4x^2-2x+1)$$

Resolve into factors

17 a^3+64 18 $27x^3+a^3$ 19 $8a^3+1$

20 $1+27k^3$. 21 $125a^3+27x^3$ 22 x^3+64y^3

23 $64a^3+125b^3$ 24. $343x^3+8$ 25 x^6+y^6.

106 Formula VII $(a-b)(a^2+ab+b^2)=a^3-b^3$, [§ 88, Ex (ii), 2].

That is, *the* difference *of two quantities into the sum of their squares increased by their product = the* difference *of their cubes.*

REMARK The student should notice that this Formula is obtained from Formula VI, by putting $-b$ for $+b$

Examples

Ex 1 Multiply x^2+3x+9 by $x-3$ [§ 88, Ex (ii), 7]

Required product $=(x^2+3x+3^2)(x-3)$

$=x^3-3^3=x^3-27$; [. here $a=x$, $b=3$].

Ex 2 Multiply $25a^2+10ab+4b^2$ by $5a-2b$ [§ 88, Ex. (ii), 15].

Required product $=\{(5a)^2+5a(2b)+(2b)^2\}(5a-2b)$

$=(5a)^3-(2b)^3=125a^3-8b^3$.

Ex. 3. Write down the value of $(9x^2+6xy+4y^2)(3x-2y)$.

Since $9x^2=(3x)^2$, $6xy=(3x)(2y)$ and $4y^2=(2y)^2$, the proposed expression is of the form $(a^2+ab+b^2)(a-b)$ Hence

value required $=(3x)^3-(2y)^3=27x^3-8y^3$.

Multiply

4. x^2+xy+y^2 by $x-y$ 5 $a^2+2ax+4x^2$ by $a-2x$

6 $4a^2+6ab+9b^2$ by $2a-3b$ 7 $1+2x^2+4x^4$ by $1-2x^2$

8 a^2+a+1 by $a-1$ 9 $1+x^2+x^4$ by x^2-1

10 $4x^2+6x+9$ by $2x-3$ 11 $9x^4+6x^2yz+4y^2z^2$ by $3x^2-2yz$.

Write down the value of

12 $(x^2+2x+4)(x-2)$ 13 $(x^2+4x+16)(x-4)$

14. $(4m^2+18m+81)(2m-9)$ 15 $(9a^2+15ab+25b^2)(3a-5b)$

16 Simplify $(x^2+2xy+4y^2+x+2y)(x-2y)-(x^3-4y^2)$

17 Simplify $(a^2+ax+x^2)(a-x)+a^2(a+2x)$

Cor *Conversely*, $a^3-b^3=(a-b)(a^2+ab+b^2)$

Thus an expression of the form a^3-b^3 can be resolved into factors

Ex 18 Resolve $27x^3-1$ into factors

$$27x^3-1=(3x)^3-1^3=(3x-1)\{(3x)^2+3x\cdot 1+1^2\}$$
$$=(3x-1)(9x^2+3x+1)$$

Resolve into factors

19 x^3-8y^3 20. $27a^3-b^3$ 21 $64k^3-1$ 22 $512x^3-27$.

23 $54m^3-2$ 24 $a^3x^3-64y^3$ 25 $8a^3c^6-b^3$

107 Formula VIII $(x+a)(x+b)=x^2+(a+b)x+ab$

[§ 88, Ex (i), 7]

Cor Conversely $x^2+(a+b)x+ab=(x+a)(x+b)$ Thus an expression of the form $x^2+(a+b)x+ab$ can be resolved into factors Similarly the converse of the Formulæ in §§ 108, 109 are true

Remark This general formula includes the formulæ of §§ 108 and 109 as particular cases

Examples

Ex 1 Write down the product of $x+4$ and $x+5$

Since $4+5=9$ and $4\times5=20$, therefore required product
$$=x^2+9x+20.$$

Ex. 2 Write down the product of $ax+1$ and $ax+2$

Here $1+2=3$ and $1\times2=2$,

therefore required product $=(ax)^2+3(ax)+2=a^2x^2+3ax+2$

Ex 3 Multiply $a+b+2$ by $a+b+5$.

Put $a+b=x$, thus required product

$$=(x+2)(x+5)=x^2+(2+5)x+2\times 5$$
$$=x^2+7x+10=(a+b)^2+7(a+b)+10$$
$$=a^2+b^2+2ab+7a+7b+10$$

Write down the value of

4. $(x+3)(x+7)$ 5 $(x+8)(x+11)$ 6 $(x+3a)(x+5a)$
7. $(3x+2)(3x+4)$ 8 $(5x+1)(5x+10)$ 9 $(mx+3)(mx+4)$
10 $(2x+a)(2x+b)$ 11 $(x^2+c)(x^2+d)$ 12 $(2x^3+3)(2x^3+9)$

Multiply

13 $x+a+1$ by $x+a+2$. 14 x^2+2x+3 by x^2+2x+4.
15 x^2-xy+4 by x^2-xy+7 16 $2a^2-ab+3$ by $2a^2-ab+8$

108 Formulæ IX (i) $(x+a)(x-b)=x^2+(a-b)x-ab$.
(ii) $(x-a)(x+b)=x^2-(a-b)x-ab$.

[These results may be proved by direct multiplication, or thus —
Substitute $-b$ for b in Formula VIII; thus

$$(x+a)\{x+(-b)\}=x^2+\{a+(-b)\}x+a(-b),$$

that is, $$(x+a)(x-b)=x^2+(a-b)x-ab$$

Similarly by putting $-a$ for a, we may prove (ii).]

Examples

(i) $(x+a)(x-b)=x^2+(a-b)x+ab$.

Ex 1 Write down the product of $x+3$ and $x-2$

Since $3-2=1$ and $3\times 2=6$, therefore required product $=x^2+x-6$

Ex. 2. Find the product of $ma+4$ and $ma-9$

Put $ma=x$; thus

$$(ma+4)(ma-9)=(x+4)(x-9)=x^2+(4-9)x-4\times 9$$
$$=x^2-5x-36=m^2a^2-5ma-36$$

Ex. 3 Multiply x^2+2x+5 by x^2+2x-9

Put $x^2+2x=a$, thus required product

$$=(a+5)(a-9)=a^2+(5-9)a-5\times 9$$
$$=a^2-4a-45=(x^2+2x)^2-4(x^2+2x)-45$$
$$=x^4+4x^3+4x^2-4x^2-8x-45$$
$$=x^4+4x^3-8x-45$$

Examples

Simplify

13. $(x+2y-1)(x+2y-3)$.
14. $(2a-b-4)(2a-b-7)$.
15. $(ax+by-4)(ax+by-8)$.
16. $(3x^2-4x-6)(3x^2-4x-8)$.
17. $(x^2-3y^2+4z^2)(x^2-2y^2+4z^2)$.
18. $(a^2-c^2+3ab)(a^2-d^2+3ab)$.

110 Formula XI. $(ax+b)(cx+d)=acx^2+(bc+ad)x+bd.$

$$[(ax+b)(cx+d)=(ax+b)cx+(ax+b)d$$
$$=(acx^2+bcx)+(adx+bd)$$
$$=acx^2+bcx+adx+bd$$
$$=acx^2+(bc+ad)x+bd \text{ (§ 77, Cor.)}]$$

Examples

Ex. 1. Multiply $2x+3$ by $5x+4$.

Required product $=2\cdot5x^2+(3\cdot5+2\cdot4)x+3\times4$
$=10x^2+23x+12$.

Ex. 2. Multiply $5x-6$ by $4x-7$.

Here substitute 5 for a, -6 for b, 4 for c, and -7 for d, thus required product $=5\cdot4x^2+(-6\times4+5\times-7)x+(-6)(-7)$
$=20x^2-59x+42$.

Find the value of

3. $(3x+2)(5x+4)$.
4. $(2x+3)(3x+2)$.
5. $(4x+5)(8x-3)$.
6. $(5x-6)(3x+4)$.
7. $(3x-8)(2x-3)$.
8. $(9x-1)(2x+5)$.
9. $(4a-3)(3a-4)$.
10. $(5x+9)(7x-6)$.
11. $(2x-15)(8x-7)$.

111 Formula XII. $(x+a)(x+b)(x+c)$

$=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$ [Ex. 4, § 89]

Examples

Ex. 1. Find the product of $x+3$, $x+4$ and $x+5$.

Required product $=x^3+(3+4+5)x^2+(4\cdot5+5\cdot3+3\cdot4)x+3\cdot4\cdot5$
$=x^3+12x^2+47x+60$.

Find the value of

2. $(m+1)(m+2)(m+3)$.
3. $(x+2)(x+5)(x+8)$.
4. $(x+7)(x+1)(x+6)$.
5. $(x+1)(x+3)(x+9)$.
6. $(x+a)(x+2a)(x+3a)$.
7. $(x+5a)(x+3a)(x+a)$.
8. $(5x+2)(5x+3)(5x+4)$.
9. $(ax+3)(ax+9)(ax+11)$.

112 Formula XIII $(x-a)(x-b)(x-c)$
$$=x^3-(a+b+c)x^2+(bc+ca+ab)x-abc \quad [\text{Ex. } 5, \S\, 89]$$

REMARK This formula is obtained from Formula XII, by putting $-a$, $-b$, $-c$, for a, b and c respectively.

Ex 1 Find the product of $x-8$, $x-5$ and $x-9$

Required product $=x^3-(8+5+9)x^2+(5\cdot 9+9\cdot 8+8\cdot 5)x-8\cdot 5\cdot 9$
$$=x^3-22x^2+157x-360$$

Find the value of

2 $(x-1)(x-2)(x-3)$

3 $(x-2)(x-5)(x-9)$

4 $(x-8)(x-10)(x-11)$

5 $(x-3)(x-6)(x-12)$.

6 $(x-a)(x-3a)(x-5a)$

7 $(x-4a)(x-2a)(x-8a)$

8 $(3x-1)(3x-4)(3x-10)$

9 $(ax-3)(ax-1)(ax-7)$

Note We may use Formula XII, to find other products of a similar nature

Ex 10 Find the product of $x+7$, $x-3$ and $x-10$

Required product $=(x+7)\{x+(-3)\}\{x+(-10)\}$
$$=x^3+\{7+(-3)+(-10)\}x^2$$
$$+\{(-3)(-10)+(-10)7+7(-3)\}x+7(-3)(-10)$$
$$=x^3+(7-3-10)x^2+(30-70-21)x+210$$
$$=x^3-6x^2-61x+210.$$

Ex 11 Find the product $(x-2)(x+5)(x-8)$

Required product $=\{x+(-2)\}(x+5)\{x+(-8)\}$
$$=x^3+\{(-2)+5+(-8)\}x^2$$
$$+\{5(-8)+(-8)(-2)+(-2)5\}x+(-2)5(-8)$$
$$=x^3+(-2+5-8)x^2+(-40+16-10)x+80$$
$$=x^3-5x^2-34x+80$$

Find the product

12 $(x+1)(x-2)(x+7)$

13 $(x-1)(x+2)(x+3)$

14. $(x+4)(x-5)(x-6)$

15. $(x-8)(x-7)(x+9)$

16 $(l+1)(l-3)(l+4)$

17 $(x+6)(x+8)(x-1)$

18 $(y-3)(y+4)(y-5)$

19. $(m+1)(m-7)(m-10)$

20 $(p-1)(p-8)(p+11)$

21 $(z-1)(z+3)(z-4)$

22 $(h-3)(h-4)(h+6)$.

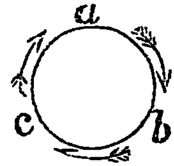
23 $(x+a)(x+2a)(x-3a)$

24 $(y-3m)(y-2m)(y+m)$

25 $(x-4a)(x+2a)(x-6a)$

26 $(h+2m)(h-3n)(h-4n)$

***113 Cyclic order** Let three letters a, b, c be placed in order round the circumference of a circle as shewn in the annexed diagram. If we start from any letter a and go round the circle in the direction of the arrows, we see that a is followed by b, b by c and c by a, that is, the letters follow one another in the order abc, similarly if we start from b, the order of the letters is bca, and if from c, the order is cab When a, b, c follow one another in this way, they are said to be in *cyclic order*



Hence the products of every two of the letters a, b, c, when written in cyclic order are ab, bc, ca if we start with a, but the products ab, ac, bc are not in cyclic order Similarly if we start with b, the same products, written in cyclic order, are bc, ca, ab Again the differences of every two of the letters a b, c, starting with b, are $b-c$, $c-a$, $a-b$, when written in cyclic order, but not so are $b-c$, $a-b$, $a-c$ And so on

It is highly desirable to observe cyclic order in arranging the letters of an expression, for then the work will gain in simplicity and elegance, and thus much labour and trouble will be saved [See § 180, *post*] We shall observe cyclic order in the following formulæ

***114 Formula XIV**

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=a^3+b^3+c^3-3abc.$$

[This Formula may be established by ordinary multiplication; or thus · $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$

$$=(a+b+c)\{(a^2-ab+b^2)+(c^2-ca-bc)\}$$
$$=(a+b+c)(a^2-ab+b^2)+c(c-a-b)(a+b+c)$$
$$=(\overline{a+b}+c)(a^2-ab+b^2)+c(c-\overline{a+b})(c+\overline{a+b})$$
$$=(a+b)(a^2-ab+b^2)+c(a^2-ab+b^2)+c\{c^2-(a+b)^2\}$$
$$=a^3+b^3+c\{(a^2-ab+b^2)+c^2-(a^2+2ab+b^2)\}$$
$$=a^3+b^3+c(c^2-3ab)=a^3+b^3+c^3-3abc\ ]$$

Cor Conversely $a^3+b^3+c^3-3abc$

$$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$

Thus expressions of the form $a^3+b^3+c^3-3abc$ can be resolved into factors We shall return to this subject in § 130

Examples.

Ex 1. Multiply $x^2+4y^2+9z^2-2xy+6yz+3zx$ by $x+2y-3z$

Put $x=a$, $2y=b$ and $-3z=c$, thus required product

$$=\{x^2+(2y)^2+(-3z)^2-x(2y)-2y(-3z)-(-3z)x\}(x+2y-3z)$$
$$=(a^2+b^2+c^2-ab-bc-ca)(a+b+c)=a^3+b^3+c^3-3abc$$
$$=x^3+(2y)^3+(-3z)^3-3x(2y)(-3z)=x^3+8y^3-27z^3+18xyz$$

Write down the value of

2 $(a+b-c)(a^2+b^2+c^2+bc+ca-ab)$

3 $(2x-y+z)(4x^2+2xy+y^2+z^2+yz-2zx)$

4 $(3x-2y-4z)(9x^2+6xy+4y^2-8yz+16z^2+12zx)$

5 $(1-x+2y)(1+x-2y+x^2+2xy+4y^2)$

6 $(-x-2y+3)(x^2-2xy+4y^2+3x+6y+9)$

7 $(3a-4b-2)(9a^2+16b^2+12ab+6a-8b+4)$

8 $(x+y-1)(x^2-xy+y^2+x+y+1)$ [§ 88, Ex (iii), 8]

***115 Formula XV** $(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab$ [§ 99, Ex 4]

Cor 1 Conversely $a^2+b^2+c^2+2bc+2ca+2ab=(a+b+c)^2$

Cor 2 Also $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c)$

[For $(bc+ca+ab)^2=(bc)^2+(ca)^2+(ab)^2+2(ca)(ab)+2(ab)(bc)+2(bc)(ca)$

$$=b^2c^2+c^2a^2+a^2b^2+2a^2bc+2ab^2c+2abc^2$$

$$=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c)\ ]$$

Examples

Ex. 1 Expand $(2m-3n-p)^2$

Given expn. $=(2m)^2+(-3n)^2+(-p)^2+2(2m)(-3n)+2(2m)(-p)+2(-3n)(-p)$

$$=4m^2+9n^2+p^2-12mn-4mp+6np$$

[For examples for exercise see §§ 99, 100]

Ex. 2 Resolve $4p^2+q^2+16r^2-4pq+16pr-8qr$ into factors, and find its value, when $p=4$, $q=12$ and $r=1$

Given expn $=(2p)^2+(-q)^2+(4r)^2+2(2p)(-q)+2(2p)(4r)+2(-q)(4r)$

$=(2p-q+4r)^2$ [*Cor* 1]

$=(2\times4-12+4\times1)^2=(8-12+4)^2=0$

Otherwise thus —Arranging according to the powers, say, of p, we have given expn $=4p^2-4p(q-4r)+(q^2+16r^2-8qr)$

$$=(2p)^2-2(2p)(q-4r)+(q-4r)^2$$

$$=\{2p-(q-4r)\}^2=(2p-q+4r)^2=\&c$$

Ex. 3 If $x=b+c$, $y=c-a$, $z=a-b$, prove that

$x^2+y^2+z^2-2xy-2xz+2yz=4b^2$ [*Cal*, 1888]

Left side $=x^2+(-y)^2+(-z)^2+2x(-y)+2x(-z)+2(-y)(-z)$

$$=(x-y-z)^2=(b+c-c+a-a+b)^2=(2b)^2=4b^2$$

Otherwise thus.—Arrange according to powers of some *one* letter, say x, thus

Left side $= x^2 - 2x(y+z) + (y^2+z^2+2yz) = x^2 - 2x(y+z) + (y+z)^2$
$= \{x-(y+z)\}^2 = (x-y-z)^2 = \&c$

Ex. 4. Prove that $\{(y-z)(z-x)+(z-x)(x-y)+(x-y)(y-z)\}^2$
$= (y-z)^2(z-x)^2 + (z-x)^2(x-y)^2 + (x-y)^2(y-z)^2$

Put $a=y-z$, $b=z-x$, $c=x-y$, thus $a+b+c=y-z+z-x+x-y=0$;
and $\therefore$ left side $=(ab+bc+ca)^2 = a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)$
$= a^2b^2+b^2c^2+c^2a^2$, $\quad a+b+c=0$,
$= (y-z)^2(z-x)^2+(z-x)^2(x-y)^2+(x-y)^2(y-z)^2$.

Find the value of

5. $x^2+y^2+z^2-2xy+2xz-2yz$, when $x=5$, $y=9$, $z=3$
6 $x^2+y^2+4z^2-2xy+4xz-4yz$, when $x=8$, $y=10$, $z=12$.
7 $9m^2+4n^2+16p^2-12mn-24mp+16np$, when $m=2$, $n=3$, $p=-4$.
8 $x^2+y^2-2xy-4x+4y+16$, when $x=23$, $y=11$
9 $x^2+y^2-2xy+2x-2y-1$, when $x=16$, $y=3$.
10 $2x^2+4y^2+4xy-6x-12y-9$, when $x=17$, $y=8$
11. $4a^2+3y^2-4ab-16a+8b+25$, when $a=4$, $b=6$.
12 $x^2+y^2+z^2+2xy+2yz+2zx$, when $x=b+c-3a$, $y=c+a-3b$, and $z=a+b-3c$.
13. $x^2+y^2+4z^2+2xy-4zx-4yz$, when $x=b+c-2a$, $y=c+a-2b$, and $-z=a+b-c$.
14 $bc+ca+ab$, when $a+b+c=2$, $a^2+b^2+c^2=1$
15. $a^2+b^2+c^2$, when $a+b+c=3$, $bc+ca+ab=2$.
16 $a+b+c$, when $a^2+b^2+c^2=9$, $bc+ca+ab=8$.

Resolve into factors

17. $x^2+y^2+z^2+2yz-2zx-2xy$
18. $1+a^2+b^2+2a-2b-2ab$
19. $x^4-2ax^3+a^2x^2+2x^2-2ax+1$
20 $x^4+2ax^3-a^2x^2-2a^3x+a^4$
21. If $x=b-2c$, $y=c-2a$, $z=a-2b$, shew that
$$x^2+y^2+z^2+2yz+2zx+2xy=(a+b+c)^2.$$
22 Shew that $(a-b)^2+(b-c)^2+(c-a)^2+2(a-b)(b-c)$
$+2(b-c)(c-a)+2(c-a)(a-b)=0$

23 Shew that $x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2$.
$$=2xy(z-x)(z-y)+2yz(x-y)(x-z)+2zx(y-z)(y-x)$$

24 Prove that $\{(y-z)^2+(z-x)^2+(x-y)^2\}^2$
$$=4(y-z)^2(z-x)^2+4(z-x)^2(x-y)^2+4(x-y)^2(y-z)^2$$

25 If $x=2a-b-c$, $y=2b-c-a$, $z=2c-a-b$, shew that
$$(x^2+y^2+z^2)^2=4(y^2z^2+z^2x^2+x^2y^2)$$

***116 Formula XVI.** $\begin{cases}(b-c)+(c-a)+(a-b)=0\\ a(b-c)+b(c-a)+c(a-b)=0.\\ (b^2-c^2)+(c^2-a^2)+(a^2-b^2)=0\end{cases}$

[The student can easily prove these simple results]

Examples.

Simplify

1 $(a+1)(b-c)+(b+1)(c-a)+(c+1)(a-b)$

2 $(x+a)(b-c)+(x+b)(c-a)+(x+c)(a-b)$

3 $a^2(b+c)(b-c)+b^2(c+a)(c-a)+c^2(a+b)(a-b)$

4 $(x+y-z)(x-y)+(y+z-x)(y-z)+(z+x-y)(z-x)$

5 $(x^2-a^2)(b^2-c^2)+(x^2-b^2)(c^2-a^2)+(x^2-c^2)(a^2-b^2)$

6 $(x-b-c)(b-c)+(x-c-a)(c-a)+(x-a-b)(a-b)$

7 $(2a-b-c)(b-c)+(2b-c-a)(c-a)+(2c-a-b)(a-b)$

8 $(la+b+c)(b-c)+(lb+c+a)(c-a)+(lc+a+b)(a-b)$.

9. $\{ma-n(b+c)\}(b-c)+\{mb-n(c+a)\}(c-a)$
$$+\{mc-n(a+b)\}(a-b)$$

10 $\{\lambda a+l(b+c)+m\}(b-c)+\{\lambda b+l(c+a)+m\}(c-a)$
$$+\{\lambda c+l(a+b)+m\}(a-b)$$

***117 Formula XVII**

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b)$$

Arranging according to the descending powers of a, we have

$$\begin{aligned}a^2(b-c)+b^2(c-a)+c^2(a-b)&=a^2(b-c)-a(b^2-c^2)+bc(b-c)\\&=(b-c)\{a^2-a(b+c)+bc\}\\&=(b-c)(a-c)(a-b)\ [\S\ 109]\\&=-(b-c)(c-a)(a-b),\quad (a-c)=-(c-a)\end{aligned}$$

It is easy to see that

$$a^2(b-c)+b^2(c-a)+c^2(a-b)\ \ldots\ \ldots\ \ldots\ (\text{i})$$

and
$$bc(b-c)+ca(c-a)+ab(a-b)\ \ldots\ \ldots\ (\text{ii})$$

are, when the brackets are removed, only *different forms* of the expression

$$a^2b-ab^2+b^2c-bc^2+c^2a-ca^2\ \ldots\ \ldots\ \ldots\ (\text{iii}).$$

Thus (i), (ii) and (iii) are equal to one another. Also by removing the brackets it will be seen that

$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2) \qquad \ldots \quad \text{(iv)}$$
$$=-(a^2b-ab^2+b^2c-bc^2+c^2a-ca^2)$$

Thus (iv) is equal to the first three in *absolute value*, but differs from them *only in sign*.

Hence Formula XVII may be written in the following forms

$$\left.\begin{aligned} &a^2(b-c)+b^2(c-a)+c^2(a-b)\\ \text{or } &bc(b-c)+ca(c-a)+ab(a-b)\\ \text{or } &a^2b-ab^2+b^2c-bc^2+c^2a-ca^2 \end{aligned}\right\}=-(b-c)(c-a)(a-b) \quad \text{(i)}.$$

Also $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(b-c)(c-a)(a-b)$ (ii)

The student will notice that we have here observed *cyclic order* throughout He will see later on the advantage of this arrangement

Examples

Ex. 1 Simplify $(a^2-bc)(b-c)+(b^2-ca)(c-a)+(c^2-ab)(a-b)$

Given expression

$=a^2(b-c)-bc(b-c)+b^2(c-a)-ca(c-a)+c^2(a-b)-ab(a-b)$

$=\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}-\{bc(b-c)+ca(c-a)+ab(a-b)\}$

$=0$

Ex. 2 Simplify $a(a+1)(b-c)+b(b+1)(c-a)+c(c+1)(a-b)$

Expn. $=(a^2+a)(b-c)+(b^2+b)(c-a)+(c^2+c)(a-b)$

$=a^2(b-c)+a(b-c)+b^2(c-a)+b(c-a)+c^2(a-b)+c(a-b)$

$=\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}+\{a(b-c)+b(c-a)+c(a-b)\}$

$=-(b-c)(c-a)(a-b)+0$ [§ 116] = &c.

Ex 3 Shew that $(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)$
$+(c-a)(x-c)(x-a)=(a-b)(b-c)(a-c)$.

Left side $=(a-b)\{x^2-(a+b)x+ab\}+(b-c)\{x^2-(b+c)x+bc\}$
$+(c-a)\{x^2-(c+a)x+ca\}$ [§ 109]

$=x^2\{(a-b)+(b-c)+(c-a)\}-x\{(a^2-b^2)+(b^2-c^2)+(c^2-a^2)\}$
$+\{ab(a-b)+bc(b-c)+ca(c-a)\}$

$=x^2\times 0-x\times 0+\{ab(a-b)+bc(b-c)+ca(a-b)\}$ [§ 116]

$=-(b-c)(c-a)(a-b)=$ &c

Ex 4 Resolve into factors

$$(b-c)(b+c-2a)^2+(c-a)(c+a-2b)^2+(a-b)(a+b-2c)^2$$

Assume $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, thus

$$x-y=b+c-2a-(c+a-2b)=3(b-a)=-3(a-b),$$

similarly $\quad y-z=-3(b-c)$ and $z-x=-3(c-a)$

Hence $3(b-c)(b+c-2a)^2=-(y-z)x^2=-x^2(y-z)$,

$$3(c-a)(c+a-2b)^2=-(z-x)y^2=-y^2(z-x),$$

$$3(a-b)(a+b-2c)^2=-(x-y)z^2=-z^2(x-y),$$

$$\begin{aligned}3 \text{ times proposed expn } &= -x^2(y-z)-y^2(z-x)-z^2(x-y)\\ &=(y-z)(z-x)(x-y)\\ &=-3(b-c)\times-3(c-a)\times-3(a-b)\\ &=-27(b-c)(c-a)(a-b),\end{aligned}$$

whence proposed expression $=-9(b-c)(c-a)(a-b)$

5 Resolve $a(b+c)(b-c)+b(c+a)(c-a)+c(a+b)(a-b)$

6 Simplify

$$(a^2-bc+a)(b-c)+(b^2-ca+b)(c-a)+(c^2-ab+c)(a-b)$$

Resolve into factors

7 $(x^2+x+1)(y^2-z^2)+(y^2+y+1)(z^2-x^2)+(z^2+z+1)(x^2-y^2)$

8 $(x-a)(x-b)(y-z)+(y-a)(y-b)(z-x)+(z-a)(z-b)(x-y)$.

9 $bc(b-c)(x+a)^2+ca(c-a)(x+b)^2+ab(a-b)(x+c)^2$

10 $(b-c)(b+c)^2+(c-a)(c+a)^2+(a-b)(a+b)^2$

11 $(1+ca)(1+ab)(b-c)+(1+ab)(1+bc)(c-a)$
$+(1+bc)(1+ca)(a-b)$

12 $a(b-c)(x-b)(x-c)+b(c-a)(x-c)(x-a)+c(a-b)(x-a)(x-b)$

13 $(b-c)(b+c-a)^2+(c-a)(c+a-b)^2+(a-b)(a+b-c)^2$

14 If $x=b-c$, $y=c-a$, $z=a-b$, shew that

$$xyz=a(b+c)x+b(c+a)y+c(a+b)z$$

*118 Formula XVIII

$$(a^2+ab+b^2)(a^2-ab+b^2)=a^4+a^2b^2+b^4$$

[Left side $=(a^2+b^2+ab)(a^2+b^2-ab)=(a^2+b^2)^2-(ab)^2$
$=a^4+b^4+2a^2b^2-a^2b^2=a^4+a^2b^2+b^4$]

Conversely $\quad a^4+a^2b^2+b^4=(a^2+ab+b^2)(a^2-ab+b^2)$

118*a* We collect below important **Formulæ for ready reference**

1 $(a+b)^2=a^2+2ab+b^2$

2 $(a-b)^2=a^2-2ab+b^2$

2 (1) $(a+b)^2=(a-b)^2+4ab$;

(2) $(a-b)^2=(a+b)^2-4ab$;

(3) $$a^2+b^2=(a+b)^2-2ab=(a-b)^2+2ab$$
$$=\tfrac{1}{2}(a+b)^2+\tfrac{1}{2}(a-b)^2 ;$$

(4) $$ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2.$$

3. $(a+b)(a-b)=a^2-b^2$

4. $$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$
$$=a^3+b^3+3ab(a+b) ;$$
$$a^3+b^3=(a+b)^3-3ab(a+b).$$

5. $$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$
$$=a^3-b^3-3ab(a-b) ;$$
$$a^3-b^3=(a-b)^3+3ab(a-b)$$

6. $(a+b)(a^2-ab+b^2)=a^3+b^3.$

7. $(a-b)(a^2+ab+b^2)=a^3-b^3.$

8. $(x+a)(x+b)=x^2+(a+b)x+ab$

9. $$\left.\begin{aligned}&(x+a)(x-b)=x^2+(a-b)x-ab.\\&(x-a)(x+b)=x^2-(a-b)x-ab\end{aligned}\right\}$$

10. $(x-a)(x-b)=x^2-(a+b)x+ab$

11. $(ax+b)(cx+d)=acx^2+(bc+ad)x+bd$

12. $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc.$

13. $(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(bc+ca+ab)x-abc$

14. $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=a^3+b^3+c^3-3abc$

15. $(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab.$

16. $$\begin{cases}(b-c)+(c-a)+(a-b)=0 ;\\ a(b-c)+b(c-a)+c(a-b)=0 ,\\ (b^2-c^2)+(c^2-a^2)+(a^2-b^2)=0\end{cases}$$

17. $$\left.\begin{aligned}&a^2(b-c)+b^2(c-a)+c^2(a-b)\\&bc(b-c)+ca(c-a)+ab(a-b)\\&-a(b^2-c^2)-b(c^2-a^2)-c(a^2-b^2)\end{aligned}\right\}=-(b-c)(c-a)(a-b)$$

18. $(a^2+ab+b^2)(a^2-ab+b^2)=a^4+a^2b^2+b^4.$

***119 Substitution.** We shall in the present article draw the attention of the student to the fact, which he has perhaps already noticed, that *each letter in a formula may stand for a single letter as well as for an expression* Hence from a result already established, we can deduce *new results*, by substituting for its letters other letters or

expressions Thus the *principle of substitution* is a very important element in the Science of Algebra and gives to algebraical results an immense scope for development The truth of this remark will be seen from the following illustrations

In Formula I, put $2x-y=a$ and $y=b$, thus we get

$$\{(2x-y)+y\}^2=(2x-y)^2+2(2x-y)y+y^2\ ;$$

therefore $$(2x-y)^2+2(2x-y)y+y^2=\{(2x-y)+y\}^2=(2x)^2=4x^2$$

Again in the same formula, put $x+y=a$ and $x-y=b$, thus we have

$$\{(x+y)+(x-y)\}^2=(x+y)^2+2(x+y)(x-y)+(x-y)^2\ ,$$

$$(x+y)^2+2(x+y)(x-y)+(x-y)^2=\{(x+y)+(x-y)\}^2$$
$$=(2x)^2=4x^2$$

By the same substitutions, we have from Formula II

$$\{(x+y)-(x-y)\}^2=(x+y)^2-2(x+y)(x-y)+(x-y)^2\ ,$$

thus $$(x+y)^2-2(x+y)(x-y)+(x-y)^2=\{(x+y)-(x-y)\}^2$$
$$=(2y)^2=4y^2$$

Again put $ax+by-cz=A$, $ax-by+cz=B$, thus from Formula III we get $\{(ax+by-cz)+(ax-by+cz)\}\{(ax+by-cz)-(ax-by+cz)\}$

$$=(ax+by-cz)^2-(ax-by+cz)^2\ ,$$

or $$(ax+by-cz)^2-(ax-by+cz)^2=2ax\times 2(by-cz)=4ax(by-cz)$$

By the same substitutions again, we have from Formula IV

$$\{(ax+by-cz)+(ax-by+cz)\}^3=(ax+by-cz)^3+(ax-by+cz)^3$$
$$+3(ax+by-cz)(ax-by+cz)\{(ax+by-cz)+(ax-by+cz)\}\ ;$$

thus $$(ax+by-cz)^3+(ax-by+cz)^3+6ax\{a^2x^2-(by-cz)^2\}$$
$$=(2ax)^3=8a^3x^3$$

And so on

Examples

Simplify

1 $(x-2a)^2+4a(x-2a)+4a^2$ 2 $(a+1)^2-6(a^2-1)+9(a-1)^2$

3 $(a-x)^3+6a(a^2-x^2)+(a+x)^3$

4 $8(x-y)^3-(x-2y)^3-6x(x-y)(x-2y)$.

5 $(a+b+c)^2-2\{(a+b)^2-c^2\}+(a+b-c)^2$

6 $(x-y)^2-(x-y)(x-z)+(x-z)^2$

7 $(3x+5)^3-3\times 5x(3x+5)(2x-5)+(2x-5)^3$

Prove that

8 $(a+b)^2+2(a+b)c+c^2=a^2+2a(b+c)+(b+c)^2$

9 $(a-b)^2-2(a-b)c+c^2=a^2-2a(b+c)+(b+c)^2$

Prove that

10. $a^3-3a^2(b-c)+3a(b-c)^2-(b-c)^3$
$$=(a-b)^3+3(a-b)^2c+3(a-b)c^2+c^3$$

11. $a^3+3a^2(b-c)+3a(b-c)^2+(b-c)^3$
$$=(a+b)^3-3(a+b)^2c+3(a+b)c^2-c^3$$

12. $(a+b+2c)^3-(b+c)^3-(c+a)^3=3(b+c)(c+a)(a+b+2c)$

13. $(a-b)^3-(b-c)^3-(a-2b+c)^3=3(a-b)(b-c)(a-2b+c)$.

14. $(a+b-c-d)^2=(a+b+c+d)^2-4(a+b)(c+d)$

15. $(a+b)^2+(c+d)^2=(a+b+c+d)^2-2(a+b)(c+d)$

Multiply

16. $(a+1)^2+2(a+1)+1$ by a^2-4a+4

17. $(a+b)^2-2(a+b)b+b^2$ by $(a+b)^2+2(a+b)b+b^2$

18. $(a+b)^2-(a+b)+1$ by $a+b+1$.

19. $(x^2+2x+1)-3y(x+1)+9y^2$ by $x+3y+1$

20. $(4a^2-4ab+b^2)+(2a-b)(a+b)+(a+b)^2$ by $a-2b$

21. $(a-b)^2-(a-b)(c-d)+(c-d)^2$ by $a-b+c-d$.

Simplify

22. $(x+1)(x+2)-(x-1)(x-2)$

23. $a(a+b)(a+c)-b(a+c)(a-c)$

24. $(x+1)(x-2)+(x+2)(x-3)-2(x+3)(x-4)$

25. $(x+1)(x+2)(x-3)-(x-1)(x-2)(x+3)$

26. $(a+b)^2-(a+b)(a-b)-\{a(2b-a)-b(2a-b)\}$

27. $\{x(x+a)-a(x-a)\}\{x(x-a)-a(x+a)\}$

28. $\{a(x+y)+b(x-y)\}\{a(x-y)-b(x+y)\}$

29. $(3x+y)(3x-y)+(3y+2z)(3y-2z)+(2z-3x)(2z+3x)$

30. $(a+b+c)(b+c-a)+(c+a-b)(a+b-c)$.

31. $(x^2+xy+y^2)(x+y)-(x^2-xy+y^2)(x-y)$

Shew that

32. $(x+y)^2-2(x+y)y+2y^2=x^2+y^2$

33. $(1+a^2)(1+b^2)-(1-ab)^2=(a+b)^2$.

34. $(a+3b)^2+3(a-b)^2=(a-3b)^2+3(a+b)^2$.

35. $(a+2)(b+2)+2(a-1)(b-1)=(a-2)(b-2)+2(a+1)(b+1)$

36. $(2x+1)^2+(x+2)^2=(x-2)^2+4x(x+3)+1$

37. $(x-6)^2+(x+1)^2+2x(x-1)=(2x-3)^2+28$.

38. $3x^2+(x+1)^2+2x-y^2=(2x+y+1)(2x-y+1)$.

39. $(2a-3b)^2+5(a+b)^2-10(a+b)b-4b^2=3a(3a-4b)$

Prove that

40 $(a+b)^2+(b+c)^2+(c+a)^2=2(a^2+b^2+c^2+bc+ca+ab)$

41 $(a-b)^2+(b-c)^2+(c-a)^2=2(a^2+b^2+c^2-bc-ca-ab)$

42 $(a-b)^2-(b-c)(c-a)=(b-c)^2-(c-a)(a-b)$
$$=(c-a)^2-(a-b)(b-c).$$

43 $(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)$
$$=bc+ca+ab-a^2-b^2-c^2$$

Simplify

44. $(a+b+c)\{(b-c)^2-(c-a)(a-b)\}$

45. $(a+b+c)\{(a-b)^2+(a-b)(c-a)+(c-a)^2\}$

46 $(a+b+c)\{(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)\}$

Prove that

47 $(x+y)^2+(y+z)^2+(z+x)^2+(x-y)^2+(y-z)^2+(z-x)^2$
$$=4(x^2+y^2+z^2).$$

48 $(x+y)^2+(y+z)^2+(z+x)^2-(x-y)^2-(y-z)^2-(z-x)^2$
$$=4(yz+zx+xy)$$

49. $(x+y)(x+z)-x^2=(y+z)(y+x)-y^2=(z+x)(z+y)-z^2$

50 $(y+z)^2+(z+x)^2+(x+y)^2-x^2-y^2-z^2=(x+y+z)^2.$

51 $(b+c)(b+c-a)+(c+a)(c+a-b)+(a+b)(a+b-c)$
$$=2(a^2+b^2+c^2)$$

52 $(b-c)(b+c-a)+(c-a)(c+a-b)+(a-b)(a+b-c)=0$

Simplify

53 $(a+b)^2(b+c-a)(c+a-b)+(a-b)^2(a+b+c)(a+b-c).$

54 $(b-c)(c-a)(a-b)-\{a^2(c-b)-[c^2(a-b)-b^2(c+a)]\}$

Shew that

55. $(a+b)^2(a^2+2ab-b^2)+(b+c)^2(b^2+2bc-c^2)+(c+a)^2(c^2+2ca-a^2)$
$$=4a^2b(a+b)+4b^2c(b+c)+4c^2a(c+a).$$

56 $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$
$$=2bc(b^2-c^2)+2ca(c^2-a^2)+2ab(a^2-b^2).$$

57. $(b-c)(a^2+b^2+c^2+bc)+(c-a)(b^2+c^2+a^2+ca)$
$$+(a-b)(c^2+a^2+b^2+ab)=-(b-c)(c-a)(a-b)$$

58. $(1-ax+a^2)(1-ay+a^2)-(1-a^2)^2=a^2(xy+4)-a(1+a^2)(x+y)$

59. $(a+b)^2+(b+c)^2+(c+a)^2+2(a+b)(b+c)+2(b+c)(c+a)$
$$+2(c+a)(a+b)=4(a+b+c)^2.$$

60 $(2x-y)^2+(2y-z)^2+(2z-x)^2+2(2x-y)(2y-z)+2(2y-z)(2z-x)$
$$+2(2z-x)(2x-y)=(x+y+z)^2.$$

Shew that

61. $4(x+y+z)^2=(y+z)^2+(z+x)^2+(x+y)^2+2(y+z)(z+x)$
$+2(z+x)(x+y)+2(x+y)(y+z)$

62 $8(x+y+z)^3=(x+2y)^3+(x+2z)^3+6(x+2y)(x+2z)(x+y+z)$

63. $8(x+y+z)^3=(x+y)^3+(x+y+2z)^3+6(x+y)(x+y+z)(x+y+2z)$

64 $(b-c)^2(b+c-2a)+(c-a)^2(c+a-2b)+(a-b)^2(a+b-2c)$
$=(2a-b-c)(2b-c-a)(2c-a-b).$

65 $a(b-c)(1+ab)(1+ac)+b(c-a)(1+bc)(1+ba)$
$+c(a-b)(1+ca)(1+cb)=-abc(b-c)(c-a)(a-b).$

66 Given $a+b=m$ and $a^3+b^3=n^3$, find ab in terms of m and n.

67 Given $p-q=a$ and $p^3-q^3=b^3$, find pq in terms of a and b.

68 If $x-y=m$, shew that $x^3=m^3+3mxy+y^3$.

69 If $x+y=2ma$, $xy=m^2a^2-n^2b^2$, shew that x^3+y^3
$=2m^3a^3+6mn^2ab^2$.

70 If $x+y=2a$ and $x-y=2b$, find x^3+y^3 in terms of a and b.

71. If $x-y=p$ and $xy=qr$, find $x+y$ and x^3-y^3 in terms of p, q and r

72. If $x+y=a$ and $xy=b^2$, express $x-y$ and x^3+y^3 in terms of a and b

73. If $a-b=-6$ and $ab=91$, find the value of a^3+b^3

74 If $a+b=26$ and $ab=165$, find the value of a^3-b^3

75 If $x-y=2$ and $x^3-y^3=14$, find the value of x^2-xy+y^2

76. If $x+y=1$ and $x^3+y^3=7$, find the value of x^2+xy+y^2.

***119a Multiplication by employing Brackets** Much of the labour of multiplication is often saved, and the work neatly performed by arranging the multiplicand and multiplier according to the powers some *one* letter (called the *symbol of reference*).

Examples

Ex. 1 Multiply $bc+ca+ab$ by $a+b+c$

Arrange multiplicand and multiplier in powers of a

$$\begin{array}{l}
a(b+c)+bc \\
\underline{a+(b+c)} \\
a^2(b+c)+abc \\
\underline{\qquad\qquad +a(b+c)^2+bc(b+c)} \\
a^2(b+c)+a(b^2+c^2+3bc)+bc(b+c) \\
=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc
\end{array}$$

Ex 2. Multiply $x^2+ax-bx-ab$ by $x-a+b$

Here consider x as the symbol of reference

$$\begin{array}{l} x^2+(a-b)x-ab \\ \underline{x-(a-b)\qquad\qquad} \\ x^3+(a-b)x^2-abx \\ \underline{\quad -(a-b)x^2-(a-b)^2x+ab(a-b)} \\ x^3 \qquad\qquad\quad -(a^2-ab+b^2)x+ab(a-b) \end{array}$$

Ex 3 Multiply $x^2-xy+y^2+x+y+1$ by $x+y-1$ [Ex 8, § 88, (iii)]

Here consider x as the symbol of reference

$$\begin{array}{l} x^2-(y-1)x+(y^2+y+1) \\ \underline{x+(y-1)\qquad\qquad\qquad\quad} \\ x^3-(y-1)x^2+(y^2+y+1)x \\ \underline{\quad +(y-1)x^2-(y^2-2y+1)x+(y^3-1)} \\ x^3 \qquad\qquad\qquad +3yx \qquad +(y^3-1) \\ =x^3+y^3+3xy-1 \end{array}$$

Ex 4 Multiply $a^2+bc-ca-ab$ by $b^2-c^2-ca+ab$

Consider a as the symbol of reference

$$\begin{array}{l} a^2-a(b+c)+bc \\ \underline{a(b-c)+(b^2-c^2)\qquad\qquad\qquad} \\ a^3(b-c)-a^2(b^2-c^2)+abc(b-c) \\ \underline{\qquad\quad +a^2(b^2-c^2)-a(b+c)(b^2-c^2)+bc(b^2-c^2)} \\ a^3(b-c) \qquad\qquad -a(b-c)\{(b+c)^2-bc\}+b^3c-bc^3 \\ =a^3(b-c) \qquad\quad\; -a(b^3-c^3) \qquad\qquad\quad +b^3c-bc^3. \end{array}$$

Find the product of

5. $x^2+ax+bx+ab$ and $x-c$
6. $ax^2+ax+bx+a$ and $x-1$
7. $a^2+bc+ca+ab$ and $a-b+c$
8. $x^2-ax-bx+ab$ and $x+a-b$
9. $(a^2-a+1)x^2+(a-1)x+1$ and $(a+1)x-1$.
10. $(x-1)a^2-(x-1)a+3$ and $(x^2+x+1)a-(x+1)$
11. $a^2+b^2+c^2+bc-ca+ab$ and $a-b+c$
12. $x^2-xy+y^2-2x+y+1$ and $x+y-1$
13. $a^3-(m-1)a^2+(m+1)a-2$ and $(m+1)a^2+a-m$

Find the product of

14 $(a^2+ab+b^2)x^3-(a+b)x-ab$ and $(a-b)x^2+2x-1$.

15 $2x^3-(a+b)x^2+abx-a+b$ and $(a-b)x^2+abx+a+b$.

16 $bc(b-c)+ca(c-a)+ab(a-b)$ and $bc+ca+ab$

* **119b Division by employing Brackets** The labour of division is considerably shortened and the operation much neatly performed by arranging the dividend and divisor according to the powers of some *one* letter (called the *symbol of reference*).

Examples

Ex 1 Divide $a^3-ab^2-b^2c+bc^2+c^2a+2ca^2+abc$ by $a^2+bc+ca+ab$

Arrange dividend and divisor in powers of a.

$$
\begin{array}{l}
a^2+a(b+c)+bc\,\big)\,a^3+2a^2c-a(b^2-bc-c^2)-bc(b-c)\,\big(\,a-(b-c)\\
\qquad\qquad\qquad\quad \underline{a^3+a^2(b+c)+abc}\\
\qquad\qquad\qquad\quad -a^2(b-c)-a(b^2-c^2)-bc(b-c)\\
\qquad\qquad\qquad\quad \underline{-a^2(b-c)-a(b^2-c^2)-bc(b-c)}.
\end{array}
$$

Ex. 2 Divide $x^3+y^3+3xy-1$ by $x+y-1$ [Ex 25, § 95]

Arrange dividend and divisor according to the descending powers of x

$$
\begin{array}{l}
x+(y-1)\,\big)\,x^3+3xy+(y^3-1)\,\big(\,x^2-(y-1)x+(y^2+y+1)\\
\qquad\qquad\quad \underline{x^3+(y-1)x^2}\\
\qquad\qquad\quad -(y-1)x^2+3xy\\
\qquad\qquad\quad \underline{-(y-1)x^2-(y-1)^2x}\\
\qquad\qquad\qquad\qquad (y^2+y+1)x+(y^3-1)\\
\qquad\qquad\qquad\qquad \underline{(y^2+y+1)x+(y^3-1)}.
\end{array}
$$

Ex 3 Divide $(a-b)x^3-(x-b)a^3+(x-a)b^3$ by $x^2-(a+b)x+ab$

Consider x as the *symbol of reference*

$$
\begin{array}{l}
x^2-(a+b)x+ab\,\big)\,(a-b)x^3-(a^3-b^3)x+ab(a^2-b^2)\,\big(\,(a-b)x+(a^2-b^2)\\
\qquad\qquad\qquad\quad \underline{(a-b)x^3-(a^2-b^2)x^2+ab(a-b)x}\\
\qquad\qquad\qquad\qquad\qquad (a^2-b^2)x^2-(a+b)(a^2-b^2)x^*+ab(a^2-b^2)\\
\qquad\qquad\qquad\qquad\qquad \underline{(a^2-b^2)x^2-(a+b)(a^2-b^2)x+ab(a^2-b^2)}
\end{array}
$$

[*since $-\{a^3-b^3+ab(a-b)\}=-(a-b)(a+b)^2=-(a+b)(a^2-b^2)$]

Ex. 4 Divide $(x^3-1)a^3-(x^3+x^2-2)a^2+(4x^2+3x+2)a-3(x+1)$ by $(x-1)a^2-(x-1)a+3$

Here take a as the *symbol of reference* .

$$(x-1)a^2-(x-1)a+3)\,(x^3-1)a^3-(x^3+x^2-2)a^2+(4x^2+3x+2)a-3(x+1)\,(\,(x^2+x+1)a-(x+1)$$
$$\underline{(x^3-1)a^3-(x^3-1)a^2+(3x^2+3x+3)a}$$
$$-(x^2-1)a^2+(x^2-1)a-3(x+1)$$
$$\underline{-(x^2-1)a^2+(x^2-1)a-3(x+1)}$$

Divide

5. $a^3+b^3-3a^2+3a-1$ by $a+b-1$
6. $a^3+b^2(2a+b)+ab(2a+c)-c^3$ by $a+b-c$
7. $x^3-a^2(x-b)+(a-b)bx-ab^2$ by $(x+a)(x-b)$
8. $a^3(b+c)+b^3(c+a)+c^3(a+b)+abc(a+b+c)$ by $bc+ca+ab$
9. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$ by $(x-a)(x-b)$.
10. $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a^2(b-c)+b^2(c-a)+c^2(a-b)$
11. $x^3+(1-a-b)x^2-2bx-a^2+b^2$ by $x-a-b$
12. $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$ by $bc+ca+ab$
13. $a^4-(x^2-y-z)a^2-(y-z)ax+yz$ by a^2+ax+y
14. $x^3-2ax^2+(a^2-ab-b^2)x+a^2b+ab^2$ by $(x-a)(x+b)$
15. $x^3+(4ab-b^2)x-(a-2b)(a^2+3b^2)$ by $x-a+2b$ [Ex. 73, § 95]
16. $(a^3-1)x^3+(2a^2+a)x^2+2ax+1$ by $(a-1)x+1$
17. $a^3+b^3+c^3-3abc$ by $a+b+c$
18. $1+x^3-8y^3+6xy$ by $1+x-2y$ [Ex 50, § 95]
19. $(a+b)(x+y)(ax+by)-bx-ay-1$ by $ax+by-1$ [Ex 79, § 95]
20. $x^4-a^4+2ax^3+2na^3x-(n^2-1)a^2x^2$ by $x^2+a^2-(n-1)ax$ [Ex. 74, § 95]
21. $x(x-1)a^3+(x^3+2x-2)a^2+(3x^2-x^3)a-x^4$ by $a^2x+2a-x^2$
22. $x^3(a+1)-xy(x-y)(a+b)-y^3(b-1)$ by $x(a+1)-y(b-1)$.
23. $(2x-y)^2a^4-(x+y)^2a^2x^2+2(x+y)ax^4-x^6$ by $(2x-y)a^2-(x+y)ax+x^3$

CHAPTER IX.

Resolution of Expressions into Factors

120 Composition and Resolution of Algebraic Expressions From Chapter VI, it is seen that a product is *composed*

of two or more factors, and may, therefore, be called a *Composition* or Compound of *elements*, which are its component factors. When these *elements* or *elementary factors*, as they are called, have to be found out, we *decompose* or *resolve* the product The process by which the component factors are found is called the Resolution of Expressions into Factors. Hence Multiplication and Resolution are two *inverse* processes

It is to be pointed out here that we cannot resolve into factors any expression* that we may come across Indeed it is beyond the range of this work to resolve into factors expressions of a degree higher than the second We shall therefore give a systematic method of resolving Quadratics *i e*, expressions of the form ax^2+bx+c [§ 128], and confine our attention, in the case of expressions of a higher degree, to those only that can be resolved by comparing them with the Formulæ given in the preceding Chapter

actors of Expressions of the form $an+bn+cn+$

§ 77, *Cor* we have $an+bn+cn+\ldots=n(a+b+c+\ldots)$
ssions of the given form are resolved by *taking out the*
ion to all the terms, and enclosing the rest in a bracket.

Examples.

Ex. 1 Resolve $2ax+3xy$ into factors

Given expression $=x(2a+3y)$; ∴ required factors are x and $2a+3y$.

Ex 2 Resolve $xy+yz-y^2$ into factors

Given expression $=y(x+z-y)$, ∴ the factors are y and $x+z-y$.

Ex 3. Resolve $6a^3x^2-9a^2x^2+15abxy$ into factors

Given expression $=3ax(2a^2x-3ax+5by)$; ∴ &c

Resolve into factors

4 $ab+ax$ 5 $m+mx$ 6. $3r^3-15rs$. 7. a^2b+ab^2.

8 $ab-bc+abc$ 9. $8a^2-6ab-4ac$ 10 $20pq-15p^2r+55rp$.

11. $3a^2b-3ab^4$. 12 $2a^2b^2+3ab$ 13 $12p^3q-3pq^2$.

14 $8x^2-10mx^5$. 15 $18m^3-6m^5n$ 16. $3x^2y+xy+2xy^2$

17 $4a^3b-16a^2b+20ab^2$. 18. $2x^3y^3-3rx^2y+2rxy^2$.

19 $6x^2y-9x^2y^2+12x^3y^2$. 20. $81x^6y^4+63x^4y^3$

21. $24m^4x^6z^3-42m^2x^3y^5$ 22. $7a^3xy^3+14a^2x^2y^2-21a^2xy^3$

* In this Chapter, "expression" means a "*rational* and *integral* expression," that is, one which is free from radical signs, and in which the symbol of reference does not occur in the denominator of any term

Ex 23 Resolve $a(x+1)+2(x+1)$ into factors

Here $x+1$ is the common factor, put it $=m$, thus

given expression $=ma+2m=m(a+2)=(x+1)(a+2)$

Ex 24 Resolve $a^3x+ab^2x+a^2by+b^3y$ into factors

Given expression $=(a^3x+ab^2x)+(a^2by+b^3y)$

$$=ax(a^2+b^2)+by(a^2+b^2)=(a^2+b^2)(ax+by)$$

The same result also may be obtained by a *suitable rearrangement and grouping* of the terms, thus

given expression $=a^3x+a^2by+ab^2x+b^3y=(a^3x+a^2by)+(ab^2x+b^3y)$

$$=a^2(ax+by)+b^2(ax+by)=(ax+by)(a^2+b^2)$$

Resolve into factors

25 $ab+bc+cd+ad$

26 $m^2+m+n+mn$

27 $ax+a+x+1$

28 $2ax-4ay-bx+2$…

29 $1-x-y+xy$.

30 $a^2-ab-ac+bc$

31 $x^2-mx-my+xy$

32 a^3+a^2+a+1

33 $1-x+x^2-x^3$

34 $f^2g-cfh+fgh-c$…

35. $ab+b^2c+a^2c+abc^2$

36 $acx^2-bcx-adx+bd$

37 $abq-bcq^2+acq-a^2$

38 $xyz^2-x^3z-y^3z+x^2y^2$

39 $(a+b)x-(a+b)y+(a+b)z$

40 $5(p-q)mp+10(p-q)mq-25(p-q)m^2$

Ex 41 Resolve $4m^4n-6m^3n^2+4m^2n^3-6mn^4$ into factors

Given expression $=2mn(2m^3-3m^2n+2mn^2-3n^3)$,

and the expression within the bracket

$$=(2m^3-3m^2n)+(2mn^2-3n^3)$$
$$=m^2(2m-3n)+n^2(2m-3n)$$
$$=(2m-3n)(m^2+n^2),$$

given expression $=2mn(2m-3n)(m^2+n^2)$

Resolve into factors

42 $xyz+bxz+cxy+bcx$

43 $a^2b+3ab-4a^2-12a$

44 $m^3xy-m^2y^2-m^2x^2+mxy$

45 $8m^2n^2-24m^2np-16mn^3+48mn^2p$

46 $4a^3gh-2a^2cfg-8a^2bfh+4abcf^2$

Ex 47 Resolve $bc+a(a+b+c)$ into factors

Given expression $=bc+a^2+ab+ac=(a^2+ab)+(ac+bc)$

$$=a(a+b)+c(a+b)=(a+b)(a+c)$$

Resolve into factors

48. $x(x-a-b)+ab$ 49. $x(x+ma-1)-ma$.

50. $ab+c\{a-b(b+c)\}$ 51. $ab(1-c^2)+c(b^2-a^2)$

52. $ax^2+(ac-b)x-bc$ 53. $a^2-aq(b+c)+bcq^2$

122. Factors of Expressions of the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$. We know that

$$a^2+2ab+b^2=(a+b)^2 \text{ and } a^2-2ab+b^2=(a-b)^2.$$

Examples.

Ex. 1. Resolve $x^2+4xy+4y^2$ into factors.

Given expression $=x^2+2(x)(2y)+(2y)^2=(x+2y)^2$,

∴ required factors are $x+2y$ and $x+2y$.

[illegible] se by the method of § 121 —

[illegible] $y+4y^2=x^2+2xy+2xy+4y^2=(x^2+2xy)+(2xy+4y^2)$

$=x(x+2y)+2y(x+2y)=(x+2y)(x+2y)$.

[illegible] Resolve $4a^2-4ab+b^2$ into factors.

[illegible] n expression $=(2a)^2-2(2a)(b)+b^2=(2a-b)^2$.

Ex. 3. Resolve $45m^4\pm30m^3n+5m^2n^2$ into factors.

Given expression $=5m^2(9m^2\pm6mn+n^2)=5m^2(3m\pm n)^2$, ∴ &c.

Resolve into component factors:

4. $1+2a+a^2$ 5. $9x^2-6x+1$ 6. $9a^3+30a^2x+25ax^2$.

7. $16a^4-8a^2b+b^2$ 8. $36x^2y^2-12xy+1$. 9. $36a^4x^2-48a^3x+16a^2$

10. $25a^2x^4+20abxy^2+4b^2y^4$ 11. $3x^3y-18x^2y^2+27xy^3$

12. $4a^2+28a+49$ 13. p^4-20p^2+100 14. $169m^2+52m+4$.

Ex. 15. Resolve $(a+b)^2+2(a+b)c+c^2$ into factors.

Here put $a+b=x$, thus the given expression

$$=x^2+2xc+c^2=(x+c)^2=(a+b+c)^2.$$

Ex. 16. Resolve $x^2-2xy+y^2+x-y$ into factors.

Given expression $=(x^2-2xy+y^2)+(x-y)$

$=(x-y)^2+(x-y)$

$=(x-y)(x-y+1)$ [§ 121]

Ex. 17. Resolve $4mp^2+4m^2p+m^3-2mp-m^2$ into factors.

Given expression $=(4mp^2+4m^2p+m^3)-(2mp+m^2)$

$=m(4p^2+4mp+m^2)-m(2p+m)$

$=m(2p+m)^2-m(2p+m)$

$=m(2p+m)(2p+m-1)$ [§ 121]

Ex 18. Resolve $(x+2y)^2-2(x+2y)(x+z)+(x+z)^2$ into factors

Put $x+2y=a$ and $x+z=b$, thus the given expression

$$=a^2-2ab+b^2=(a-b)^2=\{(x+2y)-(x+z)\}^2$$
$$=(x+2y-x-z)^2=(2y-z)^2$$

Resolve into component factors

19 $(2x+y)^2-2(2x+y)z+z^2$ 20 $a^2+2(ax-ay)+(x-y)^2$

21 $4m^2+4m+4(2m+1)q+4q^2+1$

22 $(2a+3b)^2x^2-(4a+6b)xy+y^2$ 23 $1-2x+x^2-a+ax$

24 $a^2x^2-2axy-ax+y^2+y$ 25 $3ax^3+6ax^2y+3axy^2-ax^2-axy$.

26 $16a^4x^2-8a^2bx^2+b^2x^2-4a^2+b$ 27 $a^2+b^2+2(bc+ca+ab)$.

28 $(a+b)^2+2(a+b)(a-b)+(a-b)^2$

29 $(p+q)^2-2(p+q)(r-s)+(r-s)^2$

30 $(ax+1)^2+2(a+x)(ax+1)+(a+x)^2$

*31 $(x+y+z)^2-2(x+y+z)z+z^2$

123 Factors of Expressions of the form a^2 ... have seen [§ 102] how expressions of this form can b... We shall here give a few more examples

Examples

Ex 1 Resolve $32a^3b^3-18abx^2$ into factors.

Given expression $=2ab(16a^2b^2-9x^2)=2ab\{(4ab)^2-(3x)^2$

$$=2ab(4ab+3x)(4ab-3x)$$

Ex. 2 Find the value of $571\times571-429\times429$

Reqd value $=(571)^2-(429)^2=(571+429)(571-429)$

$$=1000\times142=142000$$

Ex 3 Find the square of 839

Now $$(839)^2=(839)^2-(39)^2+(39)^2$$
$$=(839+39)(839-39)+(39)^2$$
$$=878\times800+(39)^2=702400+(39)^2,$$

and $$(39)^2=(39)^2-1+1=(39+1)(39-1)+1$$
$$=40\times38+1=1521$$

required square $=702400+1521=703921$

Ex. 4 Find the square of 875

Now $$(875)^2=(875)^2-(25)^2+(25)^2$$
$$=(875+25)(875-25)+625$$
$$=900\times850+625=765625$$

Resolve into factors

5 $4a^4-b^4$. 6 $36a^6-x^4$. 7. $25a^8-9b^{10}$.

8. m^4-16 9 $16a^4-1$. 10. $1-x^8$.

11. a^8-x^8 12 $x^{16}-a^{16}$. 13 x^3-4xy^2

14 $50a^3-2a$ 15. $3-48a^2x^2$ 16 $32m^2n^2-2$.

17. $3a^3-3ax^2$. 18 $20a^3x^2-5ab^2x$. 19 $50x^3-32xy^2$.

20. $3x-12x^3$. 21. $50a^4b^2x^2-8a^2x^4y^2$. 22. $81x^5-16xy^4$.

23. $3x^3-\frac{1}{3}xy^2$. 24. $2a^5-\frac{1}{8}ax^4$ 25 $162a^6b^2-32a^2b^6$.

Find the value of

26. $(354)^2-(254)^2$. 27 $(879)^2-(121)^2$. 28 $(487)^2-(394)^2$.

29. $(9728)^2-(9727)^2$. 30 $(5218)^2-(4882)^2$. 31 $(64)^2$.

32 $(85)^2$. 33 $(145)^2$ 34. $(193)^2$

35 $(416)^2$. 36 $(989)^2$. 37 $(9999)^2$.

[illegible]ve into factors

[illegible]$)^2-z^2$ 39. $m^2-(2n+q)^2$ 40 $9y^2-4(2b-3x)^2$.

[illegible]$4y)^2-1$. 42 $4a^2-9(b-3c)^2$ 43 $16-9(5z+6u)^2$.

[illegible]$by)^2-1$ 45. $25(3a+b)^2-4m^2$ 46 $25p^2-4(3q-1)^2$

[illegible]$-2y)^2-9z^2$. 48. $4(lx-my)^2-81x^2y^2$. 49. $(5r-5s)^2-25t^2$

50 $(a+b)^2-(c+d)^2$ 51 $(3x-2y)^2-(1+2z)^2$.

52. $(3x+8)^2-(3x-8)^2$ 53 $(3a-5b)^2-(3a+5b)^2$

54 $(7a-6b)^2-(6a-7b)^2$ 55 $(3ax+by)^2-(3by-ax)^2$

56. $(a^2-ab+b^2)^2-(a^2-ab-b^2)^2$ 57 $(a^2-ax+x^2)^2-(a^2+ax+x^2)^2$.

58. $(2a+b-3c)^2-(a-3b+2c)^2$ 59 $(a-2b-4c)^2-(3a+2b+6c)^2$.

60. $(x^2+xy)^2-(xy+y^2)^2$. 61 $(1+xy)^2-(x+y)^2$

62. $(x^2+y^2)^2-x^2(x-y)^2$. 63. $(mx+y)^2-(x+my)^2$

64 $(pr-qs)^2-(ps-qr)^2$ 65. $(x+a)^4-(x-a)^4$.

66 $(ax+by)^4-(bx-ay)^4$. 67. $(x^2-z^2)^2-y^2(2x+y)^2$.

Ex 68. Resolve $4b^2c^2-(b^2+c^2-a^2)^2$ into factors

Given expression $=(2bc)^2-(b^2+c^2-a^2)^2$

$$=(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)$$
$$=\{(b+c)^2-a^2\}\{a^2-(b-c)^2\}$$
$$=(b+c+a)(b+c-a)(a+b-c)(a-b+c)$$
$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

69 $(a^2+b^2-c^2)^2-4a^2b^2$ 70 $4a^2c^2-(a^2-b^2+c^2)^2$

71. $(b^2+c^2)^2-(b^2-c^2)^2-(a^2-b^2-c^2)^2$.

72 $4(ad+bc)^2-(a^2-b^2-c^2+d^2)^2$

124. Factors of expressions which can be put into the form a^2-b^2. These expressions when expressed as the difference of two squares are resolved as in § 123.

Ex. 1. Resolve $x^2+2ax+a^2-1$ into factors.

Given expression $=(x^2+2ax+a^2)-1$

$$=(x+a)^2-1=(x+a+1)(x+a-1).$$

Ex. 2. Resolve $x^2+y^2-z^2-u^2+2(xy+zu)$ into factors.

Given expression $=(x^2+2xy+y^2)-(z^2-2zu+u^2)$

$$=(x+y)^2-(z-u)^2$$
$$=\{(x+y)+(z-u)\}\{(x+y)-(z-u)\}$$
$$=(x+y+z-u)(x+y-z+u).$$

Resolve into factors:

3. $4a^2-4ab+b^2-c^2$. 4. $1-2y+y^2-z^2$.

5. $1+2ax+a^2x^2-x^2$. 6. $1-a^2-b^2-2ab$.

7. $x^2-y^2+2xz+z^2$. 8. $4p^2-4pq-4+q^2$.

9. $a^2-z^2-y^2+2yz$. 10. $2ax-a^2x^2+a^2y^2-1$.

11. $d^2+9c^2-4a^2-6cd$. 12. $9x^2-25y^2-64-80y$.

13. $x^4-(p^2+2)x^2y^2+y^4$. [Bom., 1888].

14. $a^2x^4+(2ab-c^2)x^2y^2+b^2y^4$. 15. $x^2-y^2+2(x-2y)-3$.

16. $(a^2-b^2)(x^2-y^2)+4abxy$. 17. $a^2+4ax+4x^2-y^2-6y-9$.

18. $x^2+y^2-z^2+2xy-2z-1$. 19. $2(ab+xy)+a^2+b^2-x^2-y^2$.

20. $a^2+b^2-c^2-d^2-2(ab-cd)$. 21. $4a^2-b^2-9c^2+d^2+2(2ad-3bc)$.

✓22. $x^3+2xy^2-xy^2-xz^2$.

Ex. 23. Resolve x^4+4y^4 into factors.

Given expression $=(x^4+4x^2y^2+4y^4)-4x^2y^2$

$$=(x^2+2y^2)^2-(2xy)^2$$
$$=\{(x^2+2y^2)+2xy\}\{(x^2+2y^2)-2xy\}$$
$$=(x^2+2xy+2y^2)(x^2-2xy+2y^2).$$

Otherwise thus:—Since $a^2+b^2=(a+b)^2-2ab$ [§ 101],

we have $x^4+4y^4=(x^2)^2+(2y^2)^2$

$$=(x^2+2y^2)^2-2(x^2)(2y^2)$$
$$=(x^2+2y^2)^2-(2xy)^2$$
$$=(x^2+2xy+2y^2)(x^2-2xy+2y^2).$$

Ex. 24. Resolve $a^4+a^2b^2+b^4$ into factors. [See § 119]

Given expression $=(a^4+2a^2b^2+b^4)-a^2b^2$

$=(a^2+b^2)^2-(ab)^2$

$=\{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\}$

$=(a^2+ab+b^2)(a^2-ab+b^2)$.

Or thus: $a^4+a^2b^2+b^4=\{(a^2)^2+(b^2)^2\}+a^2b^2$

$=\{(a^2+b^2)^2-2a^2b^2\}+a^2b^2$ [§ 101]

$=(a^2+b^2)^2-a^2b^2=$ &c.

Ex. 25. Resolve x^4-7x^2+9 into factors.

We may proceed as usual, or perhaps thus:—Since $a^2+b^2=(a-b)^2+2ab$ [§ 101], we have

$x^4-7x^2+9=\{(x^2)^2+3^2\}-7x^2$

$=\{(x^2-3)^2+6x^2\}-7x^2$

$=(x^2-3)^2-x^2=(x^2+x-3)(x^2-x-3)$.

[illegible] into factors:

26. [illegible] 27. $1+4x^4$ 28. $4a^4+b^4$

29. [illegible]$+81$ 30. $4x^4+16y^4$ 31. $81x^4+64y^4$

32. x^4+324a^4. 33. x^4+x^2+1. 34. x^8+x^4+1.

35. $x^4+a^2x^2+a^4$ [C.U., 1897] 36. x^4+2x^2+9

37. x^4-15x^2+9. 38. x^4-26x^2+25 39. $x^4-3a^2x^2+a^4$

40. $x^4-9x^2y^2+16y^4$ 41. $x^4+5x^2y^2+9y^4$. 42. $4a^4+8a^2b^2+9b^4$.

43. $4a^4-11a^2b^2+9b^4$. 44. $4a^4-13a^2b^2+9b^4$.

45. $9x^4-31a^2x^2+25a^4$. 46. $9x^4+21a^2x^2+25a^4$

47. $16x^4-25x^2+1$. 48. a^4-18a^2+1

49. $64a^4-33a^2b^2+b^4$. 50. $16x^4+33x^2y^2+81y^4$.

51. x^8-16x^4 [C.U., 1887]. 52. $(x+1)^4+(x^2-1)^2+(x-1)^4$.

53. $(x+a)^4+(x^2-a^2)^2+(x-a)^4$.

Note. Sometimes expressions are proposed which may be transformed into the form $a(a^2-b^2)+a(a+b)$ or $a(a^2-b^2)+a(a-b)$.

Ex. 54. Resolve $a^2+2ab+b^2-c^2+a+b-c$.

Given expression $=\{(a^2+2ab+b^2)-c^2\}+(a+b-c)$

$=\{(a+b)^2-c^2\}+(a+b-c)$

$=(a+b-c)(a+b+c)+(a+b-c)$

$=(a+b-c)(a+b+c+1)$.

Resolve into factors

55 $(x^2-3xz)-(y^2-3yz)$ 56 $(x+a)(x+b)-(y+a)(y+b)$

57 $(x^4-2x^3y)-(y^4-2xy^3)$ 58 $(a-b)(b-c)+(a-d)(c-d)$.

59 $1+(b-a^2)x^2-abx^3$ 60 $a^2x^4-y^2-ax^2z^2+yz^2$.

61 $(a+2b)^2-b^2-3a-9b$ 62 $p^2+2pqr+q^2r^2-mp-mqr$

63 $x^2-2xy+y^2-z^2+2x-2y-2z$ 64 $x^2-y^2-z^2+2yz+x+y-z$.

65 $a^2+a-b^2-b-c^2-c+2bc$. 66 $4b^2-9(a^2+x^2-2ax)-2b-3(a-x)$.

124*a* Factors of Expressions of the form $a^3+3a^2b+3ab^2+b^3$ or $a^3-3a^2b+3ab^2-b^3$

We have seen [§ § 103, 104] how such expressions can be resolved into factors We shall now give a few more examples

Ex. 1 Resolve $a^3+2a^2b+2ab^2+b^3$ into factors [See § 124b, Ex. 6].

Given expression $=a^3+3a^2b+3ab^2+b^3-(a^2b+ab^2)$

$$=(a+b)^3-ab(a+b)=(a+b)\{(a+b)^2-$$

$$=(a+b)(a^2+ab+b^2).$$

Ex 2 Resolve $(x+y)^3-3(x+y)^2+3(x+y)-1$ into fa

Put $x+y=a$, thus given expression

$$=a^3-3a^2+3a-1=(a-1)^3=(x+y-1)^3.$$

Note $(x+y)^3-3(x+y)^2+3(x+y)-1$

$$=\{(x+y)^3-1\}-3(x+y)\{(x+y)-1\}$$

Thus the given expression may be resolved as in § 124b, *post*

Resolve into factors

3 $a^3-a^2b-ab^2+b^3$ 4 $(x^3+3xy^2)^2-(3x^2y+y^3)^2$

5. $(x^3+12x)^2-(6x^2+8)^2$ 6 $(1-3x)^2-(3x^2-x^3)^2$

7 $(x-2a)^3+3a(x-2a)^2+3a^2(x-2a)+a^3$

8 $(x+a)^3-3(x+a)^2(a-1)+3(a-1)^2(x+a)-(a-1)^3$

124*b* Factors of Expressions of the form a^3+b^3 or a^3-b^3

We have seen in §§ 105 and 106, how these expressions can be resolved into factors Here we shall give some examples of greater difficulty.

Ex. 1 Resolve $8a^3+\frac{1}{27}b^3$ into factors

$$8a^3+\tfrac{1}{27}b^3=8a^3+\frac{b^3}{27}=(2a)^3+\left(\frac{b}{3}\right)^3$$

$$=\left(2a+\frac{b}{3}\right)\left\{(2a)^2-2a\times\frac{b}{3}+\left(\frac{b}{3}\right)^2\right\}$$

$$=\left(2a+\frac{b}{3}\right)\left(4a^2-\tfrac{2}{3}ab+\frac{b^2}{9}\right).$$

Ex. 2. Resolve $(a^2+b^2)^3-8a^3b^3$ into factors.

Put x for a^2+b^2, and y for $2ab$; because $(2ab)^3=8a^3b^3$, the given expression is reduced to the form x^3-y^3. Hence

Given expression $=\{(a^2+b^2)-2ab\}\{(a^2+b^2)^2+(a^2+b^2)(2ab)+(2ab)^2\}$

$=(a-b)^2(a^4+2a^3b+6a^2b^2+2ab^3+b^4)$.

Ex. 3. Resolve x^6-y^6 into factors.

Given expression $=(x^3)^2-(y^3)^2=(x^3+y^3)(x^3-y^3)$

$=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$.

Otherwise thus: given expression $=(x^2)^3-(y^2)^3$

$=(x^2-y^2)\{(x^2)^2+x^2y^2+(y^2)^2\}$

$=(x^2-y^2)(x^4+2x^2y^2+y^4-x^2y^2)$

$=(x^2-y^2)\{(x^2+y^2)^2-(xy)^2\}$

$=(x^2-y^2)(x^2+y^2+xy)(x^2+y^2-xy)$

$=(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$.

Ex. 4. Resolve $8a^4+27ab^3$ into factors.

Given expression $=a(8a^3+27b^3)=a\{(2a)^3+(3b)^3\}$

$=a(2a+3b)\{(2a)^2-2a\cdot3b+(3b)^2\}$

$=a(2a+3b)(4a^2-6ab+9b^2)$.

Ex. 5. Resolve $x^3-3x^2y+3xy^2-y^3-z^3$ into factors.

Given expression $=(x^3-3x^2y+3xy^2-y^3)-z^3$

$=(x-y)^3-z^3$

$=(x-y-z)\{(x-y)^2+(x-y)z+z^2\}$

$=(x-y-z)(x^2-2xy+y^2+xz-yz+z^2)$.

Ex. 6. Resolve $a^3+2a^2b+2ab^2+b^3$ into factors. [See Art. 11.]

Given expression $=(a^3+b^3)+2ab(a+b)$

$=(a+b)(a^2-ab+b^2)+2ab(a+b)$

$=(a+b)(a^2-ab+b^2+2ab)$

$=(a+b)(a^2+ab+b^2)$.

Ex. 7. Resolve $(2a+3b)^3+8a^3+27b^3$ into factors.

Given expression $=(2a+3b)^3+(2a)^3+(3b)^3$

$=(2a+3b)^3+(2a+3b)\{(2a)^2-(2a)(3b)+(3b)^2\}$

$=(2a+3b)^3+(2a+3b)(4a^2-6ab+9b^2)$

$=(2a+3b)\{(2a+3b)^2+4a^2-6ab+9b^2\}$

$=(2a+3b)(8a^2+6ab+18b^2)$

$=2(2a+3b)(4a^2+3ab+9b^2)$.

Resolve into elementory factors

8 $x^3+\frac{1}{8}y^3$ 9 $\frac{1}{4}a^3-54x^6$ 10 $(ax+by)^3+1$

11 $x^3-8(y+z)^3$ 12. $64(a+b)^3+27c^3$ 13 $(a+b)^3x^3-8c^3y^3$.

14 $(x-2y)^3+y^3$. 15 $(a^2-bc)^3+8b^3c^3$. 16 $x^6+(x^2-2a^2)^3$

17 $8x^3-(x^2+1)^3$ 18 $(x^4+a^4)^3+27a^{12}$ 19 x^4y+8xy^4

20. $2a^8-128a^2x^6$ 21 a^9+b^9 22 $(x+y+z)^3+(x-y+z)^3$.

23 $x^3-3x^2y+3xy^2-y^3-z^3$ 24 $x^3-y^3+3y^2-3y+1$.

25. $a^3+3a^2b+3ab^2+2b^3$ 26 $x^3-27a^3+(x-3a)^3$

27 $8x^3-y^3+8x-4y$ 28 $a^3-2a^2b+2ab^2-b^3$

29 $2x^3-3x^2y+3xy^2-2y^3$ 30 $(a+b)^3+(a+b)-2$

31 $8y^3+(x+2y)^3+x^3$ 32 $(a+b)^3x^3-\{(a+b)x+1\}^3+1$.

33 $a^3+a(b+c)(a+b+c)+(b+c)^3$

125 Factors of Expressions of the form [illegible] **found by inspection** The general form of q[illegible] pressions is ax^2+bx+c, and when the coefficient of x^2 [illegible] x^2+px+q Thus *a quadratic in its general form consis*[illegible] *only, viz*, the first term which contains x^2, the second [illegible] contains x, and the third or *last* term which does not conta[illegible]

We have $x^2+(a+b)x+ab=(x+a)(x+b)$ [§ 107] . .. (1),

$x^2-(a+b)x+ab=(x-a)(x-b)$ [§ 109] . ....(2),

$x^2+(a-b)x-ab=(x+a)(x-b)$ [§ 108] .... (3)

A little consideration will shew that each of these expressions is of the form x^2+px+q, for in (1), $p=a+b$, $q=ab$; in (2), $p=-(a+b)$, $q=ab$, and in (3), $p=a-b$, $q=-ab$ Hence in resolving into factors expressions of the form x^2+px+q, we have only to see whether q is the *product of two quantities such that their algebraical sum is p*.

Again it is easy to see that the left side of (1) corresponds to the quadratic x^2+px+q, that of (2) corresponds to x^2-px+q, and that of (3) corresponds to x^2+px-q or x^2-px-q according as a is greater or less than b Hence it follows from (1) and (2), that when q (the last term) is *positive*, a and b (the second terms of the required factors) *have both the same sign*, and *that sign is the sign of p* (the coefficient of x), and from (3) that when q is *negative*, a and b have *opposite signs and the greater of the two has the sign of p*

Examples.

Ex 1 Resolve x^2+5x+6 into factors.

Here we must find two numbers, such that their product $=6$ and sum $=5$ Now pairs of numbers, of which the product is 6, are (1)

1 and 6, (ii) 2 and 3 We reject the first pair and take the second, for $2+3=5$ Hence the second terms (*i e* the a and the b) of the required factors are 2 and 3,

$$\therefore x^2+5x+6=(x+2)(x+3).$$

Note If we take the first pair of numbers, we see that $x^2+7x+6=(x+1)(x+6)$.

Ex 2. Resolve $x^2-15x+36$ into factors.

Here we want to find two numbers, such that their product $=36$ and sum $=-15$ Pairs of numbers whose product is 36 are (i) 1 and 36, (ii) 2 and 18, (iii) 3 and 12, (iv) 4 and 9, (v) 6 and 6 Now we must select that pair whose sum is 15, and we take the third pair 3 and 12 Moreover the last term 36 being *positive*, 3 and 12 must each have the *same sign*, which is here $-$, as the co-efficient of [illegible] has the $-$ sign Thus the second terms of the required factors [illegible] nd -12,

$$\therefore x^2-15x+36=(x-3)(x-12)$$

[illegible] Resolve $x^2+8x-48$ into factors

[illegible] have to find two numbers whose product is -48 and [illegible] sum (or difference) $+8$ Pairs of numbers, whose product is [illegible] (i) 1 and 48, (ii) 2 and 24, (iii) 3 and 16, (iv) 4 and 12, (v) 6 and 8, and we take the fourth pair, 4 and 12, for their difference is 8 Also the last term -48 being *negative*, the numbers we select must have *opposite* signs, the greater of the two having the $+$ sign, as the co efficient of x has the sign $+$. Hence the second terms of the required factors are $+12$ and -4

$$\therefore x^2+8x-48=(x+12)(x-4)$$

Ex. 4 Resolve $x^2-13x-48$ into factors.

We have to find a pair of numbers whose product $=-48$ and algebraic sum (or difference) $=-13$. As shewn in the last example, there are 5 pairs of numbers whose product is 48; and of these we choose the third pair, 3 and 16, for their difference is 13 And the last term -48 being *negative*, the numbers we choose must have *opposite* signs, the greater being *negative* as the co efficient of x is *negative* Thus the second terms of the required factors are $+3$ and -16

$$\therefore x^2-13x-48=(x+3)(x-16).$$

Note It is easy to see that if $x+a$ and $x+b$ be the factors of x^2+px+q (i), those of $x^2+pxy+qy^2$ (ii)

will be $x+ay$ and $x+by$, that is, *the factors of* (ii) *can be derived from those of* (i) *by writing* ay *for* a *and* by *for* b

For from (i), $p=a+b$, $q=ab$; $\therefore$ from (ii)

$$x^2+pxy+qy^2=x^2+(a+b)xy+aby^2=(x+ay)(x+by)$$

Thus from Ex 1, $x^2+5xy+6y^2=(x+2y)(x+3y)$,

from Ex 4, $x^2-13xy-48y^2=(x+3y)(x-16y)$, and so on

From the above proof it is clear that to resolve (ii) *independently*, we have to find two numbers a and b such that $ab=q$ and $a+b=p$

Ex 5 Resolve $a^2+ax-156x^2$ into factors.

Reasoning as in Ex 3, we find that

$$-156=(+13)(-12), \text{ and } +1=+13-12,$$
$$\therefore a^2+ax-156x^2=(a+13x)(a-12x)$$

Ex 6 Resolve $x^2-2xy-63y^2$ into factors

Reasoning as before (see Ex 4), we see that

$$-63=(+7)(-9), \text{ and } -2=+7-9,$$
$$x^2-2xy-63y^2=(x+7y)(x-9y)$$

Ex 7 Resolve $x^2+2ax+(a^2-1)$ into factors [Ex. 1, §

Now $a^2-1=(a+1)(a-1)$, and $2a=(a+1)+(a-1)$,

$$\text{given expression}=(x+a+1)(x+a-1)$$

Ex 8 Resolve $x^2-(a+b)x+(a+1)(b-1)$ into factors

Given expression $=x^2-\{(a+1)+(b-1)\}x+(a+1)(b-1)$

$$=\{x-(a+1)\}\{x-(b-1)\}=(x-a-1)(x-b+1)$$

Ex 9 Resolve $x^2-2abx-a^4-a^2b^2-b^4$ into factors

Given expression $=x^2-2abx-(a^4+a^2b^2+b^4)$

Now $a^4+a^2b^2+b^4=(a^2-ab+b^2)(a^2+ab+b^2)$,

and $\qquad -2ab=(a^2-ab+b^2)-(a^2+ab+b^2)$,

$$\therefore \text{given expression}=\{x+(a^2-ab+b^2)\}\{x-(a^2+ab+b^2)\}$$
$$=(x+a^2-ab+b^2)(x-a^2-ab-b^2)$$

Ex 10 Resolve $39x-3x^2-120$ into factors.

Given expression $=3(13x-x^2-40)=-3(x^2-13x+40)$

$$=-3(x-5)(x-8)=3(x-5)(8-x)$$
$$\text{or } =3(5-x)(x-8)$$

Resolve into factors

11	$x^2+7x+12$	12	$a^2+7a+10$	13.	$x^2+10x+21$
14.	$p^2+24p+80$	15.	$z^2+17z+42$	16	$x^2+9x+20$.
17	a^2-4a+3	18	$a^2-9a+20$.	19	x^2-6x+8
20	$a^2-7a+12$	21	x^2-5x+6.	22.	$x^2-27x+170$

Resolve into factors

23.	$x^2+4x-32$	24.	$a^2-4a-21$	25.	a^2+a-72
26	$x^2-4x-32.$	27.	$x^2+6x-40$	28	$x^2+19x-42.$
29.	x^2+x-42	30	$x^2-3x-54.$	31	$x^2-15x-54.$
32.	$m^2+4m-96$	33	$m^2-29m-96$	34.	$m^2-20m-96$
35.	$a^2+2a-120$	36	$a^2-19a-120$	37	$p^2+7p-144.$
38.	$x^2+8x-48$	39	$x^2+2x-48$	40	$x^2-13x-48$
41	$k^2+19k-20$	42	$l^2-23l-78$	43.	$l^2-7l-78$
44	$x^2+x-156$	45.	$x^2-20x-156$	46	$a^2b^2+15ab+36$
47	$a^2b^2-20ab+36$	48	$x^2y^2-3xy-18$	49.	$p^2q^2+14pq-32.$
50	$m^2n^2-6mn-16$	51	$a^2x^2+5ax-36$	52	$a^2-3ab-54b^2.$
[illegible]	[illegible] $5xy-100y^2.$	54	$x^2+2xy-24y^2.$	55	$x^2-10xy-24y^2$
[illegible]	[illegible] $b-60b^2$	57	$l^2+18lm+45m^2$	58	$a^2+17ab-60b^2.$
[illegible]	[illegible] $mn-128n^2$	60	$m^2-28mn-128n^2.$	61	$p^2+pq-380q^2.$
[illegible]	[illegible] $24x-5x^2.$	63	$64-16a-195a^2$	64	$1+2m-24m^2.$
65.	[illegible] $-k-420$	66.	$x^2+x-650$	67.	$q^2+105q+2000.$
68	$1+3x-18x^2$	69	$1-7x-18x^2$	70	$9+12x-32x^2$
71.	$9-42x-32x^2$	72	$9-48x-28x^2$	73.	$25-90m+72m^2.$
74.	$25+30x-16x^2.$	75.	$25-75x-16x^2$	76	$m^2-5m-300.$
77.	$m^2-5mn-50n^2$	78	$x^2+2ax-80a^2$	79.	$1+3xy-4x^2y^2$
80.	$1+55xy+750x^2y^2$	81	$k^2+4k-192$	82	$x^2-6xy-40y^2.$
83	$a^2+9a-486$	84	$x^2-2x-360$	85	$x^2+26x-560$
86	$24x-x^2-128$	87.	$72+6x-x^2.$	88	$80-16x-x^2.$
89	$x^2+\frac{7}{2}x+\frac{5}{9}$	90	$x^2-\frac{10}{3}x+1$	91	$x^2-\frac{1}{2}x-5.$

92 $a^2+(z+2)a+2z.$ 93 $x^2+(1+a)xy+ay^2$

94. $y^2+(2x-1)y+x(x-1)$ 95 $x^2-(2a+1)x+a^2+a-6$

96 $x^2-2(y-2)x+(y-1)(y-3).$ 97 $x^2+(y+1)x-(y-2)(2y-1)$

98 $x^2-4abx-(a^2-b^2)^2$

126. Factors of Expressions of the form ax^2+bx+c, found by inspection An expression of this form can be resolved as in the last article by reducing it to the form x^2+px+q, *which is done by multiplying it by the coefficient of x^2*

Examples

Ex 1. Resolve $6x^2+23x+20$ into factors.

Multiply the given expression by 6, thus

$$\begin{aligned}6\times \text{given expn.} &= 6\times 6x^2+6\times 23x+6\times 20\\ &=(6x)^2+23(6x)+120\\ &=X^2+23X+120, \text{ where } X=6x,\\ &=(X+15)(X+8)\\ &=(6x+15)(6x+8), \text{ replacing } X \text{ by } 6x, \quad \ldots \text{ (A)}\\ &=3(2x+5)\times 2(3x+4)=6(2x+5)(3x+4),\end{aligned}$$

$$\therefore \text{ given expn} =(2x+5)(3x+4)$$

Note 1 From the above process, we see that 120 is the product of 6 (coefficient of x^2) and 20 (last term), and that 23 (coefficient of x) is the sum of two numbers whose product is 120. Also we see from [illegible]

$$6\times(6x^2+23x+20)=(6x+15)(6x+8),$$

or

$$6x^2+23x+20=(6x+15)(6x+8)\div 6$$

Thus we first find the factors $6x+15$ and $6x+8$, the fir[illegible] which are *each* $6x$, and the second terms are 15 and 8, obtaine[illegible]ing 6×20 into two factors, and then divide the product of th[illegible] by 6 to get the factors of the given expression. Hence we see th[illegible]

$ax^2+bx+c=(ax+\text{one factor of } ac)(ax-\text{other factor of } ac)\div a$, the sum of the factors being $=b$.

Similarly it will be seen that

$ax^2+bx+c=(ax-\text{one factor of } ac)(ax-\text{other factor } ac)\div a$,

$ax^2+bx-c=(ax+\text{greater factor of } ac)(ax-\text{smaller factor of } ac)\div a$,

$ax^2-bx-c=(ax-\text{greater factor of } ac)(ax+\text{smaller factor of } ac)\div a$, the *difference* of the factors in the last two cases being $=b$.

Hence the rule :—*Multiply the last term by the coefficient of x^2, and resolve this product into two factors, whose sum or difference will be equal to the coefficient of x, according as the product is positive or negative.*

Ex 2. Resolve $3x^2-14x+8$ into factors.

We may proceed as in Ex 1, or according to the above rule thus.

$3\times 8=24$ which is *positive*, and the two factors, which make up 24 and whose *sum* is 14, are 12 and 2,

$$3x^2-14x+8=(3x-12)(3x-2)\div 3=(x-4)(3x-2)$$

Ex 3. Resolve $2x^2+x-6$ into factors.

Proceed as in Ex 1, or thus :—

$2\times(-6)=-12$, which is *negative*, and the two factors, which make up 12 and whose *difference* is 1, are 4 and 3,

$$\therefore\ 2x^2+x-6=(2x+4)(2x-3)\div 2=(x+2)(2x-3)$$

Ex. 4 Resolve $12x^2-7x-45$ into factors.

Proceeding according to the above rule we have $12\times(-45)=-540$, which is negative; the two factors which make up 540 and whose difference is 7, are 27 and 20;

$$\therefore\ 12x^2-7x-45=(12x-27)(12x+20)\div 12=(4x-9)(3x+5).$$

Resolve into component factors

5. $4x^2+11x+6$.	6. $5a^2+19a-4$	7. $6x^2+x-12$
8. $6x^2-35x+36$	9 $3x^2-10x-25$.	10. $4x^2+23x-72$
11. $8y^2-6y-35$	12. $15x^2+4x-96$.	13. $12x^2-17x-5$.
14. $10x^2+3x-4$.	15 $9m^2+9m-28$	16 $8x^2+61x-24$.
17. $15x^2-26x-21$.	18. $8x^2+14x-15$.	19. $12x^2-17x-40$.
20. $8x^2+10x-7$.	21. $6x^2+11x-10$.	22 $4x^2-13xy-12y^2$.
23. [illegible] $3x-27x^2$.	24. $12x^2-x-6$.	25. $12x^2-32x+5$
[illegible] $x-21$.		27. $24p^2-62pq+35q^2$.
[illegible] $-10x^2$.		29 $9x^2y^2+52xy-12$.
[illegible] $ax-12a^2$.		31. $6-11x-10x^2$.
[illegible] $34ab+15b^2$		33 $12x^2+31xy+20y^2$.
34. [illegible] $-28ab-5b^2$		35 $18m^2+9mn-35n^2$.
36. $12x^2-41ax+24a^2$		37. $42x^2-41x-20$.
38. $56y^2+99yz-45z^2$.		39. $24x^2-37xy-72y^2$.

Note 2. We have

$$(ax-b)(bx+a)=abx^2+(a^2-b^2)x-ab.$$
$$(ax-b)(bx-a)=abx^2-(a^2+b^2)x+ab.$$
$$(ax+b)(bx-a)=abx^2+(b^2-a^2)x-ab$$
$$\text{or }=abx^2-(a^2-b^2)x-ab.$$

From the right side of these identities, we learn that when the coefficient of x^2 is *the same as the last term* (irrespective of signs), the last term ab is the product of two numbers, the *sum or difference of whose squares* is the coefficient of x. Hence an expression of the form mx^2+nx-m, can be resolved into two factors of the form $(ax\ \ b)$ and $(bx\ \ a)$; the proper signs being supplied as in § 125.

Ex. 40 Resolve $14x^2+53x+14$ into factors.

Here 14 is the product of two numbers, the sum of whose squares is 53. Now $14=2\times7$ and $53=2^2+7^2$;

$$\therefore\ 14x^2+53x+14=(2x+7)(7x+2).$$

Ex. 41 Resolve $15x^2-224x-15$ into factors.

Here $-15=1\times(-15)$ and $-224=1^2-(15^2)$;

$$\therefore\ 15x^2-224x-15=(15x+1)(x-15).$$

Resolve into component factors

42	$6x^2+13x+6$	43	$15x^2+34x+15$	44	$12x^2+25x+12$
45	$24x^2-73x+24$	46	$12x^2-145x+12$	47	$40x^2-89x+40$
48	$8x^2+63x-8$	49	$12m^2-7m-12$	50	$15x^2-16x-15$
51	$12x^2-45x-12$	52	$16x^2+255x-16$	53	$11x^2+120x-11$
54	$9x^2+80x-9$	55	$6x^2-35x+6$	56	$45l^2-56l-45$
57	$8m^2-65mn+8n^2$	58	$36x^2+21x-36$	59	$6x^2-5xy-6y^2$
60	$14x^2-35xy+14y^2$	61	$7x^2-48xy-7y^2$	62	$40x^2-39xy-40y^2$

The method of resolving quadratic expressions by inspection given in the last two articles are often practically very useful We shall, however, give in § 128, *a general method of resolution applicable to all cases of quadratics*

***127 Expression of a Quadratic as the diffe[illegible]f two squares** Every quadratic can be expressed as th[illegible] of two squares by the method of *completing the square*[illegible] two ways by which we can complete the square—[illegible] Method and *Sridhara's* Method

Common Method

We know that $x^2+2ax+a^2$ is a perfect square, where the last term is *the square of* a *which is half the coefficient of* x Thus a quadratic of the form x^2+px may be made a complete square $(x+\frac{p}{2})^2$ by adding the square of $\frac{1}{2}p$ to it Hence we have

$$(a)\qquad x^2+px=x^2+px+\left(\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2=\left(x+\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2$$

If the quadratic be of the form ax^2+bx, we have

$$(\beta)\qquad ax^2+bx=a\left\{x^2+\frac{b}{a}x\right\}=a\left\{x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right\}$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right\}$$

Thus to express a quadratic as the difference of two squares, we *reduce it to the form* x^2+px, *and add and subtract the square of half the coefficient of* x

*Sridhara's Method**

This method is often convenient, as it enables us *to avoid fraction in completing the square* Here we *multiply and divide the given*

* This ingenious method is due to SRIDHARACHARYA, a celebrated Hindu algebraist and is commonly known as the "HINDU METHOD" Some writer erroneously ascribes it to BHASKAR, who, however, himself lays no claim to it (see *Vijaganita*, § 131)

expression by four times the coefficient of x^2, and add and subtract the square of the coefficient of x.

Multiply and divide x^2+px by 4 times 1, the coefficient of x^2; thus

$$(\alpha)\quad x^2+px=(x^2+px)\times 4\div 4=\tfrac14(4x^2+4px)$$
$$=\tfrac14(4x^2+4px+p^2-p^2)=\tfrac14\{(2x+p)^2-p^2\}.$$

Multiply and divide by 4 times a, the coefficient of x^2; thus

$$(\beta)\quad ax^2+bx=(ax^2+bx)\times 4a\div 4a=\frac{1}{4a}(4a^2x^2+4abx)$$
$$=\frac{1}{4a}\left\{4a^2x^2+4abx+b^2-b^2\right\}=\frac{1}{4a}\left\{(2ax+b)^2-b^2\right\}.$$

REMARK. We may use the *first* method when the quadratic is of the form x^2+px+q, and the *second* method when it is of the form ax^2+bx+c.

EXAMPLES

(i) Express x^2+8x as the difference of two squares.

Common Method

$$x^2+8x=x^2+8x+(\tfrac82)^2-(\tfrac82)^2=(x^2+8x+16)-(4)^2=(x+4)^2-(4)^2.$$

Sridhara's Method.

$$x^2+8x=(x^2+8x)\times 4\div 4=\tfrac14(4x^2+32x)=\tfrac14(4x^2+32x+64-64)$$
$$=\tfrac14\{(2x+8)^2-(8)^2\}$$

(ii) Express $3x^2-16x$ as the difference of two squares.

Common Method

$$3x^2-16x=3\{x^2-\tfrac{16}{3}x\}$$
$$=3\{(x^2-\tfrac{16}{3}x+(\tfrac83)^2-(\tfrac83)^2\},\ \because\ \tfrac12 \text{ of } \tfrac{16}{3}=\tfrac83,$$
$$=3\{(x-\tfrac83)^2-(\tfrac83)^2\}$$

Sridhara's Method

$$3x^2-16x=(3x^2-16x)\times 12\div 12=\tfrac1{12}(36x^2-192x)$$
$$=\tfrac1{12}\{36x^2-192x+(16)^2-(16)^2\}$$
$$=\tfrac1{12}\{(6x-16)^2-(16)^2\}$$

(iii) Put $x^2-13x+40$ as the difference of two squares.

$$x^2-13x+40=x^2-13x+(\tfrac{13}{2})^2-(\tfrac{13}{2})^2+40$$
$$=\{x^2-13x+(\tfrac{13}{2})^2\}-\{(\tfrac{13}{2})^2-40\}$$
$$=\{x^2-13x+(\tfrac{13}{2})^2\}-\tfrac94=(x-\tfrac{13}{2})^2-(\tfrac32)^2.$$

Note. Hence $x^2-13x+40=(x-\frac{13}{2}+\frac32)(x-\frac{13}{2}-\frac32)=(x-5)(x-8)$. Thus the given expression is resolved into factors.

(iv) Exhibit $(x+a)(x+3a)$ as the difference of two squares

$$(x+a)(x+3a)=x^2+4ax+3a^2$$
$$=x^2+4ax+\left(\frac{4a}{2}\right)^2-\left(\frac{4a}{2}\right)^2+3a^2$$
$$=x^2+4ax+(2a)^2-a^2=(x+2a)^2-a^2.$$

(v) Express $(x+3a)(x+5a)(x+7a)(x+9a)$ as the difference of two square quantities [*Cal.*, 1887]

$$\text{Given expn } =(x+3a)(x+9a)\times(x+5a)(x+7a)$$
$$=(x^2+12ax+27a^2)\times(x^2+12ax+35a^2)$$
$$=(x^2+12ax)^2+(27a^2+35a^2)(x^2+12ax)+945a^4,$$
taking x^2+12ax as one term,
$$=X^2+62a^2X+945a^4, \text{ where } X=x^2+12\ldots$$
$$=X^2+62a^2X+(31a^2)^2-(31a^2)^2+945\ldots$$
$$=(X+31a^2)^2-16a^4=(x^2+12ax+31\ldots$$

Note We have seen [§ 101 and 124] how a product of t... factors, can be expressed as the difference of two squares I... way, a quadratic can be expressed as the difference of two squ...

Example. Express $x^2-13x+40$ as the difference of two squares [Ex. (iii)]

Now $x^2-13x+40=x(x-13)+40$, and since

$$ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2 \text{ [§ 101]},$$

we have $x(x-13)=\left\{\frac{x+(x-13)}{2}\right\}^2-\left\{\frac{x-(x-13)}{2}\right\}^2$

$$=(x-\tfrac{13}{2})^2-(\tfrac{13}{2})^2,$$

$$\therefore\ x^2-13x+40=(x-\tfrac{13}{2})^2-(\tfrac{13}{2})^2+40=(x-\tfrac{13}{2})^2-(\tfrac{3}{2})^2.$$

Examples

Express as the difference of two squares

1. x^2+12x
2. x^2+5x
3. x^2-20x
4. x^2-3x
5. $x(x-18)$
6. $x(x+13)$
7. x^2+4ax
8. $x(x-2a)$.
9. $x^2+30x+200$.
10. $x^2-15x+54$
11. $x^2-20x-96$
12. $(x+3)(x-4)$
13. $(x-2m)(x+3m)$
14. $(x+p)(x+p-2q)$
15. $(x+y)^2-6y(x+y)$
16. $(x+q)^2-(x+q)$
17. $x(x+1)(x+2)(x+3)$
18. $(x-a)x(x+a)(x+2a)$

***128 General method of resolving Quadratics** We *express the proposed quadratic* [§ 125] *as the difference of two squares* by the method explained in § 127, and thus resolve it into *two* factors, each of the *first degree*

Examples

Ex. 1. Resolve $x^2+4x-21$ into factors

Common Method

$$x^2+4x-21=x^2+4x+(\tfrac{4}{2})^2-(\tfrac{4}{2})^2-21=(x+2)^2-5^2$$
$$=(x+2+5)(x+2-5)=(x+7)(x-3)$$

Sridhara's Method

$$x^2+4x-21=\tfrac{1}{4}(4x^2+16x-84)=\tfrac{1}{4}(4x^2+16x+16-16-84)$$
$$=\tfrac{1}{4}\{(2x+4)^2-(10)^2\}=\tfrac{1}{4}(2x+4+10)(2x+4-10)$$
$$=\tfrac{1}{4}(2x+14)(2x-6)=(x+7)(x-3)$$

[illegible]solve $4x^2-5x-21$ into factors

$$[illegible]\;5x-21=4(x^2-\tfrac{5}{4}x-\tfrac{21}{4})$$
$$=4\{x^2-\tfrac{5}{4}x+(\tfrac{5}{8})^2-(\tfrac{5}{8})^2-\tfrac{21}{4}\}$$
$$=4\{(x-\tfrac{5}{8})^2-(\tfrac{25}{64}+\tfrac{21}{4})\}$$
$$=4\{(x-\tfrac{5}{8})^2-(\tfrac{19}{8})^2\}=4(x-\tfrac{5}{8}+\tfrac{19}{8})(x-\tfrac{5}{8}-\tfrac{19}{8})$$
$$=4(x+\tfrac{7}{4})(x-3)=(4x+7)(x-3)$$

(2) $4x^2-5x-21=\frac{1}{16}(64x^2-80x-336)$

[multiplying and dividing by 16]

$$=\tfrac{1}{16}(64x^2-80x+25-25-336)$$

[adding and subtracting 5^2]

$$=\tfrac{1}{16}\{(8x-5)^2-(19)^2\}=\tfrac{1}{16}(8x-5+19)(8x-5-19)$$
$$=\tfrac{1}{16}(8x+14)(8x-24)=(4x+7)(x-3)$$

Ex. 3 Resolve $19x-10-6x^2$ into factors

$$19x-10-6x^2=-(6x^2-19x-10)$$
$$=-\tfrac{1}{24}(144x^2-456x+240)$$
$$=-\tfrac{1}{24}\{[144x^2-456x+(19)^2]-(19)^2+240\}$$
$$=-\tfrac{1}{24}\{(12x-19)^2-(11)^2\}$$
$$=-\tfrac{1}{24}(12x-19+11)(12x-19-11)$$
$$=-\tfrac{1}{24}(12x-8)(12x-30)$$
$$=-\tfrac{1}{24}\times 4(3x-2)\times 6(2x-5)=(2-3x)(2x-5)$$

Ex. 4 Resolve $x^2-y^2-3z^2-2zx+4yz$ into factors

Consider this as an expression in x, thus

$$\text{Given expression}=x^2-2zx-(y^2+3z^2-4yz)$$
$$=x^2-2zx+z^2-(y^2+4z^2-4yz)$$
$$=(x-z)^2-(y-2z)^2$$
$$=(x+y-3z)(x-y+z)$$

[Resolve this by considering it as an expression in y, and also in z.]

Ex. 5 Resolve $a^2+4ab-2ac-5b^2+2bc$ into factors

Arrange in powers of a, thus

$$\text{Given expression}=a^2+2(2b-c)a-(5b^2-2bc)$$
$$=a^2+2(2b-c)a+(2b-c)^2-(2b-c)^2-(5b^2-2bc)$$
$$=\{a^2+2(2b-c)a+(2b-c)^2\}-(9b^2-6bc+$$
$$=(a+2b-c)^2-(3b-c)^2$$
$$=(a+2b-c-3b+c)(a+2b-c+3b-$$
$$=(a-b)(a+5b-2c)$$

Ex. 6 Resolve $2x^2-xy-6y^2+9x+17y-5$ into facto

$$\text{Given expression}=2x^2-(y-9)x-(6y^2-17y+5)$$
$$=\tfrac{1}{8}\{16x^2-8(y-9)x-(48y^2-136y+40)\}$$
$$=\tfrac{1}{8}\{16x^2-8(y-9)x+(y-9)^2-(y-9)^2-(48y^2-136y+40)\}$$
$$=\tfrac{1}{8}\{(4x-y+9)^2-(7y-11)^2\}$$
$$=\tfrac{1}{8}(4x-y+9+7y-11)(4x-y+9-7y+11)$$
$$=\tfrac{1}{8}\times 2(2x+3y-1)\times 4(x-2y+5)$$
$$=(2x+3y-1)(x-2y+5)$$

The same result will of course follow by considering it as an expression in y

Resolve into factors

7. $x^2+28x+192$
8. $k^2-21k+90$
9. $a^2+9a-162$
10. $x^2-18x+80$
11. $a^2-a-380$
12. $3x^2-20x-32$
13. $12a^2-23a+10$
14. $30-31x+5x^2$.
15. $12+17x-7x^2$
16. $10x^2-19xy-15y^2$
17. $6x^2+23ax-18a^2$
18. $4x^2-51ax+36a^2$
19. $12-71x-60x^2$
20. $15x^2+32x-775$
21. $24z^2-406z+1715$.
22. $y^2-y-8930$
23. $8x^2+513x-10935$
24. x^2+2x-y^2-4y-3
25. x^2-6x-y^2+2y+8

Resolve into factors

26. $a^2-6ab+5b^2+4bc-c^2$. 27. $x^2+10xy+9y^2+8yz-z^2$.

28. $a^2-2ab-4ac+2bc+3c^2$. 29. $a^2+2ab-2ac-3b^2+2bc$.

30. $x^2-4xy+3y^2+4x-10y+3$. 31. $3x^2-2ax-a^2+4ac-3c^2$.

32. $xy+2y^2+x+5y+3$. 33. $2xy-6x+3y^2-5y-12$.

34. $x^2-2y^2-z^2-xy-3yz$. 35. $15xy+11x+3x^2-5y-4$.

36. $2y^2-5xy+2x^2-ay-ax-a^2$. 37. $x^2+4xy+3y^2+4ax+6ay+3a^2$.

38. $3x^2-6ax+15xy-x-5y+2a$.

Note We have ax^2+bx+c

$$=(ax^2+bx+c)\times 4a\div 4a=\frac{1}{4a}(4a^2x^2+4abx+4ac)$$

[illegible] $\left\{4a^2x^2+4abx+b^2-b^2+4ac\right\}=\frac{1}{4a}\left\{(2ax+b)^2-(b^2-4ac)\right\}$

[illegible] $+b+\sqrt{b^2-4ac})(2ax+b-\sqrt{b^2-4ac})$

[illegible] c is the most general form of quadratic expressions in x, [illegible] use this equality as a **Formula**, from which particular [illegible] by *substitution*

Example To resolve $3x^2-14x+8$ into factors

Here $a=3$, $b=14$, $c=8$,

$\therefore$ given expression $=\frac{1}{12}(6x-14+\sqrt{196-4\cdot 3\cdot 8})(6x-14-\sqrt{196-4\cdot 3\cdot 8})$

$=\frac{1}{12}(6x-14+10)(6x-14-10)$

$=\frac{1}{12}(6x-4)(6x-24)=(3x-2)(x-4)$.

129 Factors of expressions, reduced to the form ax^2+bx+c The transformed expression evidently becomes a quadratic and therefore the methods of §§ 125—128 will apply

Ex 1. Resolve x^4-13x^2+36 into factors.

Assume $x^2=X$, thus $x^4=(x^2)^2=X^2$, therefore

given expression $=X^2-13X+36=(X-4)(X-9)$

$=(x^2-4)(x^2-9)$, replacing X by x^2,

$=(x+2)(x-2)(x+3)(x-3)$.

Ex. 2 Resolve $8x^4+2x^2-45$ into factors.

Put $x^2=X$, thus $x^4=(x^2)^2=X^2$; therefore

given expression $=8X^2+2X-45=(2X+5)(4X-9)$

$=(2x^2+5)(4x^2-9)=(2x^2+5)(2x+3)(2x-3)$.

* That is, one which contains *only two powers* of some letter or expression, *one* of which is the *square* of the other.

Ex 3 Resolve x^6+5x^3-24 into factors

Assume $x^3=X$, thus $x^6=(x^3)^2=X^2$, therefore

given expression $=X^2+5X-24=(X-3)(X+8)$

$=(x^3-3)(x^3+8)=(x^3-3)(x+2)(x^2-2x+4).$

Ex. 4 Resolve $4x^8+7x^4-36$ into factors

Given expression $=4X^2+7X-36$, when $X=x^4$,

$=(4X-9)(X+4)$

$=(4x^4-9)(x^4+4)$

$=(2x^2+3)(2x^2-3)(x^2+2x+2)(x^2-2x+2)$

[§ 124, Ex 26]

Ex 5 Resolve $(a^2+3a)^2-16(a^2+3a)-36$ into factors

Assume $a^2+3a=X$, thus

given expression $=X^2-16X-36=(X+2)(X-18)$

$=(a^2+3a+2)(a^2+3a-18)$

$=(a+1)(a+2)(a-3)(a+6)$

Ex. 6 Factorise $2(a^2+3a+3)^2+3(a^2+3a+3)-5$

Put $a^2+3a+3=X$, thus

given expression $=2X^2+3X-5=(2X+5)(X-1)$

$=\{2(a^2+3a+3)+5\}\{(a^2+3a+3)-1\}$

$=(2a^2+6a+11)(a^2+3a+2)$

$=(2a^2+6a+11)(a+1)(a+2)$

Ex 7 Factorise $(x^2+7x+4)(x^2+7x+6)-48$

Let $x^2+7x=X$, therefore

given expression $=(X+4)(X+6)-48$

$=X^2+10X+24-48$

$=X^2+10X-24=(X+12)(X-2)$

$=(x^2+7x+12)(x^2+7x-2)$

$=(x+3)(x+4)(x^2+7x-2)$

Ex 8. Factorise

$$2(x^2-ax+a^2)^2+5(x^2-ax+a^2)(x^2+a^2)-3(x^2+a^2)^2$$

Put $x^2-ax+a^2=X$ and $x^2+a^2=Y$, therefore

given expression $=2X^2+5XY-3Y^2=(2X-Y)(X+3Y)$

$=\{2(x^2-ax+a^2)-(x^2+a^2)\}$

$\times\{x^2-ax+a^2)+3(x^2+a^2)\}$

$=(x^2-2ax+a^2)(4x^2-ax+4a^2)$

$=(x-a)^2(4x^2-ax+4a^2)$

Ex. 9 Factorise $x(x+1)(x+2)(x+3)-15$.

Given expression $=x(x+3)\times(x+1)(x+2)-15$

$=(x^2+3x)(x^2+3x+2)-15$

$=(x^2+3x)^2+2(x^2+3x)-15$

$=X^2+2X-15$, where $X=x^2+3x$,

$=(X-3)(X+5)$

$=(x^2+3x-3)(x^2+3x+5)$

Resolve into factors

10. x^4+11x^2-12.
11. $6x^4+5x^2-1$
12. $15x^4-34x^2y^2+15y^4$.
13. $3x^4-11x^2+8$
14. $2x^4+x^2y^2-3y^4$.
15. $8x^6+7x^3-1$.
16. [illegible] $19x^3-216$
17. $3x^6-x^3y^3-2y^6$.
18. [illegible] $3a^4-16$
19. $a^8+3a^4x^4-4x^8$.
20. [illegible] $+3(a^2+2a)+2$
21. $(x^2+5x)^2+10(x^2+5x)+24$.
22. [illegible] $-(x^2+3x)-6$.
23. $(x^2+4x)^2-2(x^2+4x)-15$
24. [illegible] $)^2+4(x^2-6x)-5$
25. $(x^2+2x)^2-11(x^2+2x)+24$
26. [illegible] $3x)^2-2(x^2-3x)-8$
27. $(x^2-9x)^2-2(x^2-9x)-80$
28. $(x^2-10x)^2+13(x^2-10x)-264$
29. $(x^4-2x^2)^2+4(x^2-2)x^2+3$
30. $3(x^2-7x)^2+28(x^2-7x)-96$
31. $(3x^2-2x-10)^2+6(3x^2-2x-10)+8$
32. $(x^2+5x+7)^2-4(x^2+5x+7)+3$.
33. $(2x+3y)^2+3(2x+3y)(3x+2y)+2(3x+2y)^2$.
34. $(x^2+y^2)^2-8(x^4-y^4)-48(x^2-y^2)^2$.
35. $(4a+x)^2-5(4a+x)(a-3x)+6(a-3x)^2$
36. $(2a-3b)^2+2(2a-3b)(a-2b)-24(a-2b)^2$.
37. $(2a+3b)^2-9(4a-b)(2a+3b)-90(4a-b)^2$.
38. $(x^2-8y^2)^2+2xy(x^2-8y^2)-8x^2y^2$.
39. $(a^2+ab)^2-3(ab-b^2)(a^2+ab)-4(ab-b^2)^2$.
40. $(a^2-7ab)^2-3(a^2-7ab)(ab-3b^2)-10(ab-3b^2)^2$.
41. $2(x^2-3x+9)^2+5(x^2-3x+9)(x^2+9)-3(x^2+9)^2$.
42. $6(2x^2+xy-y^2)^2-(2x^2+xy-y^2)(x^2-xy-2y^2)-12(x^2-xy-2y^2)^2$.
43. $(a^2+6a+2)(a^2+6a-4)-27$
44. $\{3(x^2-5x)+4\}\{(x^2-5x)-3\}+18(x^2-5x-1)$
45. $x(x+3)(x+6)(x+9)+56$.
46. $(x+1)(x+4)(x+7)(x+10)-10$

Examples.

Ex. 1 Factorise $1+8x^3+18xy-27y^3$.

Given expn. $=1^3+(2x)^3+(-3y)^3-3\cdot 1\cdot 2x(-3y)$

$=a^3+b^3+c^3-3abc$ [where $a=1$, $b=2x$, $c=-3y$]

$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$

$=(1+2x-3y)(1+4x^2+9y^2+6xy+3y-2x)$.

Or we may proceed directly, thus —Given expression

$=(1+2x)^3-3\cdot 1\cdot 2x(1+2x)-27y^3+18xy$ [§ 103, *Cor.*]

$=\{(1+2x)^3-(3y)^3\}-\{6x(1+2x)-18xy\}$

$=(1+2x-3y)\{(1+2x)^2+3y(1+2x)+9y^2\}-6x(1+2x-3y)$

$=(1+2x-3y)(1+4x^2+9y^2+6xy+3y-2x)$

Ex. 2. If $x=\cdot 43$ and $y=\cdot 57$, find the value of $x^3+y^3+3xy+1$

[illegible] expression $=(x^3+y^3+3xy-1)+2$

$=(x+y-1)(x^2-xy+y^2+x+y+1)+2$;

[illegible] $x+y-1=\cdot 43+\cdot 57-1=1-1=0$;

[illegible] expression $=0\times(x^2-xy+y^2+x+y+1)+2$

$=0+2$ [§ 18, *Note* 1] $=2$.

Ex. 3. Shew that $(y+z)^3+(z+x)^3+(x+y)^3-3(y+z)(z+x)(x+y)$

$=2(x^3+y^3+z^3-3xyz)$

Given expression $=\frac{1}{2}(y+z+z+x+x+y)\{(z-y)^2+(x-z)^2+(y-x)^2\}$

[$\because$ $(z+x)-(x+y)=z-y$, &c.]

$=(x+y+z)\times 2(x^2+y^2+z^2-yz-zx-xy)$

$=2(x^3+y^3+z^3-3xyz)$

Ex. 4. If $x=a+b+c$, shew that $(x+a)^3+(x+b)^3+(x+c)^3$

$-3(x+a)(x+b)(x+c)=4(a^3+b^3+c^3-3abc)$

Given expression $=\frac{1}{2}(3x+a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$

[$\because$ $(x+a)-(x+b)=a-b$, &c.,]

$=2(a+b+c)\times 2(a^2+b^2+c^2-bc-ca-ab)$

[$\because$ $x=a+b+c$]

$=4(a^3+b^3+c^3-3abc)$

Ex. 5 Prove that $(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$

Here $(b-c)+(c-a)+(a-b)=0$ [§ 116],

$\therefore$ $(b-c)^3+(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b)=0$ [Cor.],

whence $(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$.

Ex 6 If $a+b+c=0$, shew that

$$(2a-b)^3+(2b-c)^3+(2c-a)^3=3(2a-b)(2b-c)(2c-a).$$

Here $(2a-b)+(2b-c)+(2c-a)=a+b+c=0$ by hypothesis,

$\therefore$ $(2a-b)^3+(2b-c)^3+(2c-a)^3-3(2a-b)(2b-c)(2c-a)=0$ [Cor],

whence $(2a-b)^3+(2b-c)^3+(2c-a)^3=3(2a-b)(2b-c)(2c-a)$

7. Factorise $a^3-3ab+b^3+1$ 8 Factorise $a^3+8b^3+6ab-1$

9. Factorise $x^3+3axy+y^3-a^3$

10. Factorise $(y-z)^3+(z-x)^3+(x-y)^3$

11. Factorise $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$

12. Factorise $(2x-y)^3-(x+y)^3+(2y-x)^3$

13 If $x=b+c$, $y=c+a$, $z=a+b$, shew that

$$x^3+y^3+z^3-3xyz=2(a^3+b^3+c^3-3abc)$$

14 If $a=y+z-x$, $b=z+x-y$, $c=x+y-z$, then

$$a^3+b^3+c^3-3abc=4(x^3+y^3+z^3-3xyz).$$

15 Prove that $(x-2y)^3+(2y-1)^3+(1-x)^3=3(x-2y)(2y$ [illegible]

16 Prove that

$(ax-by)^3+(by-cz)^3+(cz-ax)^3=3(ax-by)(by-cz)(cz-$ [illegible]

17 Prove that $(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3$

$$=3(b+c-2a)(c+a-2b)(a+b-2c).$$

18 Find the value of $a^3+b^3+3abc-c^3$, when $a=02$, $b=08$ and $c=10$

19 If $x=\frac{41}{75}$, $y=\frac{34}{75}$, find the value of $x^3+y^3+3xy-1$

20 If $x=(b-c)(a-d)$, $y=(c-a)(b-d)$, $z=(a-b)(c-d)$, find the value of $x^3+y^3+z^3-3xyz$ [See *App*]

21 If $s=a+b+c$, shew that $(3a-s)^3+(3b-s)^3+(3c-s)^3$

$$=3(3a-s)(3b-s)(3c-s)$$

22 If $2s=a+b+c$, shew that

$$(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)=\tfrac{1}{2}(a^3+b^3+c^3-3abc)$$

23. Shew that

$$a^3(bz-cy)^3+b^3(cx-az)^3+c^3(ay-bx)^3=3abc(bz-cy)(cx-az)(ay-bx)$$

24 Shew that $(3a-b-c)^3+(3b-c-a)^3+(3c-a-b)^3$

$$-3(3a-b-c)(3b-c-a)(3c-a-b)=16(a^3+b^3+c^3-3abc).$$

25 Shew that the value of $x^2+y^2+z^2-yz-zx-xy$ will not change if x, y, z be *increased* or *decreased by a constant quantity*, e g, if $x=x+d$, $y=y+d$, $z=z+d$, or if $x=x-d$, $y=y-d$, $z=z-d$.

131 Miscellaneous Examples The methods employed in the following examples should be carefully studied.

Examples (i).

The factors of an expression can be found by a *suitable rearrangement and grouping of the terms* [§ 121, Ex 24]

Ex 1 Resolve $4x^3-3y^3-xy(4y-3x)$ into factors

Given expression $=4x^3-3y^3-4xy^2+3x^2y$, and rearranging and grouping its terms, we have

(i) $(4x^3-4xy^2)+(3x^2y-3y^3)$,

(ii) $(4x^3+3x^2y)-(4xy^2+3y^3)$

Take (i), thus the given expression

$$=4x(x^2-y^2)+3y(x^2-y^2)$$
$$=(x^2-y^2)(4x+3y)=(x+y)(x-y)(4x+3y)$$

[illegible]y by taking (ii), the same factors can be found

[illegible] into factors

[illegible]$+bx+ac$ 3 $a^3+a^2b+ab^2+b^3$.

4 $x(6x^2-8y^2)-y(3x^2-4y^2)$ 5 $xy(y^2-2x^2)+x^4-2y^4$

6. $a(a+b)-c(b+c)$ 7 $acx^3+bcx^2+adx+bd$

8. $a^3-a^2c-ab^2+b^2c$ 9 $x^4-a^3x+bx^3-a^3b$

10 $abx^2+(a+b)x+1$ 11. $ab(x^2+1)+(a^2+b^2)x$

12. $ab(x^2+y^2)+(a^2+b^2)xy$ 13. $ay(x^3+b^3)+bx(by^2+a^2x)$.

14 $x^2(x+2y)-y^2(2x+y)$ 15 $x^3(x+2y)-y^3(2x+y)$

16 $a^3(x^6-1)-x^3(a^6-1)$ 17 $(ax+by)^2+(bx-ay)^2$

18. $3x^2-6ax+15xy-x-5y+2a$ [§ 128, Ex 38]

Examples (ii)

The factors of an expression, in which *one of the letters* occurs only in the *first power*, are generally obvious when the expression is rearranged according to the powers of that letter

Ex 1. Resolve $1+(b-a^2)x^2-abx^3$ into factors

Here b occurs *only* in the *first* power Arrange, therefore in powers of b, thus we have

$$(1-a^2x^2)+b(x^2-ax^3)$$
$$=(1+ax)(1-ax)+bx^2(1-ax)$$
$$=(1-ax)(1+ax+bx^2),$$

Ex 2. Resolve $(a^2-b^2)x^2-(2a^2-b^2)bx+a^2b^2$ into factors

Putting $a^2=m$, we see that m occurs only in the first power. Arranging according to powers of m, i e, a^2, we have

$$a^2(x^2-2bx+b^2)-(b^2x^2-b^3x)$$
$$=a^2(x-b)^2-b^2x(x-b)$$
$$=(x-b)\{a^2(x-b)-b^2x\}$$
$$=(x-b)(a^2x-b^2x-a^2b)$$

Ex. 3. Resolve $x^2-axy-2ax+a^2y+a^2$ into factors.

Arrange according to powers of y, thus the given expression

$$=y(a^2-ax)+(x^2-2ax+a^2)$$
$$=ay(a-x)+(a-x)^2=(a-x)(ay+a-x).$$

Ex. 4 Factorise $a^2-2ab-4ac+2bc+3c^2$ [§ 128, Ex 28]

Arrange in powers of b, thus the given expression

$$=(a^2-4ac+3c^2)-2b(a-c)$$
$$=(a-c)(a-3c)-2b(a-c)$$
$$=(a-c)(a-2b-3c)$$

Ex. 5 Factorise $x^4-(3a^2+b^2)x^2+(2a^2-b^2)ax+2a^2b^2$.

Here as in Ex 2, we must arrange in powers of b^2; thus the given expression

$$=x^4-3a^2x^2+2a^3x-b^2(x^2+ax-2a^2)$$
$$=x(x^3-3a^2x+2a^3)-b^2(x^2+ax-2a^2)$$
$$=x\{(x^3-a^3)-3a^2(x-a)\}-b^2(x^2+ax-2a^2)$$
$$=x(x-a)(x^2+ax-2a^2)-b^2(x^2+ax-2a^2)$$
$$=(x^2+ax-2a^2)(x^2-ax-b^2)$$
$$=(x-a)(x+2a)(x^2-ax-b^2)$$

Resolve into factors

6. $ax+bc+bx+ac$ [Ex (1) 1] 7. $x^2-(a+b+c)x+ab+ac$

8 $a^3-a^2c-ab^2+b^2c$ [Ex (1) 8] 9 $x^4-a^3x+bx^3-a^3b$ [Ex (1) 9]

10 $acx^3+bcx^2+adx+bd$ [Ex (1) 7]

11 $x^2+2ax-2ab-b^2$ 12 $x^3+(a-b)x^2+(1-ab)x+a$.

13. $x^3-2(a+c)x^2+(3a+4c)ax-6a^2c$

14 $1-2ax-(c-a^2)x^2+acx^3$

15 $15xy+11x+3x^2-5y-4$ [§ 128, Ex. 35]

16. $x^2+2xz-6xy-2yz+5y^2$ 17. $a^2+ab-3ac-2b^2+3bc$.

18. $a^2-13ab-20bc+36b^2+5ac$

Resolve into factors

19. $x^3-(2a+b)x^2+(2ab+a^2)x-a^2b$

20. $x^3+3mx^2+(3m^2-n^2)x+m(m^2-n^2)$

21. $x^4+(p-a)x^3-(ap+q+1)x^2-(p-aq)x+q$.

22. $a^3-a^2(5b^2+c^2)+ab^2(4b^2+5c^2)-4b^4c^2$.

23. $x^4-px^3-(2p^2+q^2)x^2+pq^2(x+2p)$.

Examples (III)

We may rearrange and group the terms of an expression, which is of the *second degree in any one* of the letters, by writing it according to the powers of that letter

Ex 1. Factorise $a^2-2ab-4ac+2bc+3c^2$. [Ex (II), 4]

[illegible] this as a quadratic in a, and arrange accordingly; thus t[illegible]pression

$$=a^2-2(b+2c)a+c(2b+3c)$$
$$=a^2-\{c+(2b+3c)\}a+c(2b+3c)$$
$$=(a-c)(a-2b-3c) \quad [\S\ 125]$$

[Otherwise proceed as in § 128 by completing the square.]

Ex 2. Factorise $x^2-2(a+1)x-3a(a-2)$

Given expression $=x^2-(2a+2)x-3a(a-2)$

$$=x^2-\{2a-(a-2)\}x-3a(a-2)$$
$$=(x-3a)(x+a-2).$$

[Try this example by the method of § 128.]

Ex 3. Factorise $x^2-2(a+b)x-ab(a-2)(b+2)$

Given expression $=x^2-(2a+2b)x-(ab-2b)(ab+2a)$

$$=x^2-\{(ab+2a)-(ab-2b)\}x-(ab+2a)(ab-2b)$$
$$=\{x-(ab+2a)\}\{x+(ab-2b)\} \quad [\S\ 125]$$
$$=(x-2a-ab)(x-2b+ab)$$

[Try by the method of § 128.]

Ex 4. Factorise $a^2+3b^2-4c^2-4ab+4bc$

Arrange according to powers of a, thus given expression

$$=a^2-4ab+(3b^2+4bc-4c^2)$$
$$=a^2-4ab+(3b-2c)(b+2c)$$
$$=a^2-\{(3b-2c)+(b+2c)\}a+(3b-2c)(b+2c)$$
$$=\{a-(3b-2c)\}\{a-(b+2c)\}$$
$$=(a-3b+2c)(a-b-2c)$$

Resolve into factors

5. $x^2-(a+b+c)x+ab+ac$ [Ex (ii) 7]

6. $a(a+b)-c(b+c)$ [Ex (i) 6]   7. $x^2+2ax-2ab-b^2$ [Ex (ii) 11]

8. $x^2+xy-6y^2-x+2y$ 9. $a^2+ab-3ac-2b^2+3bc$ [Ex (ii) 17]

10. $x^2-y^2-x+3y-2$ 11. $a^2+3a(2b-1)+2(6b-5)$

12. $a^2+2a(b+c)-3b(b-2c)$ 13. $x^2+2(a+1)x-3a(5a+2)$

14. $x^2+(a-b)x-ab(a+1)(b+1)$ 15. $x^2-5ax+6(a+2b)(a-3b)$

16. $x^2-(a^2+b^2)x-ab(a^2-b^2)$

17. $x^2-4xy+3y^2+4x-10y+3$ [§ 128, Ex 30]

18. $3x^2+(5b-11a)x+3a(2a-5b)$

Examples (iv)

Ex 1 Resolve x^3-3x^2-6x+8 into factors

The given expression, when rearranged, may be following form —

(i) $(x^3+8)-(3x^2+6x)$,

(ii) $x(x^2-3x+2)-8(x-1)$,

(iii) $(x^3-4x^2)+(x^2-4x)-(2x-8)$

Hence

(i) Given expn $=(x+2)(x^2-2x+4)-3x(x+2)$

$=(x+2)(x^2-5x+4)=(x+2)(x-1)(x-4)$

(ii) Given expn $=x(x-1)(x-2)-8(x-1)$

$=(x-1)(x^2-2x-8)=(x-1)(x+2)(x-4)$

(iii) Given expn $=x^2(x-4)+x(x-4)-2(x-4)$

$=(x-4)(x^2+x-2)=(x-4)(x-1)(x+2)$

[Observe that the *three* factors of 8 *must* be 1, 2 and 4, and since it has a + sign, *two* of these factors must have the *same sign*, and consequently the third, the sign + Hence if the proposed expression has 3 linear factors, they must be of the form $x\pm1$, $x\pm2$ and $x\pm4$]

Ex 2 Resolve x^3-5x^2+2x+8 into factors

The given expression, when rearranged, becomes

(i) $(x^3+1)-(5x^2-2x-7)$,

(ii) $x(x^2-5x+6)-4(x-2)$,

(iii) $(x^3-4x^2)-(x^2-4x)-(2x-8)$

Therefore

(i) given expn $=(x+1)(x^2-x+1)-(x+1)(5x-7)$

$=(x+1)(x^2-6x+8)=(x+1)(x-2)(x-4)$.

(ii) given expn $= x(x-2)(x-3) - 4(x-2)$

$$=(x-2)(x^2-3x-4)=(x-2)(x+1)(x-4)$$

[Take (iii) and find the factors]

Ex. 3 Resolve $x^3-7x^2-14x+48$ into factors.

By rearranging the terms, the expression is put in the following three forms :—

(i) $(x^3+27)-(7x^2+14x-21)$;

(ii) $x(x^2-7x-8)-6(x-8)$;

(iii) $(x^3-8x^2)+(x^2-8x)-(6x-48)$.

(i) given expn $=(x^3+27)-7(x^2+2x-3)$

$$=(x+3)(x^2-3x+9)-7(x+3)(x-1)$$

$$=(x+3)(x^2-10x+16)=(x+3)(x-2)(x-8).$$

(ii) given expn. $=x(x-8)(x+1)-6(x-8)$

$$=(x-8)(x^2+x-6)=(x-8)(x-2)(x+3)$$

[Find the factors taking the third form]

Ex. 4. Resolve $2x^3-13x^2+27x-18$ into factors.

We may rearrange the expression in three ways, thus :—

(i) $x^2(2x-3)-(10x^2-27x+18)$;

(ii) $x(2x^2-13x+15)+6(2x-3)$;

(iii) $(2x^3-6x^2)-(7x^2-21x)+(6x-18)$.

Taking (i), the given expression

$$=x^2(2x-3)-(2x-3)(5x-6)$$

$$=(2x-3)(x^2-5x+6)=(2x-3)(x-2)(x-3).$$

Taking (ii), the given expression

$$=x(2x-3)(x-5)+6(2x-3)$$

$$=(2x-3)(x^2-5x+6)=(2x-3)(x-2)(x-3).$$

[Find the factors taking (iii)]

Ex. 5. Resolve $6x^3+17x^2-5x-6$ into factors

We may group the terms in the following ways :—

(i) $3x^2(2x+1)+(14x^2-5x-6)$;

(ii) $x(6x^2+17x+7)-6(2x+1)$;

(iii) $(6x^3+18x^2)-(x^2+3x)-(2x+6)$

Take (i) ; thus the given expression

$$=3x^2(2x+1)+(2x+1)(7x-6)$$

$$=(2x+1)(3x^2+7x-6)=(2x+1)(x+3)(3x-2)$$

Take (ii), thus given expression

$$=x(2x+1)(3x+7)-6(2x+1)$$
$$=(2x+1)(3x^2+7x-6)=(2x+1)(x+3)(3x-2).$$

[Take (iii), and find the factors]

Ex 6. Resolve $a^3-19ab^2+30b^3$ into factors

Rearranging and grouping the terms, we have

(i) $(a^3-8b^3)-(19ab^2-38b^3)$;

(ii) $a(a^2-9b^2)-10b^2(a-3b)$,

(iii) $(a^3+5a^2b)-(5a^2b+25ab^2)+(6ab^2+30b^3)$

[Here we introduce a 'false' term]

Take (i), thus the given expression

$$=(a-2b)(a^2+2ab+4b^2)-19b^2(a-2b)$$
$$=(a-2b)(a^2+2ab-15b^2)$$
$$=(a-2b)(a-3b)(a+5b)$$

[Taking (ii) and (iii), find the factors]

Ex 7 Resolve $10x^3+19x^2-9$ into factors

Rearrange and group the terms, thus we have

(i) $5x^2(2x+3)+(4x^2-9)$,

(ii) $10(x^3+1)+19(x^2-1)$,

(iii) $(10x^3+10x^2)+(9x^2+9x)-(9x+9)$

(i) gives $5x^2(2x+3)+(2x+3)(2x-3)$

$$=(2x+3)(5x^2+2x-3)=(2x+3)(x+1)(5x-3)$$

[Take (ii) and (iii) and find the required factors]

Ex. 8 Resolve x^4-4x+3 into factors.

The terms may be grouped in the following ways—

(i) $(x^4-1)-4(x-1)$,

(ii) $x(x^3-1)-3(x-1)$,

Taking (i), we have

$$(x-1)(x^3+x^2+x+1)-4(x-1)$$
$$=(x-1)(x^3+x^2+x-3)$$
$$=(x-1)\{(x^3-1)+(x^2-1)+(x-1)\}$$
$$=(x-1)(x-1)(x^2+x+1+x+1+1)$$
$$=(x-1)^2(x^2+2x+3)$$

[Take (ii) and find the factors of the given expression]

Ex. 9. Resolve $3a^4+5a^3b-8b^4$ into factors.

Arrange the expression as follows—

(i) $3a^3(a-b)+8b(a^3-b^3)$;

(ii) $3(a^4-16b^4)+5b(a^3+8b^3)$.

Take (i); thus we have

$$3a^3(a-b)+8b(a-b)(a^2+ab+b^2)$$
$$=(a-b)\{3a^3+8b(a^2+ab+b^2)\}$$
$$=(a-b)(3a^3+8a^2b+8ab^2+8b^3).$$

The second factor $=(a^3+8b^3)+2a(a^2+4ab+4b^2)$

$$=(a+2b)(a^2-2ab+4b^2)+2a(a+2b)^2$$
$$=(a+2b)(3a^2+2ab+4b^2).$$

Hence given expression $=(a-b)(a+2b)(3a^2+2ab+4b^2)$

[Find the factors by taking (ii)]

Resolve $x^4+11x^3+41x^2+61x+30$ into factors.

$$=x^2(x^2+11x+30)+(11x^2+61x+30)$$
$$=x^2(x+5)(x+6)+(x+5)(11x+6)$$
$$=(x+5)(x^3+6x^2+11x+6)$$

The second factor $=(x^3+8)+(6x^2+11x-2)$

$$=(x+2)(x^2-2x+4)+(x+2)(6x-1)$$
$$=(x+2)(x^2+4x+3)=(x+2)(x+1)(x+3)$$

∴ Given expn $=(x+5)(x+2)(x+1)(x+3)$.

Note. If we put $x=1$, or $x=-2$ or $x=4$ in x^3-3x^2-6x+8, the expression vanishes, i.e., becomes 0, and we see from Ex 1, that $x-1$, $x+2$ and $x-4$ are factors of the expression. Again from Ex 2, we see that, when $x=-1$, or $x=2$, or $x=4$, the expression x^3-5x^2+2x+8 vanishes; also it has for its factors $x+1$, $x-2$ and $x-4$. And so on. Hence we learn that *if an expression in x vanishes, when $x=a$, $x-a$ is a factor of that expression* [see § 277 *post*].

Ex 11. Resolve $12x^4-49x^3-62x^2+29x+30$ into factors.

Now 5 is a factor of 30 (see Remark, Ex 1), and by trial we find that the expression vanishes, when $x=5$, hence $x-5$ is a factor of the proposed expression, which is therefore

$$=12x^3(x-5)+11x^2(x-5)-7x(x-5)-6(x-5)$$
$$=(x-5)(12x^3+11x^2-7x-6).$$

Again the second factor vanishes, when $x=-1$; thus $x+1$ is one of its factors. Hence it

$$=12x^2(x+1)-x(x+1)-6(x+1)$$
$$=(x+1)(12x^2-x-6)=(x+1)(3x+2)(4x-3).$$

∴ Given expression $=(x-5)(x+1)(3x+2)(4x-3)$.

Note It may be remarked that if *one* factor of the *first* degree of a cubic, *two* of a biquadratic, &c , be found by trial, the remaining factor in each case must be a *quadratic*, which we resolve as in § 128

Resolve into factors

12 $x^3+7x^2+14x+8$ 13 $x^3+10x^2+29x+20$

14. $x^3+9x^2+6x-16$ 15 $x^3+x^2-17x+15$

16 $x^3+4x^2+11x+8$ 17 $x^3-3x^2-10x+24$

18 $x^3+4x^2-17x-60$ 19 $2x^3-4x^2-8x+16$

20 a^3-3a+2 21 x^3-3x^2+4

22 $x^3-31x-30$. 23 $4a^3+ab^2-b^3$

24 $4x^3+13x^2-9$ 25 $8x^3-16x^2-9$

26 $a^3-a^2b-18b^3$ 27 $a^3+9a^2b-8b^3$

28 $x^3-28xy^2+48y^3$ 29 $a^3-2a^2b-9b^3$

30 $6a^3-7a^2b+b^3$. 31 $25x^3-19x+6$.

[*The following Examples may be omitted on a first r*

32. $x^4+10x^3+35x^2+50x+24$ 33 $x^4+2x^3-13x^2-14x+24$

34. $x^4+5x^3+11x^2+13x+6$ 35 $x^4+3x^3-24x^2-28x+48$

36 $x^4-7x^3+10x^2+11x-15$. 37 $x^4+2x^3-17x^2+36x-36$

38 $x^4-13x-42$ 39 $x^4+40x-96$

40 $12x^4+x^2-1$ 41 $x^4+20x-21$

42 x^4-5x^3+54. 43 $8x^4+5x-3$

44 x^4-14x^2+1 45 $3x^4-5x^3-8$

46 $7x^4+20x^3-27$ 47 $5x^3+7x^2-x-3$

48 *Omitted* 49 $x^4-25x^2-4x+20$

50 $3x^3-11x^2y+xy^2+15y^3$ 51 $8x^3-2x^2y-5xy^3-y^3$.

52 $x^4-2x^3y+3x^2y^2-2xy^3+y^4$ 53 $a^4-2a^3b+2a^2b^2-2ab^3+b^4$

54 $24x^3-22x^2+17x-5$ 55 $2x^4+7x^3-14x^2-46x+15$

***132 Some Useful Results** The following Examples are very useful, and the results should be committed to memory as **Formulæ.**

Ex 1 Resolve $a^2(b-c)+b^2(c-a)+c^2(a-b)$ into factors

We have already shewn how to resolve this expression into factors [§ 117] We shall here give another method to resolve it, which will be found useful

Since $(b-c)+(c-a)=b-a=-(a-b)$, we have

$$\begin{aligned}
&a^2(b-c)+b^2(c-a)-c^2\{(b-c)+(c-a)\}\\
&=a^2(b-c)-c^2(b-c)+b^2(c-a)-c^2(c-a)\\
&=(b-c)(a^2-c^2)+(b^2-c^2)(c-a)\\
&=(b^2-c^2)(c-a)-(b-c)(c^2-a^2)\\
&=(b-c)(c-a)\{(b+c)-(c+a)\}\\
&=(b-c)(c-a)(b-a)=-(b-c)(c-a)(a-b).
\end{aligned}$$

Note We have seen [§ 117] that $-(b-c)(c-a)(a-b)$ is equal to three other expressions

Ex. 2. Resolve $a^3(b-c)+b^3(c-a)+c^3(a-b)$ into factors

As in the last example, we have

$$\begin{aligned}
&a^3(b-c)+b^3(c-a)-c^3\{(b-c)+(c-a)\}\\
&=a^3(b-c)-c^3(b-c)+b^3(c-a)-c^3(c-a)\\
&=(b-c)(a^3-c^3)+(b^3-c^3)(c-a)\\
&=(b-c)(c-a)\{(b^2+bc+c^2)-(c^2+ca+a^2)\}\\
&=(b-c)(c-a)\{(b^2-a^2)+c(b-a)\}\\
&=(b-c)(c-a)(b-a)(b+a+c)\\
&=-(b-c)(c-a)(a-b)(a+b+c).
\end{aligned}$$

Otherwise thus :—Arrange the given expression according to powers of a, which is thus

$$\begin{aligned}
&=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)\\
&=(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}
\end{aligned}$$

Arrange the second factor according to powers of b ; thus

the given expression $=(b-c)\{b^2(c-a)+bc(c-a)-a(c^2-a^2)\}$

$$=(b-c)(c-a)\{b^2+bc-a(c+a)\}$$

Arrange the third factor according to powers of c; thus the given expression $=(b-c)(c-a)\{c(b-a)+(b^2-a^2)\}$

$$\begin{aligned}
&=(b-c)(c-a)(b-a)(c+b+a)\\
&=-(b-c)(c-a)(a-b)(a+b+c).
\end{aligned}$$

Note. It is easy to see that

$$\begin{aligned}
&a^3b-ab^3+b^3c-bc^3+c^3a-ca^3\\
&=a^3(b-c)+b^3(c-a)+c^3(a-b)\\
&=bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)\\
&=-\{a(b^2-c^2)+b(c^3-a^3)+c(a^3-b^3)\},
\end{aligned}$$

hence each $=-(b-c)(c-a)(a-b)(a+b+c)$ ....(1)

Ex 3 Resolve $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$ into factors

We may easily put the given expression in the form

$$b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$$
$$=b^2c^2(b-c)+c^2a^2(c-a)-a^2b^2\{(b-c)+(c-a)\} \text{ [Ex 1]}$$
$$=(b-c)(b^2c^2-a^2b^2)+(c^2a^2-a^2b^2)(c-a)$$
$$=b^2(b-c)(c^2-a^2)-a^2(b^2-c^2)(c-a)$$
$$=(b-c)(c-a)\{b^2(c+a)-a^2(b+c)\}$$
$$=(b-c)(c-a)\{c(b^2-a^2)+ab(b-a)\}$$
$$=(b-c)(c-a)(b-a)\{c(b+a)+ab\}$$
$$=-(b-c)(c-a)(a-b)(bc+ca+ab)$$

As an exercise the student is asked to work this example by arranging the given expression in powers of a, b and c successively as we have done in Ex 2

Note It is easy to see that

$$a^3b^2-a^2b^3+b^3c^2-b^2c^3+c^3a^2-c^2a^3$$
$$=a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$$
$$=b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$$
$$=-\{a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)\},$$

therefore each $=-(b-c)(c-a)(a-b)(bc+ca+ab)$ (iii)

Ex 4 Resolve $(b-c)^3+(c-a)^3+(a-b)^3$ into factors [§ 130, Ex 5]

Given expn $=(b-c)^3+(c-a)^3-\{(b-c)+(c-a)\}^3$ [see Ex 1]

$$=(b-c)^3+(c-a)^3$$
$$-\{(b-c)^3+(c-a)^3+3(b-c)(c-a)(b-a)\} \text{ [§ 103]}$$
$$=-3(b-c)(c-a)(b-a)=3(b-c)(c-a)(a-b)$$

Hence $(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$ ... (iii)

Ex. 5. Resolve $(b-c)^5+(c-a)^5+(a-b)^5$ into factors [*Punj.* 893]

Given expn $=(b-c)^5+(c-a)^5-\{(b-c)+(c-a)\}^5$ [see Ex 1]

$$=(b-c)^5+(c-a)^5-\{(b-c)^5+5(b-c)^4(c-a)+10(b-c)^3(c-a)^2$$
$$+10(b-c)^2(c-a)^3+5(b-c)(c-a)^4+(c-a)^5\}$$
$$[\quad (x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5]$$
$$=-5(b-c)^4(c-a)-5(b-c)(c-a)^4$$
$$-10(b-c)^3(c-a)^2-10(b-c)^2(c-a)^3$$
$$=-5(b-c)(c-a)\{(b-c)^3+(c-a)^3\}$$
$$-10(b-c)^2(c-a)^2\{(b-c)+(c-a)\}$$
$$=-5(b-c)(c-a)\{(b-a)^3-3(b-c)(c-a)(b-a)\} \text{ [§ 103, Cor 2]}$$
$$-10(b-c)^2(c-a)^2(b-a)$$
$$=-5(b-c)(c-a)(b-a)\{(b-a)^2-3(b-c)(c-a)\}$$
$$-10(b-c)^2(c-a)^2(b-a)$$
$$=-5(b-c)(c-a)(b-a)\{(b-a)^2-(b-c)(c-a)\}$$
$$=5(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)$$

√ Ex. 6. Prove that $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$
$$=(b+c)(c+a)(a+b).$$

[Note It is easy to see that
$$a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$$
$$=a^2(b+c)+b^2(c+a)+c^2(a+b)$$
$$=bc(b+c)+ca(c+a)+ab(a+b)$$
$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2).$$
We shall denote, for shortness, each of these equal expressions by S]

Arrange according to powers of a: thus left side
$$=a^2(b+c)+a(b^2+2bc+c^2)+b^2c+bc^2$$
$$=a^2(b+c)+a(b+c)^2+bc(b+c)$$
$$=(b+c)\{a^2+a(b+c)+bc\}$$
$$=(b+c)(a+c)(a+b)\ [\S\ 107]$$
Hence $S+2abc=(b+c)(c+a)(a+b)$ ....(iii).

Ex. 7. Prove that $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$
$$=(a+b+c)(bc+ca+ab).$$
Left side $=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc$
$$=(b^2c+bc^2+abc)+(c^2a+ca^2+abc)+(a^2b+ab^2+abc)$$
[rearranging and grouping the terms]
$$=bc(b+c+a)+ca(c+a+b)+ab(a+b+c)$$
$$=(a+b+c)(bc+ca+ab)$$
Hence $S+3abc=(a+b+c)(bc+ca+ab)$ ....(iv).

Another form of this identity is
$$(a+b+c)(bc+ca+ab)-abc=(b+c)(c+a)(a+b)$$
For left side $=S+3abc-abc=S+2abc=(b+c)(c+a)(a+b)$ [Ex. 1].

√ Ex. 8 Prove that $(a+b+c)^3=a^3+b^3+c^3+3(b+c)(c+a)(a+b)$ (v)
$$(a+b+c)^3=a^3+b^3+c^3+3a^2b+3ab^2+3b^2c+3bc^2+3c^2a+3ca^2+6abc$$
$$=a^3+b^3+c^3+3S+6abc=a^3+b^3+c^3+3(S+2abc)$$
$$=a^3+b^3+c^3+3(b+c)(c+a)(a+b)\ [\text{Ex } 1].$$
Another form of this identity evidently is
$$(a+b+c)^3-a^3-b^3-c^3=3(b+c)(c+a)(a+b).$$

Example 1 Prove that $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3$
$$=3(2a+b+c)(a+2b+c)(a+b+2c)$$

Since $8(a+b+c)^3=(2a+2b+2c)^3=\{(b+c)+(c+a)+(a+b)\}^3$,
we have (by putting $x=b+c$, $y=c+a$, $z=a+b$) left side

$$=(x+y+z)^3-x^3-y^3-z^3$$
$$=3(y+z)(z+x)(x+y)$$
$$=3(2a+b+c)(a+2b+c)(a+b+2c)$$

Example 2 Resolve into factors

$$(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3.$$

Put $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, thus

$$x+y+z=(b+c-a)+(c+a-b)+(a+b-c)=a+b+c$$

Hence the given expn $=(x+y+z)^3-x^3-y^3-z^3$
$$=3(y+z)(z+x)(x+y)$$
$$=3(2a)(2b)(2c)=24abc$$

Ex 9 Prove that $a^2(b+c)+b^2(c+a)+c^2(a+b)+a^3+b^3$ [illegible]
$$=(a+b+c)\text{[illegible]}$$

Rearranging the terms, the left side

$$=a^3+a^2(b+c)+b^3+b^2(c+a)+c^3+c^2(\text{[illegible]}$$
$$=a^2(a+b+c)+b^2(b+c+a)+c^2(c+a+b)$$
$$=(a+b+c)(a^2+b^2+c^2)$$

Hence $$S+a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2). \quad \ldots\ldots \quad \text{(vi)}$$

Ex 10 Prove that $a^2(b+c)+b^2(c+a)+c^2(a+b)$
$$-a^3-b^3-c^3-2abc=(b+c-a)(c+a-b)(a+b-c).$$

Rearranging the terms, we have

Left side $=a^2(b+c)+a(b^2-2bc+c^2)+b^2c+bc^2-a^3-b^3-c^3$
$$=a^2(b+c)-a^3+a(b^2-2bc+c^2)+bc(b+c)-(b^3+c^3)$$
$$=a^2(b+c-a)+a(b-c)^2+bc(b+c)-(b+c)(b^2-bc+c^2)$$
$$=a^2(b+c-a)+a(b-c)^2-(b+c)(b^2-2bc+c^2)$$
$$=a^2(b+c-a)+a(b-c)^2-(b+c)(b-c)^2$$
$$=a^2(b+c-a)-(b-c)^2(b+c-a)$$
$$=(b+c-a)\{a^2-(b-c)^2\}$$
$$=(b+c-a)(a-b+c)(a+b-c)$$

Hence $$S-a^3-b^3-c^3-2abc=(b+c-a)(c+a-b)(a+b-c).. \quad \text{(vii)}$$

Ex 11. Prove that $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c). \quad .\text{(viii)}.$$

Arrange according to powers of a, thus

Left side $= -\{a^4 - 2(b^2+c^2)a^2 + b^4 + c^4 - 2b^2c^2\}$

$= -\{a^4 - 2(b^2+c^2)a^2 + (b^2+c^2)^2 - (b^2+c^2)^2 + (b^2-c^2)^2\}$

$= -\{(a^2 - \overline{b^2+c^2})^2 - 4b^2c^2\} = (2bc)^2 - (a^2-b^2-c^2)^2$

$= \{2bc - (a^2-b^2-c^2)\}\{(2bc + (a^2-b^2-c^2)\}$

$= \{(b+c)^2 - a^2\}\{a^2 - (b-c)^2\}$

$= (b+c+a)(b+c-a)(a+b-c)(a-b+c)$

$= (a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

Resolve into factors

12 $bc(b-c)+ca(c-a)+ab(a-b)$.

13 $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

14 $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$.

15. [illegible] $-c^3)+b(c^3-a^3)+c(a^3-b^3)$

16. [illegible] $c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)$

17. [illegible] $c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$

18. [illegible] $-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2)$

19. $(a+b+c)^3-a^3-b^3-c^3$ 20 $(b+c)(c+a)(a+b)+abc$

21 $a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc$

22. $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc$

23 $a(b-c)^3+b(c-a)^3+c(a-b)^3$.

24. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.

25. $bc(b^3-c^3)+ca(c^3-a^3)+ab(a^3-b^3)$.

26 $a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4)$.

27. $a(b-c)(bc-a^2)+b(c-a)(ca-b^2)+c(a-b)(ab-c^2)$.

28. $bc(b-c)(1-a^2)+ca(c-a)(1-b^2)+ab(a-b)(1-c^2)$.

29. $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$.

30. $(b+c)(b-c)^3+(c+a)(c-a)^3+(a+b)(a-b)^3$.

***133 Division by Resolution into Factors.** The following examples will sufficiently illustrate the method.

Examples.

Ex 1 Divide a^3-3a+2 by $a+2$.

Dividend $= a^3+8-3a-6 = (a^3+8)-(3a+6)$

$= (a+2)(a^2-2a+4)-3(a+2)$ [§ 105]

$= (a+2)(a^2-2a+4-3) = (a+2)(a-1)^2$;

$\therefore$ required quotient $= (a-1)^2$ or a^2-2a+1.

Ex 2 Divide $a^3+ab^2-a^2b-b^3$ by $a-b$

Dividend $=a(a^2+b^2)-b(a^2+b^2)=(a-b)(a^2+b^2)$,

∴ required quotient $=a^2+b^2$.

Ex. 3 Divide $8x^3+y^6$ by $2x+y^2$ [Ex 63 § 95].

$$\text{Dividend} =(2x)^3+(y^2)^3$$
$$=(2x+y^2)\{(2x)^2-(2x)y^2+(y^2)^2\} \text{ [§ 105]}$$
$$=(2x+y^2)(4x^2-2xy^2+y^4),$$

∴ required quotient $=4x^2-2xy^2+y^4$

Ex 4 Divide $x^3-3ax^2+3a^2x-a^3+b^3$ by $x-a+b$ [Ex 48, § 95]

$$\text{Dividend} =(x-a)^3+b^3 \text{ [§ 104]}$$
$$=(x-a+b)\{(x-a)^2-(x-a)b+b^2\}$$
$$=(x-a+b)\{x^2-(2a+b)x+a^2+ab+b^2\},$$

∴ required quotient $=x^2-(2a+b)x+a^2+ab+b$

Ex 5 Divide $(x^2-yz)^3+8y^3z^3$ by x^2+yz

Dividend $=(x^2-yz)^3+(2yz)^3$ [which is of the form

$$=\{(x^2-yz)+2yz\}\{(x^2-yz)^2-(x^2-yz)2yz$$
$$=(x^2-yz+2yz)(x^4-2x^2yz+y^2z^2-2x^2yz+2y^2z^2+4y^2z^2)$$
$$=(x^2+yz)(x^4-4x^2yz+7y^2z^2),$$

∴ required quotient $=x^4-4x^2yz+7y^2z^2$

Ex 6 Divide $x^5-1-5(x-1)$ by $(x-1)^2$ [*Punj*, 1893]

$$\text{Dividend} =(x-1)(x^4+x^3+x^2+x+1)-5(x-1)$$
$$=(x-1)(x^4+x^3+x^2+x-4)$$
$$=(x-1)\{(x^4-1)+(x^3-1)+(x^2-1)+(x-1)\}$$
$$=(x-1)^2\{(x^3+x^2+x+1)+(x^2+x+1)+(x+1)+1\}$$
$$=(x-1)^2(x^3+2x^2+3x+4),$$

∴ required quotient $=x^3+2x^2+3x+4$

Divide

7 $a^2(a-2b)-b^2(b-2a)$ by $a-b$

8 $(x-a)(x-b)-(y-a)(y-b)$ by $(x-a)+(y-b)$

9. $(a-b)(b-c)+(a-d)(c-d)$ by $b-d$

10 $(1+x)^2(1+y^2)-(1+x^2)(1+y)^2$ by $1-xy$

11 $x(x-1)(x-2)+y(y-1)(y-2)-6xy$ by $x+y$

12 $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$ [Ex 57, § 95]

13 $a(b^2+c^2-a^2)+b(c^2+a^2-b^2)$ by $(b+c-a)(c+a-b)$

Divide

14. $a^3+b^3-3a^2+3a-1$ by $a+b-1$.

15 $b(x^3+a^3)+ax(x^2-a^2)+a^2(x+a)$ by $(a+b)(x+a)$.

16. $(1+a)^3+(1+a)^2x+(1+a)x^2+x^3$ by $1+a+x$.

17 $(a^2+ab+b^2)^4-(a^2-ab+b^2)^4$ by $(a^2+b^2)(a^4+3a^2b^2+b^4)$.

18. $(x+y)^3+(1-x-y)3xy-1$ by $x+y-1$.

19 $(x^2-1)^4-3(x^2-1)^2+1$ by x^4-3x^2+1.

20 $(a+b)^3+(c-a)^3-(b+c)^3$ by $a^2+a(b-c)-bc$

21 $x^3+8y^3+6xy-1$ by $x+2y-1$

22 $(a-b)^2c^2+(a-b)c^3-(c^2-a^2)b^2+(c-a)b^3$ by $(a-b)c^2-(c-a)b^2$ [*Cal*, 1883]

23 [illegible] $a^4x^4+a^8$ by x^2-ax+a^2

2[illegible] 1)$(x+2)(x+3)-3$ by x^2+3x+3

2[illegible] that $(x^2+xy+y^2)^3+(x^2-xy+y^2)^3$ is divisible by $2x^2+2y^2$.

2[illegible] that $(bx+cy+az)^3-(cx+ay+bz)^3$ is divisible by $(b-c)x+(c-a)y+(a-b)z$.

27. Shew that $(x^2-1)(y^2-1)(z^2-1)+(x+yz)(y+zx)(z+xy)$ is divisible by $xyz+1$

28 Divide $x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)+4xyz$ by $1+yz+zx+xy$. [*Cal*, 1878]

Miscellaneous Examples III

1 Prove that $(a+b)cd+(c+d)ab=(a+d)bc+(b+c)ad$.

2 Prove that $x^2(x-2y)-y^2(y-2x)=(x-y)(x^2-xy+y^2)$

3. Prove that $a(a-2b)^2-b(b-2a)^2=(a-b)(a^2-7ab+b^2)$.

4. Prove that $a(a-2b)^3-b(b-2a)^3=(a-b)(a+b)^3$

5 Prove that $(a+2)^3-4(a+1)^3+6a^3-4(a-1)^3+(a-2)^3=0$

6 Shew that $(a+b)^4=2(a^2+b^2)(a+b)^2-(a^2-b^2)^2$

7 Shew that $(a^2+ab+b^2)^2-4ab(a^2+b^2)=(a^2-ab+b^2)^2$.

8 Shew that $(ax+y)(bx+y)-(x+ay)(x+by)=(ab-1)(x^2-y^2)$.

9. Shew that $(ax+by)^2+(ay-bx)^2=(a^2+b^2)(x^2+y^2)$.

10 Shew that $(x^2-ay^2)^3=(x^3+3axy^2)^2-a(3x^2y+ay^3)^2$

11 Shew that $(x-ay)^3-(y-ax)^3$
$$=(a+1)^3(x-y)^3+3(a+1)(x-y)(x-ay)(y-ax).$$

12. Divide $(a+1)^2x^3+(a+1)x^2+a^2(a-1)x-a^6$ by $(a+1)x-a^2$

13 If $a+b+c=0$, shew that

$$\text{(i) } a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc=0\,,$$

$$\text{(ii) } a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc=0$$

14. Prove that $a^3(b-c)+b^3(c-a)+c^3(a-b)$

$$+(b-c)(c-a)(a-b)(a+b+c)=0$$

15 Prove that $(a+1)^2(b-c)+(b+1)^2(c-a)+(c+1)^2(a-b)$

$$+(b-c)(c-a)(a-b)=0$$

16. Simplify $(a^2-bc)(lc+mb)+(b^2-ca)(la+mc)+(c^2-ab)(lb+ma)$

17 Prove that $(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)$

$$=(ax+by-1)^2+(ay-bx)^2$$

18. Divide $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$

$$\text{by } (b-c)^3+(c-a)^3+(a-b)^3$$

19 If $a+b+c=0$, shew that

$$a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc=0$$

20 Simplify $(a+b-1)(3a-b)-2(a+b-1)(a-b+2)$

21 Shew that $(bc+ca+ab)^2-4abc(a+c)$ is a complete square

22 Simplify $(a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b)-(a+b+c)^2$

23 Prove that $(x-y)(x+1)(y+1)-x(y+1)^2+y(x+1)^2$

$$=(x-y)(x+y+2xy) \quad \text{[}\mathit{Mad}\text{, 1887]}$$

24 Prove that $(a+d)^2(b+c)^2-(b-c)^2(a-d)^2$

$$=4ad(b^2+c^2)+4bc(a^2+d^2)$$

25. Simplify $7(a-b)(b-2a)-(2a-b)(b-7a)-6b^2$

[$\because$ $(a-b)(b-2a)=-(b-a)\times-(2a-b)=(b-a)(2a-b)$,

given expression $=(2a-b)(7b-7a-b+7a)-6b^2$

$=(2a-b)6b-6b^2=12b(a-b)$]

26 Prove that

$$(ax-by)(ax-by+1)^2-ax+by=(ax-by)^2(ax-by+2)$$

27 Divide $a^4b^2-a^2b^4+b^4c^2-b^2c^4+c^4a^2-c^2a^4$

by $\quad a^2b-ab^2+b^2c-bc^2+c^2a-ca^2$

28. If $a+b+c+d=0$, shew that

$$a^3+b^3+c^3+d^3+3(b+c)(c+a)(a+b)=0$$

29. Resolve into factors

(1) $f\{fg-(c-g)h\}-ch^2$ (2) p^4+p^3-2p

30. Find the coefficient of x^3 in the product of

$$5x^3-4x^2+3x-2 \text{ and } 6x^3+8x+3.$$

[The terms that involve x^3 may be obtained, first, by multiplying the terms of the *third* degree by those that do not involve x, and next, the terms of the *second* degree by those of the *first* degree. Therefore the terms involving x^3 in the product, are $+15x^3$, $-32x^3$ and $+18x^3$. The *algebraic sum* of these terms

$$=15x^3-32x^3+18x^3=x^3, \therefore \text{ coefficient required } =1]$$

31. Find the coefficient of x^2 in the above example.

32. Find the coefficient of x^4 in the product of

$$x^3-\frac{3x^2}{4}+\frac{2x}{3}+\frac{7}{12} \text{ and } 2x^3+\frac{2x^2}{3}-\frac{x}{2}+\frac{5}{6}$$

33. Find the coefficient of x^3 in the above example.

34. Find the coefficient of x^4 in the product of

$$x^4-ax^3+bx^2-cx+d \text{ and } x^2+px+q.$$

35. Find the coefficient of x^4 in the product of

$$1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16} \text{ and } 1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{16}.$$

36. Find the coefficient of x^3 in $(x-1)(x^2-x-1)(x^3-2x-2)$.

37. Resolve (1) $lm(x+y)(x-y)-xy(l+m)(l-m)$

$$(2)\ (1+x)^2(1+y^2)-(1+x^2)(1+y)^2.$$

38. Shew that $(ax+by)^2+(bx-ay)^2+(cx+dy)^2+(dx-cy)^2$

$$=(a^2+b^2+c^2+d^2)(x^2+y^2)$$

39. If $2s=a+b+c$, prove that

$$s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2$$

[When we find an expression involving an *abridged symbol*, as s here, it is generally advisable to effect reduction in terms of it, and when thus simplified, to substitute its value.

The given expression

$$=s^2+(s^2-2sa+a^2)+(s^2-2sb+b^2)+(s^2-2sc+c^2)$$
$$=4s^2-2s(a+b+c)+a^2+b^2+c^2$$
$$=4s^2-2s\,2s+a^2+b^2+c^2, \quad \because 2s=a+b+c,$$
$$=a^2+b^2+c^2\ ]$$

40. If $2s=a+b+c$, shew that $s^2+(s-a)(s-b)+(s-b)(s-c)$

$$+(s-c)(s-a)=bc+ca+ab$$

41. If $2s=a+b+c$, shew that

$$\{(s-a)+(s-b)\}^3=(s-a)^3+(s-b)^3+3(s-a)(s-b)c$$

42. If $2s=a+b+c$, shew that $2(s-a)(s-b)(s-c)+a(s-b)(s-c)$

$$+b(s-c)(s-a)+c(s-a)(s-b)=abc$$

43 If $s^2=bc+ca+ab$, shew that
$$(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=2abc$$

44. If $s=a+b+c$, prove that
$$s(s-2b)(s-2c)+s(s-2c)(s-2a)+s(s-2a)(s-2b)$$
$$=(s-2a)(s-2b)(s-2c)+8abc.$$

[The given expression
$$=s\{s^2-2(b+c)s+2b.2c+s^2-2(c+a)s+2c.2a+s^2-2(a+b)s+2a.2b\}$$
$$=s\{3s^2-4(a+b+c)s+(2a.2b+2b.2c+2c.2a)\}$$
$$=s\{s^2+2s^2-2(a+b+c)s-2(a+b+c)s+(2a.2b+2b.2c+2c.2a)\}$$
$$=s\{s^2+2s^2-2s^2-(2a+2b+2c)s+(2a.2b+2b.2c+2c.2a)\}$$
$$=\{s^3-(2a+2b+2c)s^2+(2a.2b+2b.2c+2c.2a)s-2a.2b.2c\}+2a.2b.2c$$
$$=(s-2a)(s-2b)(s-2c)+8abc \text{ (§ 112)}].$$

45 If $a+b=1$, prove that $(a^2-b^2)^2=a^3+b^3-ab$ [Bom.[illegible]

46 Prove that $(a-b)(1+c^2)+(b-c)(1+a^2)+(c-a)(1+$ [illegible]
$$=-(b-c)\text{[illegible]}b)$$

47 Express as the difference of two square expressions
(1) $x^2-2(a+b)x-ab(a-2)(b+2)$ [See § 131, Ex (iii) 3]
(2) $(x-a)(x-2a)(x+3a)(x+4a)$

48 Prove that $a(a+d)(a+2d)(a+3d)+d^4=(a^2+3ad+d^2)^2$.

[The given expression
$$=(a+d)(a+2d)\times a(a+3d)+d^4$$
$$=(a^2+3ad+2d^2)(a^2+3ad)+d^4 \text{ (§ 107)}$$
$$=(a^2+3ad)^2+2d^2(a^2+3ad)+d^4$$
$$=(a^2+3ad+d^2)^2 \text{ (§ 99, Cor.)}]$$

Note. If a, b, c, d be four consecutive* numbers, then it follows from this example that $abcd+1=(ad+1)^2$ that is, *the product of 4 consecutive numbers increased by 1 is a square number.*

49 Prove that $x(x-1)(x-2)(x-3)+1=(x^2-3x+1)^2$.
[See note to Ex. 48.]

50. Divide $(ax+by)^3+(ax-by)^3-(ay-bx)^3-(ay+bx)^3$
by $(a+b)^2x^2-3ab(x^2-y^2)$ [Cal., 1888]

51. If $bz-cy=\alpha$, $cx-az=\beta$, $ay-bx=\gamma$ shew that $a\alpha+b\beta+c\gamma=0$.

52. If $x^2-yz=a^2$, $y^2-zx=b^2$, $z^2-xy=c^2$, shew that
$$(a^2+b^2+c^2)(x+y+z)=a^2x+b^2y+c^2z$$

*Consecutive numbers are those which differ by unity. Thus 1, 2, 3, &c. are consecutive numbers.

53. Shew that $(x^2+y^2)^3-8x^3y^3=(x-y)^2\{(x+y)^2(x^2+y^2)+4x^2y^2\}$.

54. Prove that $(ac+bd)^2+(ad-bc)^2=(ac-bd)^2+(ad+bc)^2$
$$=(a^2+b^2)(c^2+d^2).$$

55. Prove that $\{(ac-bd)x+(ad+bc)y\}^2+\{(ac-bd)y-(ad+bc)x\}^2$
$$=(a^2+b^2)(c^2+d^2)(x^2+y^2)$$

56. If $x=a-b$, $y=b-c$, $z=c-a$, prove that
$$2(ax+by+cz)=x^2+y^2+z^2.$$

57. Shew that
$$(a+x)(b+x)-x(a+1)(b+1)+x(a-1)(b-1)-(a-x)(b-x)=0.$$

58. If $(a+b-c-d)x=cd-ab$, shew that
$$(a+x)(b+x)=(c+x)(d+x)$$

[We have $(a+b)x-(c+d)x=cd-ab$,

$\therefore (a+b)x+ab=(c+d)x+cd$, by transp.;

add x^2 to both sides and apply § 107.]

59. Prove that $(ax+by+cz)^2+(ay-bx)^2+(bz-cy)^2+(cx-az)^2$
$$=(a^2+b^2+c^2)(x^2+y^2+z^2)\ \text{[See App.]}.$$

60. Divide $(4x^3-3a^2x)^2+(4y^3-3a^2y)^2-a^6$ by $x^2+y^2-a^2$. [Lond., 1884.]

61. If $A=ax+by$, $B=bx-ay$, shew that
$$A^2-AB+B^2=(a^2-ab+b^2)x^2+(a^2-b^2)xy+(a^2+ab+b^2)y^2.$$

62. Shew that $a(b+c)(b^2+c^2-a^2)+b(c+a)(c^2+a^2-b^2)$
$$+c(a+b)(a^2+b^2-c^2)=2abc(a+b+c)$$

63. Shew that $a^3(b^2-ca)^2+b^3(c^2-ab)^2+c^3(a^2-bc)^2$
$$=3abc(b^2-ca)(c^2-ab)(a^2-bc)$$

Decompose into elementary factors

64. $(a^2-1)(b^2-1)+4ab$.

65. $x^2(x^2+3a^2)^2-a^2(3x^2+a^2)^2$.

66. $4(x-a)^3-27a^2x$.

67. $a^3-2b^3-3ab^2$.

68. $(x^2+2ax)^2+(a^2+b^2)(x^2+2ax)+a^2b^2$.

69. $(x^2+2ax)^2-(x^2+2ax)(3-a^2)+2(1-a^2)$.

70. $x^4+x^2-2ax+1-a^2$.

71. $(a^2-b^2)(x^2+y^2)+2(a^2+b^2)xy$.

72. $x^4-(a+b)x^3+(a^2b+ab^2)x-a^2b^2$.

73. $(a^2-bc)^2-(b^2-ca)^2$.

74. $m^4-n^4+2n(m^3+n^3)-(m+n)^2(m-n)^2$.

75. $a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$.

76. $4(uz-xy)^2-(u^2-x^2-y^2+z^2)^2$.

77. $16x^4+4x^2(x-1)^2+(x-1)^4$.

78. $ab(x+y)^2-(a+b)(x^2-y^2)+(x-y)^2$.

79. Shew that $\frac{375\times 375 - 025\times 025}{375 - 025} = 4$.

80. Shew that $\frac{384\times 384\times 384 - 383\times 383\times 383}{384\times 384 + 384\times 383 + 383\times 383} = 1$

81. Shew that $\frac{\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}+\frac{2}{7}\times\frac{2}{7}\times\frac{2}{7}}{\frac{3}{5}\times\frac{3}{5}-\frac{3}{5}\times\frac{2}{7}+\frac{2}{7}\times\frac{2}{7}} = \frac{31}{35}$

82. Find the value of $a^2+b^2-c^2+2ab$, when $a=03$, $b=07$, $c=\cdot 10$

83. Find the value of $x^3+y^3+3xy-1$, when $x=15$, $y=85$

84. If $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, find the value of
$$y^2+z^2-x^2+2yz.$$

85. If $A=x^2+y^2+z^2$, $B=yz+zx+xy$, shew that
$$A^3-3AB^2+2B^3=(x^3+y^3+z^3-3xyz)^2$$

86. Factorise $6x^2-(a-7)xy-(a-1)(a+2)y^2$

87. Factorise $xy+2x^2-3y^2+3xz+2yz+z^2$

88. Prove that
$$x^8+y^8+(x+y)^8-2(x^2+xy+y^2)^4+8x^2y^2(x+y)^2(x^2+xy$$ [illegible]

89. Simplify $(a+b+c)^2+(b+c-a)^2+(c+a-b)^2+(a+b-$ [illegible]

90. Simplify $(a+b+c)^2-(b+c-a)^2+(c+a-b)^2-(a+b-c)^2$

91. Simplify $(a+b+c)^2-a(b+c-a)-b(c+a-b)-c(a+b-c)$

92. Shew that $(x+y+z)(y+z-x)+(x+y-z)(x-y+z)$
$$+(y+z-x)(x+y-z)+(x+y+z)(x-y+z)=4(x+y)z$$

93. Find the value of
$$(a+b+c+d)^2+(a+b-c-d)^2+(a-b+c-d)^2+(a-b-c+d)^2$$

94. Find the value of
$$(a+b+c+d)^2-(a+b-c-d)^2+(a-b+c-d)^2-(a-b-c+d)^2.$$

95. Prove that $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2$
$$+(b+c-a)(c+a-b)(a+b-c)=4abc$$

96. If $A=a^2-bc$, $B=b^2-ca$, $C=c^2-ab$, prove that
$$\frac{A^2-BC}{a}=\frac{B^2-CA}{b}=\frac{C^2-AB}{c}=(A+B+C)(a+b+c)$$

97. Factorise $x^3+2x^2z-xy^2-xz^2+yz^2+2xyz-2z^3$

98. Shew that $a^3+b^3+c^3+6abc-a(a-b)(a-c)-b(b-c)(b-a)$
$$-c(c-a)(c-b)=(a+b+c)(bc+ca+ab)$$

99. Shew that $(a^2-bc)(b^3-c^3)+(b^2-ca)(c^3-a^3)+(c^2-ab)(a^3-b^3)$
$$=(b-c)(c-a)(a-b)(a+b+c)^2.$$

100. Shew that $(a+1)^3(b-c)^3+(b+1)^3(c-a)^3+(c+1)^3(a-b)^3$
$$=3(a+1)(b+1)(c+1)(b-c)(c-a)(a-b)$$

CHAPTER X.

SIMPLE EQUATIONS

Section I.

137 Equation and Identity When an expression *is equal* to another expression, we may view the equality to hold in two different ways · (1) it may hold for some *particular value or values* of any of the letters involved in them and (2) it may hold for *all values* of that letter In the first case, the equality constitutes what is termed an *Equation*, and in the second, what is termed an *Identity* For example, $x-1=3$ is an equation, for the value 4 only of x makes $x-1$ equal to 3 ; $x^2-5x+6=0$ is an equation, for the values 2 and 3 only, and no other values, of x make x^2-5x+6 equal to zero , and so on An EQUATION thus expresses *an equality under a condition* and is defined to be *the statement of equality of two expressions for some particular value or values of one or more of the letters involved in them*

[illegible] it is easily seen that

$$15=10+5=6\times2+3,$$
$$(a+1)x=ax+x,$$
$$(a^2-1)-(a-1)=a+1,$$
$$x^2-(a+b)x+ab=(x-a)(x-b)$$
$$\cdots\cdots\cdots\quad\cdots\cdots\quad\cdots\cdots$$

These examples shew that the first side of each of the expressions is the same as the second side, only that the latter is put into *another form* It is clear, therefore, that if *any value* whatever be given to any of the involved letters in each of the above expressions, the resulting values of the two sides will be the *same* An IDENTITY is thus defined to be *the statement of equality of two different forms of the same expression*, and therefore expresses that *the equality holds for all values of the letters involved in them*

138 Sign ≡. Two different symbols are likewise used to express these two forms of equality. Thus the symbol, =, is used to denote a *conditional equality*, or is the sign of an Equation ; and the symbol, ≡*, to denote an *identical equality* or is the sign of an Identity We shall, however, continue to use the first symbol to denote an identity as being more common

139. Side or Member of Equation, In an Equation, the two expressions, connected by the sign of equality, are called the

* First used by the German Mathematician GAUSS

SIDES or MEMBERS of the equation The left-hand expression is called the *First Side*, and the right hand expression the *Second Side* of the equation These terms are also applicable in the case of Identities

140 Variable, Root, Solution The symbol, whose value we seek, is called the *Unknown Quantity*, or more briefly the VARIABLE The last letters of the alphabet, x, y, z, u, &c, are usually employed to denote Variables

An equation is said to be *satisfied* by a value of the variable when that value makes the *two sides the same* Thus 4 *satisfies* the equation $2x+1=13-x$, because 4 standing for x, makes $2x+1$ equal to 9, also $13-x$ equal to 9

A ROOT of an equation is that value of the variable which *satisfies* the equation Thus 2 is a *root* of the equation

$$x^2-5x+6=0, \text{ as also is } 3$$

To SOLVE an equation is to find its *root* or *roots* The c[illegible]d ing noun is SOLUTION Sometimes this word is used [illegible]te the root of an equation Thus the equation $x^2-5x+6=$[illegible]d to have two *roots* or *solutions*, 2 and 3.

141 Simple Equation When the *first* power only, and no higher power, of the variable occurs in an equation, it is said to be a SIMPLE EQUATION It is also called an Equation of the FIRST DEGREE or a LINEAR EQUATION * Thus $ax+b=0$, $ax+by=c$, are Simple or Linear Equations, in one and two variables respectively [For a more complete definition, see § 221]

142 Certain necessary Axioms The following Axioms are necessary for the Solution of Equations

I If the *same* quantity, or *equal* quantities, be added to, or subtracted from, equal quantities, the results are equal Thus

If $a=b$, then (1) $a+x=b+x$, (2) $a-x=b-x$

If $a=b$ and $x=y$, then (1) $a+x=b+y$, (2) $a-x=b-y$

II If equal quantities be multiplied, or divided, by the *same* quantity or *equal* quantities, the results are equal Thus

If $a=b$, then (1) $ax=bx$, (2) $a \div x=b \div x$.

If $a=b$ and $x=y$, then (1) $ax=by$, (2) $a \div x=b \div y$

III If equal quantities be raised to the *same power*, the results are equal Thus

if $x=y$, then $x^2=y^2$ and generally $x^m=y^m$

* A term adapted from *Analytical Geometry*

IV. If the *same root* of equal quantities be extracted, the results are equal. Thus

$$\text{if } x^3=y^3 \text{ or generally } x^m=y^m, \text{ then } x=y.$$

143 Transposition of Terms Suppose $x-a=y$; add a to both sides, thus

$$x-a+a=y+a \text{ (Ax I)}, \text{ or } x=y+a,$$

that is, $-a$ is *changed* into $+a$, when it is taken from the *first* side and placed in the *second* Again if we subtract y from both sides (Ax I), we have

$$x-y=y+a-y, \text{ or } x-y=a;$$

which shews that $+y$ is *changed* into $-y$, when it is taken from the *second* side and put in the *first*

These two examples are sufficient to shew that—*Any term may be transposed from one side of an equation to the other side, if its sign be changed.*

Hence we may write *all the terms* of an equation on one side of the sign of equality and zero on the other side

Cor 1 Hence if the *same term with the same sign* occurs on both sides of an equation, it may be *removed* without the equality being affected. For then we subtract that term from both sides

Cor 2 If the *sign of every term* on *both* sides of an equation be *changed*, the equality still holds For this is practically transposing all the terms on either side to the other side

144 ; 145 [Omitted]

146. Solution of Simple Equations Rule —*Transpose all the terms involving the variable to the first side and all the others to the second side of the given equation, simplify the two sides; and divide by the co efficient (if any) of the variable, thus the required root is found*

Examples.

Ex. 1 Solve $2x+1=9$

Transpose, thus $\qquad 2x=9-1=8,$

divide by 2, thus $\qquad x=4$ [Ax II]

Ex. 2. Solve $3x-2=2x+7$.

Transpose, thus $\qquad 3x-2x=7+2,$

$$x=9.$$

Ex 3 Solve $5x+21=20x-24$

Transpose, thus $5x-20x=-24-21$,

or $-15x=-45$,

change the signs, thus $15x=45$,

divide by 15 [Ax II], thus $x=3$

Ex 4 Solve $8x-5+2x=13+2x-7x$

Remove $2x$ from both sides, thus

$$8x-5=13-7x,$$

transpose, thus $8x+7x=13+5$,

or $15x=18$,

divide by 15, thus $x=\frac{18}{15}=\frac{6}{5}$

Ex 5. Solve $4x-3=4(2x-1)+13$

$$4x-3=8x-4+13=8x+9,$$

transpose, thus $4x-8x=9+3$,

or $-4x=12$,

change the signs, thus $4x=-12$,

divide by 4, thus $x=-3$

Ex 6. Solve $3x-10(2x-3)+21=0$.

Remove the bracket, thus

$$3x-20x+30+21=0,$$

or $-17x+51=0$,

by transposition, $-17x=-51$,

changing signs, $17x=51$,

dividing by 17, $x=3$

Ex 7 Solve $ax-b=cx+d$

Transpose, thus $ax-cx=b+d$,

or $(a-c)x=b+d$,

divide by $a-c$, thus $x=\dfrac{b+d}{a-c}$

Ex 8 Solve $(x-5)^2+(2x-3)^2=5(x-2)^2$

Simplify, thus

$$x^2-10x+25+4x^2-12x+9=5(x^2-4x+4),$$

or $5x^2-22x+34=5x^2-20x+20$,

remove $5x^2$ from both sides, thus

$$-22x+34=-20x+20\,;$$

transpose, thus $\qquad -22x+20x=20-34\,;$

or $\qquad -2x=-14\,;$

$$\therefore\ 2x=14\,;$$

divide by 2, thus $\qquad x=7.$

Solve the following equations

9 $\quad x+7=15$ $\qquad$ 10 $\quad 3x-5=19$ $\qquad$ 11 $\quad x=4-15x.$

12 $\quad 5x-8=16-3x$ $\qquad$ 13. $\quad 53+30x=3-20x$

14 $\quad 3+x-15x=10x-102-3x.$ $\qquad$ 15 $\quad 10x-3(x-1)+3=90.$

16 $\quad 5-3(4-x)+4(3-2x)=0$ $\qquad$ 17 $\quad 6x+2(11-x)=3(19-x).$

18 $\quad 52x+4(3x-2)=376$ $\qquad$ 19 $\quad (3x-5)20-(4x-3)12=0.$

20. $\quad 13x-21(x-3)=10-21(3-x)$

21 [illegible]$(x-8)+13(4x-1)=15(2x-1)+13$

2[illegible] $1-2(3x-2)+3(4x-3)=4(5x-4)$

2[illegible] $3(5x-18)+3x^2=4x^2-3x$

24 $\quad 2$[illegible]$(x-3)+x(x-4)=x(3x-2)-80$

25 $\quad 15(x^2-3)+20x=x(15x+17)$

26. $\quad (x-3)(x-2)=(x-4)(x+5)$

27 $\quad (x-2)(2x-1)=2(x+1)(x+3)$

28. $\quad (8-3x)(5-x)=(3x-4)(x-6)-13$

29. $\quad (x-1)^2-(x-2)^2=5$ $\qquad$ 30 $\quad (2x+7)^2+3x(x-10)=7(x^2+8).$

31 $\quad (3x-4)^2+(2x-5)^2=13(x-6)^2.$

32 $\quad (20x+3)^2-(15x-8)^2=5(2-5x)(3-7x)+30$

33. $\quad ax+13=0$ $\qquad$ 34 $\quad 2mx-c=x$ $\qquad$ 35. $\quad 3px-2=a-1.$

36. $\quad cx+d=a-3x$ $\qquad$ 37 $\quad ax+bx-c=x+bx-d.$

38. $\quad (2m-1)x-bx=bx+m(2x-a)$ $\qquad$ 39 $\quad (a-b)x-mx=c-mx$

40 $\quad 2ax+(2+x)(a-3)+4=0$ $\qquad$ 41. $\quad 2(a+1)x-5=(2a-3)x+20.$

42 $\quad (m+n)(m-x)=m(n-x)$ $\qquad$ 43 $\quad (ax+3)^2-(ax-3)^2=12.$

CHAPTER XI.

PROBLEMS LEADING TO SIMPLE EQUATIONS.

Section I.

147. Translation into Algebraical language The student, who is supposed to have mastered the previous Chapters, can

now very easily put down in *symbols and signs of operation*, any Algebraical question, expressed in ordinary language If he has carefully studied the examples given in the several *Examination Papers* at the end of some of the foregoing Chapters, he will have no difficulty in seeing that the manner of dealing with any question in Algebra is precisely the same as that in Arithmetic, only that general symbols are here used for Arithmetical numbers Thus for instance, he knows how to solve the following Arithmetical question—*A can walk 3 miles an hour, he has walked for 4 hours, what distance has he walked?* The required distance $=(3\times 4)$ miles $=12$ miles If instead of the numbers 3 and 4 we had the general symbols a and b, the same Arithmetical question would be transformed into the following Algebraical question—*A can walk a miles an hour, he has walked for* b *hours, what distance has he walked?* Representing the required distance by x, we have $x=(a\times b)$ miles $=ab$ miles The *symbolical expression*, $x=ab$, is therefore equivalent to the proposed question, expressed in words Again, let us put into the language of Algebra the follow[illegible]te-ment—18 *years hence A's age will be 3 times his present* [illegible] x represent his present age and y what it will be after 18 [illegible]e may translate the given statement in two ways,

$$\left.\begin{array}{l}(1)\ y=3x,\\(2)\ y=x+18\end{array}\right\}\qquad(a).$$

And so on for any other verbal statement We thus see that Arithmetical and Algebraical questions are exactly similar in nature, only that the operation for resolving the latter is different from Arithmetical operation

We may now lead the student to the most interesting and at the same time, the most amusing portion of Algebra, *viz*, that which treats of the solution of Algebraical questions, commonly called problems The whole difficulty in resolving a question of this class lies in *translating* the ordinary verbal statements into the *symbolical language* of Algebra We shall therefore give some more examples of such translation for exercise

Examples

1 Find the number which is 3 times x, diminished by a

2 Twenty is divided into 2 parts one of which is x, find the other.

3 The product of two numbers is 63 and one of them is x, what is the other?

4 A father is 35 years older than his son whose age is $(x-1)$ years, what is the age of the father?

5 A walks 3 miles an hour to go to P, a distance find the distance between him and P at the end of x hours

6 A walks $1\frac{1}{2}$ miles an hour faster than B who walks $(x-1)$ miles an hour; if they start together, find the respective distances walked by each at the end of 6 hours

7. Four persons equally contributed to make up the sum £$(x+4)$, how many shillings did each give?

8. A post whose length is $12x$ has a third coloured red and a fourth coloured black, and the rest white, what is the length of this portion?

9. The age of a person is x years, how old was he 6 years back and what will be his age 6 years hence?

10 If a person travels 84 miles a week, what distance will he travel in x days?

11 A performs a journey of 25 miles in x days, what is his rate of travelling per hour?

12 In what time will a man walk 30 miles at the rate of x miles per hour?

13 If a person travels x miles in a hours, what distance does he travel in 3 hours?

14 A person gives $(x-3)$ rupees more to A than to B; if A gets y rupees, what does B get?

15 A picture and its frame together cost x rupees; if the value of the picture be $(20-y)$ rupees, find that of the frame

16 A's age is x years and B's age is 5 times what A's age will be 3 years hence, what is B's age?

17 A railway train performed a distance of a miles in x hours, it had to stop a quarter of an hour at an intermediate station, at what rate did it run?

18 A and B have each x rupees, if A pays y rupees out of his money to B, what has each then?

19 The amount in a bag is £m, if it consist entirely of half sovereigns, find their number If there be 6 half-sovereigns and the rest half-crowns, what is the number of the latter?

20 Express 38 and 83 symbolically, when x stands for 3 and y for 8

21 If x stands for 1, y for 2 and z for 3, express 231 and 312 symbolically

22 The sum of a and 15 is x, express the statement symbolically.

23 If the excess of 5 over x be denoted by y, express the statement in symbols

24 If the same number be less than x by 3 and greater than y by 4, how do you express the statement?

25 The price of a horse is represented by a certain number, find the price of the saddle, which is m rupees less than that of the horse.

148 Solution of Problems Problems are statements in words, in which are given certain quantities, called *Known Quantities* and their relations to other quantities, which latter, it is the object of the problem to find, and which are hence called *Unknown Quantities* These relations are called *Conditions of the Problem* There are no general rules for the Solution of Problems, for each particular problem requires a separate reasoning We may, however describe in general terms, the method to be pursued in the solution of problems *Represent the variables by letters* (x y, z, &c) *and then express the conditions of the given problem in the language of Algebra, after the manner described in the preceding article* This process of translation will lead to equation, on the solution of which that of the problem depends Thus, if we want to know [illegible]ge [§ 147 (*a*)] we have from (1) and (2)

$$3x=x+18,$$

an equation, solving which we can find x the age of A [see Ex 30 *post*]

We give below a collection of easy problems for exercise The *typical* ones will be worked out in full, after which will be found others of a similar nature

Examples of Problems

Ex 1 If 20 be added to a number, the sum becomes 3 times the number, what is the number?

Let x = the number,*

then $x+20$ = sum of x and 20,

also by the condition of the problem

$3x$ = sum of x and 20,

$$3x=x+20,$$

we thus get an equation, solving which, x, or the number sought, is found

By transposition $3x-x=20$,

or $2x=20$,

$$\therefore \quad x=10$$

* The symbol of quality, =, is here used to mean '*Denote*,' '*Represent*' or '*Stand for*' As it is convenient to use the sign of quality instead of these expressions, we shall always do so in the solution of problems, bearing in mind the above meaning

Ex 2 Find the number, the double of which diminished by 5 gives that number increased by 5

Let x = the number,

then $2x$ = double of the number,

and $2x-5$ = double of the number diminished by 5,

also $x+5$ = the number increased by 5.

By the condition of the problem, the last two expressions are equal;

$$2x-5=x+5,$$

an equation which gives $x=10$, the number sought

3. A certain number is equal to 570 diminished by four times the number; what is the number?

4. Find the number, five times which exceeds the double by the number which is the difference between 284 and the number

5. What is that number, 3 times which taken from 120, will leave a remainder which is equal to that number increased by 8?

6 If from 558 you subtract 8 times a certain number, the remainder will be 10 times that number, what is that number?

7 The sum of a certain number and 15, is multiplied by 12, if you now subtract twice the number from the product, the remainder will be 240 diminished by 5 times that number. Find the number

8 If 25 be added to a number, the sum will be 3 times the number, minus 15 What is the number?

9 What is that number, to which if you add 3, the sum will be the same as if you take 93 from 5 times the number?

10 Find a number, such that if its double and treble together be subtracted from 75, there will be a remainder 30

Ex 11. Find the numbers, whose sum is 31 and difference 15

Let x = the smaller number,

then, the difference between the numbers is 15,

$x+15$ = the greater number

And by the first condition of the problem, their sum is 31,

$$\therefore x+(x+15)=31.$$

or $$2x+15=31,$$

whence $$x=8,$$

∴ one of the numbers is 8, and the other is 8+15, or 23

12 Find two numbers, such that their sum may be 183, and their difference may be 13

13 Find two numbers whose difference is 15, such that if 40 be taken from the greater, the remainder is 51 diminished by the less

14 Find two numbers, whose sum is 26, such that 3 times the greater diminished by twice the less, will be equal to their difference increased by 22

15 The product of two numbers, whose difference is 7, is equal to 84 increased by the square of the smaller What are the numbers?

16 The sum of two numbers is 64, and if 4 times the greater be added to 5 times the smaller, the sum will be 285, what are the numbers?

17 The difference between two numbers is 8, and that between their squares is 288 Find the numbers

18 The difference of two numbers is 8, and if 4 times the smaller be subtracted from 5 times the greater, the remainder is [illegible] the smaller, find the numbers

19 The sum of two numbers is 40, and if 32 be taken from the greater, the remainder will be 3 times the smaller number Find the numbers

Ex 20 Divide 24 into two parts, such that the sum of the greater and 4 times the less is 33

Let x = the greater part,

$\therefore\ 24-x$ = the smaller part.

By the condition of the problem,

greater part + 4 × smaller part = 33,

$$x+4(24-x)=33,$$

or $$96-3x=33,$$

whence $x=21$, the greater part,

$\therefore$ the smaller part $=24-21=3$

21 Divide 36 into two parts, such that the sum of 10 times the less and 15, may be equal to the excess of 8 times the greater over 3

22 A horse and saddle cost Rs 1000, and the cost of the horse is 7 times that of the saddle, find the price of each

23. Divide Rs 207 between two persons, so that 5 times the share of one, together with 6 times the share of the other, may be equal to 11 times the excess of the first share over the second

24 Divide Rs 1125, among A, B and C, so that A may have 3 times as much as B, and B twice as much as C

[Represent the share of C by x]

25. A and B together counted Rs 1265, and A counted Rs. 125 more than B, what number did each count ?

26 At a municipal election 521 votes were given, and of the two candidates, the unsuccessful one had a minority of 31, how many voted for each ?

27 Two persons have between them Rs 72, and one of them has Rs 15 more than twice the amount which the other has ; how much has each ?

28 Divide 349 among three persons, so that the first may have 15 more than the second and 13 less than the third

29 Divide the number 765 into two parts, such that the excess of three times the greater over 153, may be equal to the sum of 4 times the smaller and 840

Ex. 30 Eighteen years hence A's age will be 3 times his present age ; what is his present age ? [See § 147, (a)].

$x = A$'s present age,

then $3x = A$'s age 18 years hence ;

also $x + 18 = A$'s age 18 years hence

By the condition of the problem, therefore

$$3x = x + 18,$$

whence $x = 9$ years, A's present age.

31 A father's age is 35 years and his son's age 10 years, when will the son's age be half that of his father ?

32 The ages of the father and son together amount to 52 years and the father is 30 years older than the son, what is the age of each ?

33. A father's age is 53 years ; 8 years ago, it was 3 times the son's age, what is the age of the son ?

34 The ages of A, B and C are together 4 times what B's age was 7 years ago, if A be 5 years older, and C 3 years younger than B, what is the age of each ?

Ex 35 A and B play with equal sums of money ; A gains Rs. 17, and has then twice as much as B What sum did they begin with ?

Let $x =$ required sum of money, in rupees,

then $x + 17 =$ sum of money A has, when the game is over,

and $x - 17 =$ sum of money B has, when the game is over.

By the condition of the problem, A's money is double of B's ;

$$x + 17 = 2(x - 17),$$

whence $x = 51$.

36 A and B began to play with equal sums of money, A lost Rs 12, then 15 times A's money was equal to 9 times B's, what sum had they at first?

37. A and B have 860 and 1040 rupees respectively when they begin to play After the game is over A, who is the winner, finds that 3 times his money together with 5 times B's money, amount to 7 times the money which B had at first How much did A win?

Ex 38 A labourer is engaged for 30 days, on condition that for every day he works, he shall receive 8 annas, and for every day he is idle he must pay a fine of 3 annas He receives Rs 10 3 annas in all How many days does he work?

Let $\qquad x =$ number of days, he works,

then $\qquad 30 - x = \ldots \quad . \quad .$, he is idle,

therefore he receives, as wages, $8x$ annas, and pays, as fine, $3(30-x)$ annas hence his net receipt will be $8x - 3(30-x)$, which is, by the condition of the problem, equal to Rs 10 3 annas or 163 a[illegible]

$$8x - 3(30-x) = 163,$$

whence $\qquad x = 23$

39 A workman is engaged for 40 days, on condition that for every day he idles he has to pay a fine of $6d$ He is idle for 10 days and receives £3 10s in all What were his daily wages?

*40 A man is engaged to work on condition that for every day he works, he shall get 2 rupees and, for every day he idles, shall forfeit Re 1 2 annas If he works 16 days and gets Rs 25 4 ans at the end of the stipulated time, for how many days was he engaged?

Ex 41 A bag contains Rs 365, in rupees and eight-anna bits, if the amount of the latter be less than that of the former by Rs 13, how many of each are there?

Let $\qquad x =$ amount of rupees in the bag,

then also $\qquad x =$ *number* of rupees in it,

and $\qquad x - 13 =$ amount of eight-anna bits in rupees,

$\therefore \quad 2(x-13) =$ *number* of eight-anna bits required

The total amount in the bag is Rs 365,

$$x + (x-13) = 365,$$

whence $\qquad x = 189$, the required number of rupees,

$\therefore \quad 2(x-13) = 352 =$ required number of eight-anna bits

42. A certain sum consisting of sovereigns, shillings and sixpences amounts to £5 2s The number of the shillings is 4 times that of the sovereigns and the number of the sixpences is 5 times that of the shillings. Find the number of each.

Ex 43 P and Q set out at the same time from A and B respectively to meet each other, P walking 4 miles and Q 5 miles an hour, if the distance between A and B be 27 miles, when and where will they meet?

Suppose Aa represents 4 miles and Bb 5 miles, then it is clear

A a O b B

that at the end of one hour P will be at a and Q at b, and the distance they will have jointly walked, will be Aa+Bb=4+5, or 9 miles Hence if x=number of hours after which they meet, and O the point of meeting, we have

$$4x = \text{distance } P \text{ walks} = \text{AO},$$

$$5x = \text{distance } Q \text{ walks} = \text{BO}$$

But $$\text{AO}+\text{BO}=\text{AB}=27 \text{ miles},$$

$$\therefore 4x+5x=27;$$

whence $$x=3,$$

i.e., they meet after 3 hours

And $$\text{AO}=4x=12 \text{ miles},$$

$$\text{BO}=5x=15 \text{ miles},$$

i.e., they meet at a place 12 miles from A or 15 miles from B.

44 Two persons set out from two places 26 miles apart and meet after 4 hours, if one of them walks 3 miles an hour, find the rate of the other

45. A crew, which can row at the rate of 5 miles an hour, finds that it takes 3 hours to come 27 miles down a river; what is the rate of the river?

[Note—The crew is *assisted* by the current]

46 A boat takes 4 hours to come 16 miles up a river, which flows at the rate of 3 miles an hour, what is the rate of the boat?

[Note —The boat goes, every hour, its own rate *diminished* by the rate of the river]

47 A cistern which can hold 28 maunds of water, has two supply pipes, one of which admits 3 maunds and the other 4 maunds per hour, respectively If the first be allowed to run for one hour, when the second is opened, and the two are allowed to run together, when will the cistern be filled?

149 Examination upon Chapters XI and XII.

1 Define an *Equation* and an *Identity*, and distinguish between them

2. Define the terms—*Variable*, *Root* and *Solution*.

3 When is an equation said to be *satisfied* by a number ?

4 Define a *Simple Equation* What is a Linear Equation ? If x and y are variables, is $2x-5y=3$ a simple equation ? If so, why ?

5 Define a *Problem* What is 'Condition of a Problem ?'

Miscellaneous Examples IV

1 Simplify $d-[b+c-\{a+b-(c+2b+a-d)\}]$

2 If $x=1$, $y=-2$, $z=3$, find the value of

$$\tfrac{1}{2}\{x-\tfrac{1}{3}[y-\tfrac{1}{4}(z-\overline{x-2y})]\}$$

3 Add together

$(a+b)x+(a+c)y$, $(b-c)x+(b-a)y$, and $(c-a)x+(c-b)y$.

4 Multiply $x^2+(x-y)^2-y^2$ by $x^2-(x+y)^2+y^2$

5 Resolve into factors $a^2(a+c)^2-c^2(a-c)^2$

6 . Divide $(a+b)(a+c)-(d+b)(d+c)$ by $a-d$

7 Shew that $30ab-(9a-8b)(5a+2b)-(4b-3a)(15a+4b)=4ab$

8 Find the value of $x^4-x^3+2x^2-3$ in terms of y, when $x=y-2$

9 If $a+b+c=0$, shew that $a^2-bc=b^2-ca=c^2-ab$.

10 Divide $a^8-b^8+a^2b^2(a^4-b^4)$ by the product of

$$a^2+b^2,\ a^2-ab+b^2, \text{ and } a^2+ab+b^2$$

11 Solve $x+1+2(x+3)=4(x+5)$

12 A is twice as old as B , 15 years ago he was 5 times as old What is A's age ?

---

13 Simplify $a-[3a+c-\{4a-(3b-c)\}+3b]$

14 Find the value of $a^3+b^3+c^3+3abc$, when $a=02$, $b=08$ and $c=10$

15 Add together $ax-by$, $x+y$ and $(a-1)x-(b+1)y$, and from the sum subtract $(a+1)x-(2b-1)y$

16 Multiply together $x-a$, $x-b$, $x-c$ and $x-d$, and from the product deduce the value of $(x+2)^4$.

17 Resolve into elementary factors

$$(y-z)(y+z)^2+(z-x)(z+x)^2+(x-y)(x+y)^2$$

18 Divide $a^8+a^4x^4+x^8$ by a^2+ax+x^2

19 Prove that

$$x^3+y^3+z^3-3xyz=(x+y+z)^3-3(x+y+z)(yz+zx+xy)$$

20. If $c^2=a^2+b^2$, find the value of

$$(a+b+c)(a-b+c)(a+b-c)(b+c-a)$$

21 Shew that the difference between

$$(x^2+2x+3)(3x^2+2x+1) \text{ and } (x^2-2x+3)(3x^2-2x+1)$$

is the same as that between $(x^2+4x+1)^2$ and $(x^2-4x+1)^2$.

22 Solve $5(x+1)-2=3(x-5)$

23 What is that number, which being diminished by 10 and the remainder multiplied by 10, produces the same result, as if it were diminished by 8 and the remainder multiplied by 8 ?

---

24. Simplify $2\{4x-[2y+(2x-y)-(x+y)]\}$.

25. Express $x^4-2a(a-b)x^3+(a^2+b^2)(a-b)x-a^2b^2$ with numerical coefficients, when $a=4$, $b=8$

26 From $(x+2)(x+3)$ take $(x+1)(x+4)$, and to the result add $2(x-1)(x+1)-x^2$.

27 Multiply $a^3-a^2b+2ab^2$ by $a^2+ab+2b^2$

28 Divide $(ax^2-ay^2+2bxy)^2+(by^2-bx^2+2axy)^2$ by $(x^2+y^2)^2$

29 Shew that

$$(ax+by-cz)^2-2\{b^2y^2-(cz-ax)^2\}+(by+cz-ax)^2=4(ax-cz)^2.$$

30 If $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, find the value of $\frac{1}{2}(a+b+c)(ax+by+cz)$

31. Prove that

$$(a+b)^2-(c+d)^2+(a+c)^2-(b+d)^2=2(a-d)(a+b+c+d)$$

32 If $ax+by=a$ and $bx-ay=b$, shew that $x^2+y^2=1$.

33. Solve $3(x-1)+2(x-2)=x-3$

34. At a municipal election 394 men voted, and the candidate chosen had a majority of 56. How many voted for each ?

---

35. Arrange $a^2(x+y-z)-ab(z+x-y)+b^2(z+y-x)$, bracketing the coefficients of x, y and z

36 Simplify $5x-[a-\{x+2a-(3x-7a)\}]$

37. Add together $4x^3+3x^2y-y^3$, $4x^2y-3x^3$ and $7xy^2+9y^3-2x^2y$, and find what must be subtracted from the sum to leave the remainder $2x^3-3x^2y+y^3$

38 Find the continued product of $a+x$, $a+\frac{1}{2}y$ and $a-\frac{1}{2}z$, and from the result deduce the value of $(a+b)^3$.

39. Multiply $a(a-b)+b(b-c)+c(c-a)$ by $a+b+c$.

40 Resolve $x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz$

41 Divide $(a^3+b^3)^2+2a^2b(a^3-a^2b-b^3)$ by a^2+b^2, and find the value of the quotient when $b=a+3$

42 Prove that
$$\{(x-y)^2+(y-z)^2+(z-x)^2\}^2=2\{(x-y)^4+(y-z)^4+(z-x)^4\} \quad [App.]$$

43 If $a^2+b^2=1=c^2+d^2$, shew that
$$(ac+bd)(ac-bd)=(a+d)(a-d)$$

44 If $2s=a+b+c$, shew that
$$2(s-a)(s-b)+2(s-b)(s-c)+2(s-c)(s-a)+a^2+b^2+c^2=2s^2$$

45. Solve $(x-1)(x-2)=(x-3)(x-4)$

46 Bought 12 yards of cloth for £10 14*s* For part of it I gave 19*s* a yard, and for the rest 17*s* a yard How many yards of each were bought ?

---

47 Simplify
$$2a-\{c-(a-b+2c)\}-[4b-\{3c+a-(\overline{4a+c}-5b)\}]$$

48 If $x=\frac{1}{2}$ and $x+y=x+y+z=0$, find the value of
$$(y^3-z^3)\{y^2+z^2-y(x-z)\}$$

49 Subtract $2(x-k)(y-k)$ from $(x-k)^2+(y-k)^2$, and shew that whatever value be given to k, this difference will always be the same

50 Find the product of $x-a$, $x-b$ and $x-c$, and factorise the result when $-x^2=bc+ca+ab$.

51 Resolve $\{ab-1-(a-b)x\}^2-\{a-b-(ab-1)x\}^2$

52. Divide $x^4-xy^3-x^3y+y^4$ by x^2+xy+y^2, and find the value of the quotient, when $x=a^2+ab+b^2$, $y=a^2-ab+b^2$

53 If $x=a+d$, $y=b+d$, $z=c+d$, shew that
$$x^2+y^2+c^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab$$

54 If $ax+by=1$, then will
$$ab(x^2+y^2)+(a^2+b^2)xy+(a-b)(x-y)=1$$

55 Find the value of
$$(x-a)^3+(x-b)^3+(x-c)^3-3(x-a)(x-b)(x-c), \text{ when } 3x=a+b+c$$

56 Solve $(x+1)(x-2)+(x+3)(x+4)=2(x-4)(x+2)$

57 A courier who travels 60 miles a day, had been despatched 5 days, when a second, who travels 75 miles a day, was sent to overtake him. When will he overtake the first ?

---

58 Simplify $a-\{b-(2b+c)\}+\{b-(c-2b)\}$, and find its value when $b=-\frac{1}{2}a$

59 Shew that $\frac{1}{4}(x^2+y^2)+z^2-\frac{1}{2}xy+xz-yz$ and $(y-z)^2$ become identical when $-x=y=a$

60 From the sum of $a(x-y)+b(y-z)+c(z-x)$ and $ax+by+cz$, subtract $(a-b)x+(b-c)y+(c-a)z$, and in the remainder, bracket the coefficients of x, y and z

61 Multiply $7x-3y+5z$ by $3y-5x+11z$

62. Resolve into factors $x^2(a+1)-xy(x-y)(a-b)+y^2(b+1)$.

63 What number must be added to $3x^3-13x^2+6x-5$, that it may be divisible by $3x-4$?

64 Prove that $n(n-1)(n-2)-p(p-1)(p-2)$

$$=(n-p)\{(n+p-1)(n+p-2)-np\}.$$

65 If $2s=a+b+c$, shew that

$$s^3+(s-a)(s-b)(s-c)=\tfrac{1}{2}\{(a+b)(b+c)(c+a)-abc\}.$$

66. Solve $(x-a)(x+b)-x^2=3ab$.

67 Divide the number 68 into two such parts that the difference between the greater and 84 may be equal to 3 times that between the less and 40

---

68 Find the value of

$$\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2,$$

when $a=1$, $b=3$, $c=5$

69 If $a-b=x=3$ and $a+b+x=2$, find the value of

$$(a-b)\{x^3-2ax^2+a^2x-(a+b)b^2\}.$$

70 Find the sum of $3(x-2y+3z)$, $4(y-2z+3x)$ and $5(z-2x+3y)$, and its value in terms of x, when $x=10y$ and $y=100z$.

71. Find the difference between $a(b+c)^2+b(c+a)^2+c(a+b)^2$ and $(a+b)(a-c)(b-c)+(a-b)(a-c)(b+c)-(a-b)(a+c)(b-c)$.

72. Multiply $a^2+2ab+b^2-c^2$ by $a^2-2ab+b^2+c^2$; and shew that the result may be expressed under the form $(a^2-b^2)^2-c^2(c^2-4ab)$.

73 Find the coefficient of x in the product of $x-a$, $x-2b$ and $x-3c$

74. Factorise $(a-b)x^3+(x-a)b^3-(x-b)a^3$

75. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, then

$$ad=bc.$$

76. If $2s=a+b+c$, prove that [*App*]

$$a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=a(s-a)^2+b(s-b)^2+c(s-c)^2$$

77. Shew that

$$(a+b)^2(x-y)^2+4xy(a^2+b^2)-4ab(x^2+y^2)=(a-b)^2(x+y)^2.$$

78. Solve $(x-2a)^2-(x+3c)^2=0$

79. A person bought two casks of beer, one of which held exactly 3 times as much as the other From each of these he drew 4 gallons,

and then found that there were 4 times as many gallons remaining in the first as in the other How many gallons were there in each at first?

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80 Simplify $5a-3\{a-b-2(a-b)\}$, and find its value when $a=1$, and $b=\frac{2}{3}$

81 Add together $16a^2-7ab-8b^2+3c$, $6b^2-8c+ab$ and $12ab-8a^2+5c$, and divide the sum by $a+b$

82 From $(1+x)^3+(1+x)^2y+(1+x)y^2-y^3$, take $4(1+x)^2y-2(1+x)y^2+1$, and find the expression by which this remainder must be divided that the quotient may be $x-y$

83 Multiply $3+\,01x$ by $1-x$, and find the value of the product when $x=1$.

84 Write down the coefficients of x^3 and x^5 in the product of
$$3x^4-5x^3+7x^2-9x+11 \text{ and } x^5+6x^4+12x^3-12x^2-6x+$$

85 Resolve into factors $abc(a^3+b^3+c^3)-(b^3c^3+c^3a^3+a^3b^3)$

86 Find the 5th term in the quotient of $1-x$ divided by $2-3x+5x^2$.

87 If $a^2+b^2=1=c^2+d^2$, shew that $(ac-bd)^2+(ad+bc)^2=1$

88 Prove that $(a-b)^4+(b-c)^4+(c-a)^4$
$$=2\{(a-b)^2(b-c)^2+(b-c)^2(c-a)^2+(c-a)^2(a-b)^2\}$$ [See *App*]

89 If $x=b+c$, $y=c+a$, $z=a+b$, prove that
$$x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab.$$

90 If $a^2=y+z$, $b^2=z+x$, $c^2=x+y$, and $2s=a+b+c$, shew that
$$s(s-a)(s-b(s-c)=\tfrac{1}{4}(yz+zx+xy)$$

91 If $(by-cx)^2=(b^2-ac)(y^2-cz)$, prove that
$$(bx-ay)^2=(b^2-ac)(x^2-az)$$ [See *App*]

92 Solve $(x+a)^2+(x+b)^2=2(x+c)^2$

93 A man at a party at cards, betted 3*s* to two upon every deal, after 20 deals he won 5*s* How many deals did he win?

CHAPTER XIII

HIGHEST COMMON DIVISOR.

150 Definitions When one quantity divides another without remainder, it is said to be a *Measure* of the latter Thus 2, 4 and 8 are the measures of 16; 5, a, b, $5a$, $5b$ and ab are the measures of $5ab$

For reasons given below, we shall use the word "divisor" for "measure," meaning thereby an "exact divisor."

A COMMON MEASURE or COMMON DIVISOR of two or more expressions is one which divides each of the latter without remainder. Thus a is a common divisor of $2a^2$, $3a^3b$ and a^2b^2.

The HIGHEST COMMON DIVISOR of two or more expressions is the expression of the *highest dimensions* which divides each of the given expressions without remainder. Thus a^2 is the Highest Common Divisor of $2a^2$, $3a^3b$ and a^2b^2.

The Highest Common Divisor is termed by some writers Greatest Common Measure, and by others Highest Common Factor. The corresponding abbreviations are H. C. D., G. C. M. and H. C. F.

Note 1. If A and B be two expressions whose H. C. D. is D; then *every divisor of D is a common divisor of A and B*; and conversely, *every common divisor of A and B is a divisor of D.* For let $A=amnp$ and $B=bmnp$, where a and b have no common divisor. Thus the common divisors of A and B are m, n and p, and by definition the H. C. D. of A and B is mnp, that is, $D=mnp$. Hence each of m, n and p, divides A and B, and also divides D.

Note 2. If A, B, C, D, ... be any number of expressions whereof A divides each of B, C, D, ... without remainder, then A is evidently the H. C. D. of A, B, C, D, ..., for no expressions *higher* than A can divide A.

REMARK. The Arithmetical term Greatest Common Measure is not appropriate in Algebra. Here we have to see whether the expression, found as the G. C. M. is of *highest possible degree*, without any reference to its *numerical value*. Indeed we cannot ascertain this value, unless we know the numerical values of the letters contained in the G. C. M. Hence the Algebraic G. C. M. will not always be the Arithmetical G. C. M. Thus $x+a$, which is the Algebraical G. C. M. of x^2-a^2, and $(x+a)^2$ [as the student will presently see], is not their Arithmetical G. C. M. when $x=12$, $a=4$. In this case x^2-a^2, or its value 128, is the required G. C. M. It has been thus thought safer to use the term H. C. D. in Algebra.

151. H. C. D. of Monomials. Since the required H. C. D. must be that factor of *highest dimensions which is common* to all the proposed expressions [Definition], we have the following

RULE.—*Take all the factors common to the given expressions and raise each to the lowest power in which it occurs; the product of these powers will be the required* H. C. D.

If there be *numerical* coefficients, find their G. C. M. as in Arithmetic, and put it before the H. C. D.

Examples

Ex 1. Find the H C D of a^2b^2c and a^3bd

Here the common factors are a and b, and the *lowest* power of a is a^2 and that of b is b,

$$\text{H C D required} = a^2 \times b = a^2b$$

Ex 2 Find the H C D of $2a^2x^3y^2$ and $a^3x^2z^3$

Here the common factor are a and x, and the *lowest* powers of these which occur in the given expressions are a^2 and x^2 respectively

$$\text{H C D required} = a^2 \times x^2 = a^2x^2$$

Ex 3 Find the H C D of $3ax$ and $6xy$

Here the only common factor is x, and its lowest power is x, also the G C M of 3 and 6 is 3,

$$\text{H C D required} = 3 \times x = 3x$$

Ex 4 Find the H C D of $60a^4b^3x$, $72a^2by^2$ and $84a^3b^2z$

The common factors here are a and b, the lowest powers of these occurring in the proposed expressions are a^2 and b respectively, also 12 is the G C M of the numerical co-efficients 60, 72, 84,

$$\text{H C D} = 12 \times a^2 \times b = 12a^2b$$

Find the Highest Common Divisor of

5 a^2b and ab^2. 6 $2a^2x$ and abx^2 7 $2a^3b$ and $3a^2c$

8 a^6b^2 and a^2a^6 9 $a^2b^3c^4$ and ab^2c^3 10 $75ac^2$ and $30a^2x$

11 $105ab^2x$ and $84ax$ 12 ab^3, a^2bx and a^3b^2.

13 $8xy$, $12xz$ and $20yz$ 14. $16m^2n^3$, $48m^3n$ and $80m^2p$

15. $35a^2b^3x^3y^4$ and $49a^2b^4x^4y^3$

16 $12x^3yz$, $18xy^3z^2$, $27x^2y^2z^3$ and $33x^2yz^4$

17 $5a^4b^3c^2y^2$, $15a^3c^3d^3x$, $10a^5b^4c^5y$ and $18a^2b^2c^4x^2$

18 $12m^3p^3qx^2$, $16a^3m^4x^3y$, $40a^2m^2q^2x^5$, $32b^3m^5x^4y^3$ and $20m^4px^3y^2$

Note In a similar way, we can find the H C D of a Monomial and a Binomial or a Polynomial

Ex 19 Find the H C D of $2a^2x$ and ax^2-axy.

Now $2a^2x = 2a\,ax$ and $ax^2 - axy = ax(x-y)$, and the common factors are a and x

$$\therefore \text{H C D required} = a \times x = ax$$

Find the Highest Common Divisor of

20. $6m^3n^3p^2$ and $8mn^3p^2 - 12m^2p^3$

21. $2ab^2x$ and $3a^2x + 3ab^2x$

Find the Highest Common Divisor of

22. $72m^3n^3x^3$ and $54m^4n^3x^5-36m^2n^4y^3$.

23. $108a^2m^3x^2$ and $72a^2m^2x-54a^2mx^2+90a^2m^2x$

152 H C D of expressions readily resolved into Factors The method being precisely the same as that of the last article, we here follow the same rule

Examples.

Ex. 1. Find the H. C D of a^2-ab and $ab-b^2$

Now $a^2-ab=a(a-b)$, and $ab-b^2=b(a-b)$.

The only common factor is $(a-b)$ and its lowest power which occurs in the given expressions is $(a-b)$;

$\therefore$ H C. D required $=a-b$

Ex. 2 Find the H C D of $6a^2-12ax$ and $8ax-16x^2$

Here $6a^2-12ax=6a(a-2x)$ and $8ax-16x^2=8x(a-2x)$.

Thus the highest common factor is $(a-2x)$, also the G C M. of 6 and 8 is 2;

$\therefore$ H C D required $=2(a-2x)$

Find the Highest Common Divisor of

3 a^2+ax and $ax+x^2$ 4 a^2-ax and a^3-ax^2.

5 $2a^2x-2abx$ and $2a^2x-4abx$ 6 x^2+2cx and $3ax+6ac$.

7 $ax+x^2$ and $5ax+2xy$ 8 a^2-1 and $ab-b$

9 $5x^2-10xy$ and $2xz-4yz$ 10. x^2y+xy^2-xyz and y^2z-yz^2+xyz.

Ex. 11 Find the H. C. D. of $12a^3(x+1)^2(x-2)^2$ and $15a^2b(x+1)^7(x-2)$.

The common factors, each raised to the lowest power in which it occurs in the given expressions, are a^2, $(x+1)^2$ and $(x-2)$, and the G. C M of the numerical co-efficients is 3,

$$\therefore \text{ H. C D. required } =3\times a^2\times(x+1)^2\times(x-2)$$
$$=3a^2(x+1)^2(x-2)$$

Find the Highest Common Divisor of

12. $(a+b)^3x^3$ and $(a+b)^2x^4$. 13 $4a^2(a-b)^2$ and $5a(a-b)^4$

14 $a^3(p+q)^4x$ and $2ab(p+q)x^4$

15 $4(a-1)^3(x-a)^2$ and $8(a-1)^2(x-a)^3$.

16. $18a^2b^2(x-a)^4(x^2-y^2)$, $24b^2c^2(x-a)^3(x^2-y^2)^2$, and $42a^2bc(x-a)^6(x^2-y^2)^3$.

Ex. 17 Find the H. C. D. of $a^2x^3+2a^3x^2$ and $a^2x^4-4a^4x^2$.

First expn $=a^2x^2(x+2a)$,

Second expn $=a^2x^2(x^2-4a^2)=a^2x^2(x+2a)(x-2a)$.

Hence a^2, x^2 and $(x+2a)$ are the common factors ;

H. C. D. required $=a^2\times x^2\times(x+2a)=a^2x^2(x+2a)$.

Ex. 18 Find the H. C. D. of $8x^4-8x$ and $12x^4-24x^3+12x^2$.

First expn $=8x(x^3-1)=8x(x-1)(x^2+x+1)$,

Second expn $=12x^2(x^2-2x+1)=12x^2(x-1)^2$.

The common factors taken to lowest powers are x and $(x-1)$, and the G. C. M. of 8 and 12 is 4 ;

$\therefore$ H. C. D. required $=4\times x\times(x-1)=4x(x-1)$.

Find the Highest Common Divisor of

19. x^2-y^2 and x^3-y^3.
20. x^3-xy^2 and $x^2-2xy+y^2$.
21. x^3+y^3 and $x^3+2x^2y+xy^2$.
22. x^2-4 and $x^3+6x^2+12x+8$.
23. $a^4x^2-a^2x^4$ and $a^4x^3+a^3x^4$.
24. $a^3x^4-a^3x$ and $a^2x^4-a^2x^2$.
25. $1-16x^2$ and $(1+4x)^2$.
26. a^4-x^4 and $a^4+2a^2x^2+x^4$.
27. $16(a+x)^2$ and $40(a^2-x^2)$.
28. $9(a^3+b^3)$ and $6(a^2-b^2)$.
29. $12(a^2x^2-4)$ and $16(a^2x^2+4ax+4)$.
30. $(2x^2-8ax)^2$ and $16(x^4-16a^2x^2)$.
31. $16(x^4-x^3+x^2)$ and $56(x^4+x^2+1)$.
32. $15ab(a^2-b^2)^2$ and $27a^2(a-b)(a+b)^2$.
33. $10(x+1)^2(x^2-4)$ and $15(x^2-1)(x+2)^2$.
34. $8a^2(a-x)^2$, $12ax(a^2-x^2)$ and $16a^3x^2(a^3-x^3)$.
35. $6(a^3+b^3)(a-b)^3$, $9(a^4-b^4)(a-b)^2$ and $12(a^2-b^2)^3$.
36. $8(x+1)(x^3+8)$ and $4(x+1)(x^2-5x-14)$.

Ex. 37 Find the H. C. D. of $9x^2+3x-2$ and $15x^3-14x^2+3x$.

First expn $=(3x-1)(3x+2)$,

Second expn $=x(15x^2-14x+3)=x(3x-1)(5x-3)$,

$\therefore$ H. C. D. required $=3x-1$.

Ex. 38 Find the H. C. D. of $6ac+9bc-2ad-3bd$

and $10af+16ag+15bf+24bg$.

First expn $=(6ac+9bc)-(2ad+3bd)$

$=3c(2a+3b)-d(2a+3b)=(2a+3b)(3c-d)$;

Second expn. $=(10af+15bf)+(16ag+24bg)$
$=5f(2a+3b)+8g(2a+3b)=(2a+3b)(5f+8g)$,
$\therefore$ H. C. D required $=2a+3b$

Ex. 39 Find the H C D of $2a^4x-5a^3x^2-3a^2x^3$
and $4a^3x^2+14a^2x^3+6ax^4$.

First expn. $=a^2x(2a^2-5ax-3x^2)=a^2x(2a+x)(a-3x)$,
Second expn $=2ax^2(2a^2+7ax+3x^2)=2ax^2(2a+x)(a+3x)$,
$\therefore$ H C D required $=ax(2a+x)$.

Ex 40 Find the H C D of a^2-ax, $ay-xy$ and $ax-x^2$
Now $a^2-ax=a(a-x)$, $ay-xy=y(a-x)$, $ax-x^2=x(a-x)$;
$\therefore$ H C. D required $=a-x$

Find the Highest Common Divisor of

41. x^2+5x+6 and x^2-2x-8
42. $2x^2+x-1$ and $6x^2+x-2$
43. x^2+3x+2 and x^3+2x^2+3x+6.
44. x^3+1 and x^3+mx^2+mx+1
45. a^4-x^4 and $a^4+a^3x+ax^3-x^4$.
46. $3x^3-2x^2-x$ and $6x^2-x-1$.
47. a^3-3a+2 and a^3+3a^2-4.
48. $bc-2b^2-ac+2ab$ and $2b^2-2ab+3bc-3ac$
49. $2a^2+5a-4ab-10b$ and $4a^2-6ac+10a-15c$
50. $2x^2-ax+ab-2bx$ and $4x^2-2cx+ac-2ax$
51. $2x^2+3xy+6x+9y$ and $3x^2-2xy+9x-6y$
52. $9x^2-3xy-6x+2y$ and $6x^3-4x^2-3xy^2+2y^2$.
53. $3x^2-5x+2$ and $4x^3-x-4x^2+1$
54. x^3-8x+3 and x^6+3x^5+x+3
55. $2a^4-11a^2b^2+12b^4$ and $3a^5-48ab^4$
56. $8a^2b^2-10ab^3+2b^4$ and $9a^4b-9a^3b^2+3a^2b^3-3ab^4$.
57. $2+2x$, $1-x^2$ and $1+x+y+xy$
58. x^2-9, $(x+3)^2$ and x^2+x-6.
59. a^2-c^2, $a^2-2ac+c^2$ and $a^2+ab-ac-bc$
60. $1-x^2$, x^3+1 and $1-x-2x^2$
61. x^2-x-2, x^2+x-6 and x^2-3x+2
62. $2x^2-5x+2$, $3x^2-2x-8$ and $4x^2-5x-6$
63. $px^2-(p+1)x+1$ and $qx^2-(q-1)x-1$.
64. $2x^2+xy-3y^2$ and $3x^3-x^2y-xy^2-y^3$
65. $x^6+x^2y-x^4y^2-y^3$ and $x^4-x^2y-x^2y^2+y^3$
66. $20x^4+x^2-1$ and $25x^4+5x^3-x-1$.

Find the Highest Common Divisor of

67 $x^4-(m+1)x^3+(m+1)x-1$ and $x^4-(n+1)x^3+(n+1)x-1$

68 $ay(x^3+b^3)+bx(by^2+a^2x)$ and $ax(y^3+b^3)+by(bx^2+a^2y)$

69 $1-abx^3+(b-a^2)x^2$ and $1+acx^3-(c-a^2)x^2-2ax$

70 $ab+2a^2-3b^2-4bc-ac-c^2$ and $9ac+2a^2-5ab+4c^2+8bc-12b^2$

The method of this article may be employed only in the case of expressions whose factors are *readily* found It is necessary therefore to lay down a *general rule* for finding the H C D of Polynomials The investigation of this rule depends on two *Lemmas* which we establish in the next two articles

153 Theorem *If one expression divide two others, it will divide the sum* or *difference of any integral multiples whatever of them*

Let D divide A and B, then by hypothesis $A=mD$, $B=nD$, therefore $pA=mpD$, $qB=nqD$, therefore $pA\pm qB=(mp\pm nq)D$ Thus D divides $pA\pm qB$, *i e*, the sum or difference of any integral multiples of A and B

From the above result, it is easily seen that

Corollary 1 D divides $A+B$ or $A-B$, . here $p=q=1$,

Corollary 2 D divides $A+qB$ or $A-qB$, . here $p=1$,

Corollary 3 D divides $pA+B$ or $pA-B$, here $q=1$

154 Theorem *If an expression B divide an expression A leaving a remainder R, then the* H C D *of B and R will be the* H C D of *A and B*

Let Q be the quotient when A is divided by B, thus we have [§ 96]

$$A=BQ+R \qquad \text{(i)}$$

Now every common divisor of B and R, will divide $BQ+R$ [§ 153, *Cor* 3] or A, that is, every common divisor of B and R will divide A and B

Again from (i), we have by transposition

$$R=A-BQ \qquad \text{(ii)}$$

Hence every common divisor of A and B will divide $A-BQ$ [§ 153, *Cor* 2] or R, that is, every common divisor of A and B will divide B and R

Thus B and R have exactly the same divisors as A and B, that is, the H C D of B and R is the H C D of A and B

155 H C D of two Polynomials Let A and B be two expressions, say, in x, arranged according to the *descending* powers of x, and let A be not of lower dimensions than B

Divide A by B and let p be the quotient and C the remainder; divide B by C and let q be the quotient and D the remainder; and so on

$$\begin{array}{l} B)\,A\,(p \\ \underline{pB} \\ C)\,B\,(q \\ \quad\underline{qC} \\ \quad D)\,C\,(r \\ \quad\quad\underline{rD} \\ \quad\quad\ 0 \end{array}$$

Now since A and B are arranged in descending powers of x, it is clear that each successive remainder shall be of a lower degree than the corresponding divisor [Def § 96], and consequently we must at last arrive at a stage of division where there shall be either *no remainder*, or if there be any it shall be a *constant* (i e, a quantity not involving x) In the latter case, of course, A and B *have no* H C D

Suppose, then, for the sake of simplicity, there is no remainder at the next stage of the process

Divide C by D, and let r be the quotient

Thus from § 154, the H C D of B and C will be the H. C D of A and B, and the H C D of C and D will be the H C D of B and C, therefore the H. C D of C and D will be the H C D of A and B, but the H C D of C and D is D itself, for no expression *higher* than D can divide D, thus D is the H C D required

Hence we have the following rule for finding the H C D. of two polynomials —

RULE —*Arrange the polynomials according to the* DESCENDING *powers of some common letter divide the one of a higher degree by the other, take the remainder, if any, after this division for a new divisor and the preceding divisor for dividend, and so on, until there is no remainder; the last divisor will be the* H C D *required*

If there be no remainder after the first division, then one of the proposed expressions, *viz*, that of a lower degree, will be the H C D required [§ 150, Note 2]

Note 1 If the chain of division for finding the H C D. *terminates with a zero remainder, the last divisor will be the* H C D *required*; but if it *terminates with a constant as remainder, the polynomials have no* H C D

Note 2 In the chain of division for finding the H C, D, *the* H C D *of any divisor and the corresponding dividend, will always be the* H. C D *required.*

Note 3 If $C, D,$ be the successive remainders in the process for finding the H C. D of two expressions A and B, we have

$$C=A-pB \ldots \quad \text{(i)},$$
$$D=B-qC \ \ldots \text{(ii)}.$$

Form (i), we see that every common divisor of A and B is a divisor of C, that is, a common divisor of B and C, and from (ii), that every

common divisor of B and C is a divisor of D; hence every common divisor of A and B is a divisor of D

Thus if D be the H C D of A and B, it follows that *every common divisor of two expressions is a divisor of their* H C D

This truth is of course very plain in the case of Monomials.

Examples

Ex 1 Find the H. C. D of x^2-3x+2 and x^3-4x^2+6x-4.

$$x^2-3x+2\ )\ x^3-4x^2+6x-4\ (\ x-1$$
$$\underline{x^3-3x^2+2x}$$
$$-x^2+4x-4$$
$$\underline{-x^2+3x-2}$$
$$x-2$$

Take this remainder for divisor, thus

$$x-2\ )\ x^2-3x+2\ (\ x-1$$
$$\underline{x^2-2x}$$
$$-x+2$$
$$\underline{-x+2}$$

$$\text{H C D required} = x-2$$

Ex 2. Find the H C D of x^2+2x-3 and $2x^3+5x^2-5x-6$

$$x^2+2x-3\ )\ 2x^3+5x^2-5x-6\ (\ 2x+1$$
$$\underline{2x^3+4x^2-6x}$$
$$x^2+\ x-6$$
$$\underline{x^2+2x-3}$$
$$-x-3$$

Change the sign of the remainder, which then becomes $x+3$;

$$x+3\ )\ x^2+2x-3\ (\ x-1$$
$$\underline{x^2+3x}$$
$$-x-3$$
$$\underline{-x-3}$$

$$\therefore \text{H C D required} = x+3$$

Find the Highest Common Divisor of

3 x^2-2x-3 and x^3-2x^2-2x-3

4 x^2+3x-4 and x^3+5x^2+3x-9.

Find the Highest Common Divisor of

5 $3x^2-11x-4$ and $6x^3-25x^2+3$

6 $2x^3+x^2-5x-3$ and $8x^3+6x^2-21x-18$.

7 $x^3+6x^2+2x-15$ and $x^3+5x^2-2x-10$

8. $6x^3-19x^2+13x-2$ and $12x^3-32x^2+19x-2$.

9 $x^4+x^3-x^2-2x-2$ and $x^4+2x^3-x^2-2x-3$.

10, $x^4-4x^3-30x^2-28x+17$ and $3x^4-11x^3-86x^2-81x+49$.

Ex 11 Find the H C D of $2x^3+7x^2+2x-3$ (i),
and $3x^3+8x^2-2x+3$ (ii).

The given expressions are of the same degree, hence it is immaterial which of these is considered the dividend. If we take (ii), it is evident that the first term in the quotient will be a fraction, *viz.*, $\frac{3}{2}$, to avoid which we multiply (ii) by 2, which is the coefficient of x^3 in (i) and which is *not a factor* of the divisor (i)

$$
\begin{array}{r|l}
 & 3x^3+8x^2-2x+3 \\
 & \underline{2} \\
2x^3+7x^2+2x-3\ ) & 6x^3+16x^2-4x+6\ (\ 3 \\
 & \underline{6x^3+21x^2+6x-9} \\
 & \underline{-5\,|\,-5x^2-10x+15} \\
 & x^2+2x-3.
\end{array}
$$

The remainder has a factor -5, *which is not a factor of* (i) the divisor, and since the required H C D will be the H C D. of (i), and this remainder [NOTE 2], the factor -5 cannot affect the required H C D., and may therefore be rejected The required H. C D. will therefore be the H C D of (i) and x^2+2x-3

$$
\begin{array}{rl}
x^2+2x-3\ ) & 2x^3+7x^2+2x-3\ (\ 2x+3 \\
 & \underline{2x^3+4x^2-6x} \\
 & 3x^2+8x-3 \\
 & \underline{3x^2+6x-9} \\
 & 2\,|\,2x+6 \\
 & x+3
\end{array}
$$

For the reason given above, *reject the factor* 2

$$
\begin{array}{rl}
x+3\ ) & x^2+2x-3\ (\ x-1 \\
 & \underline{x^2+3x} \\
 & -x-3 \\
 & \underline{-x-3}
\end{array}
$$

$\therefore$ required H. C D $=x+3$ [NOTE 1]

REMARK. From this example it is seen that we can *introduce* or *reject* with certain restrictions, a factor at any stage of our operation. For as the H. C. D. of two polynomials will always be the H. C. D. of any divisor and the corresponding dividend [NOTE 2], *we can multiply a dividend when necessary, by a quantity which is not a factor of the divisor,* and *vice versa.* Similarly *we can remove from a divisor, a factor which is not a factor of the dividend,* and *vice versa.* In the latter case, however, *if a factor be common to both the divisor and dividend, we may remove and reserve it, to be introduced afterwards into the H. C. D. of the remaining factors.* [See Ex. 34, below.]

Ex. 12. Find the H. C. D. of $2x^4-3x^3-2x^2+6x+3$ (i),
and $2x^4-7x^3-10x^2+x+2$ (ii).

$$
\begin{array}{r|l}
2x^4-3x^3-2x^2+6x+3 \;) & 2x^4-7x^3-10x^2+x+2 \;(\,1 \\
 & \underline{2x^4-3x^3-2x^2+6x+3} \\
 & -1 \,|\, -4x^3-8x^2-5x-1 \\
 & \quad 4x^3+8x^2+5x+1
\end{array}
$$

$$
\begin{array}{r|l}
4x^3+8x^2+5x+1 \;) & 2x^4-3x^3-2x^2+6x+3 \\
 & \underline{2} \\
 & 4x^4-6x^3-4x^2+12x+6 \;(\,x \\
 & \underline{4x^4+8x^3+5x^2+x} \\
 & \quad -14x^3-9x^2+11x+6 \\
 & \quad \underline{2} \\
 & \quad -28x^3-18x^2+22x+12 \;(\,-7 \\
 & \quad \underline{-28x^3-56x^2-35x-7} \\
 & \qquad 19 \,|\, 38x^2+57x+19 \\
 & \qquad\quad 2x^2+3x+1
\end{array}
$$

$$
\begin{array}{r|l}
2x^2+3x+1 \;) & 4x^3+8x^2+5x+1 \;(\,2x+1 \\
 & \underline{4x^3+6x^2+2x} \\
 & \quad 2x^2+3x+1 \\
 & \quad \underline{2x^2+3x+1}
\end{array}
$$

$\therefore$ H. C. D. required $=2x^2+3x+1$.

Find the Highest Common Divisor of

13. $2x^2-13x+15$ and $3x^2-13x-10$.
14. $5x^2+3x-2$ and $7x^2+4x-3$.
15. $2x^2+x-1$ and $2x^3+5x^2+4x+1$.
16. $2x^2+5x-3$ and $6x^3+31x^2+31x-24$.

Find the Highest Common Divisor of

17 $3x^2+16x-12$ and $x^3+7x^2+4x-12$.

18 x^3-6x^2+7x+4 and x^3-2x^2-7x-4.

19 x^3+4x^2-4x+5 and x^3+7x^2+9x-5.

20 $2x^3+9x^2+7x-6$ and x^3+2x^2-5x-6

21. $x^3-19x^2+119x-245$ and $3x^2-38x+119$

22 $4x^3+x-1$ and $2x^3+3x^2-1$

23. $2x^3-19x+3$ and $3x^3-10x^2+9$

24 $2x^3-12x^2-15x+7$ and $x^3-9x^2+13x+7$.

25. $2x^3+x^2-3x+6$ and $3x^3+7x^2-3x-10$

26. $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$.

27. $2x^3-x^2-x-3$ and $x^5-x^3-4x^2-3x-2$

28. $7x^3-6x^2-18x+4$ and $14x^3-19x^2-32x+28$

29. $4x^4-9x^2+6x-1$ and $6x^3-7x^2+1$

30. $3x^3+14x^2+12x+16$ and $2x^4+7x^3-4x^2-x-4$.

31 $6x^4-2x^3+7x^2-x+2$ and $6x^4-12x^3+21x^2-6x+9$.

32 $2x^4+17x^3+30x^2+8x-5$ and $x^4+4x^3-18x^2-29x-10$.

33 $x^5-x^3+4x^2-3x+2$ and $5x^4-3x^2+8x-3$

Ex. 34 Find the H C D of $2x^3+3x^2-35x$ and $10x^3-33x^2-7x$.

Evidently x is a factor of the given expressions; remove and reserve it, we have thus the two remaining factors $2x^2+3x-35$ and $10x^2-33x-7$. Hence the required H C D will be the H C D. of these two factors *multiplied by the reserved factor x* [See Remark, Ex 11]

The student will easily find the H. C. D of these factors to be $2x-7$.

$\therefore$ H C D required $=x(2x-7)=2x^2-7x$.

Ex. 35. Find the H C D of $8a^5b^2-48a^4b^3+88a^2b^4-48a^2b^5$

and $4a^4b^3-36a^3b^4+104a^2b^5-96ab^6$.

First expression $=8a^2b^2(a^3-6a^2b+11ab^2-6b^3)$.

Second expression $=4ab^3(a^3-9a^2b+26ab^2-24b^3)$

Thus evidently $4ab^2$ is a factor of *highest* degree common to the proposed expressions. Hence the required H. C. D. will be $4ab^2$ *multiplied by the* H. C. D. of $a^3-6a^2b+11ab^2-6b^3$ and

$a^3-9a^2b+26ab^2-24b^3$ [See Remark, Ex. 11]

$$
\begin{array}{r|l}
a^3-6a^2b+11ab^2-6b^3 \;) & a^3-9a^2b+26ab^2-26b^3 \;(\; 1 \\
 & \underline{a^3-6a^2b+11ab^2-6b^3} \\
-3b & \underline{-3a^2b+15ab^2-18b^3} \\
 & \quad a^2-5ab+6b^2
\end{array}
$$

$$
\begin{array}{r|l}
a^2-5ab+6b^2 \;) & a^3-6a^2b+11ab^2-6b^3 \;(\; a-b \\
 & \underline{a^3-5a^2b+6ab^2} \\
 & \quad -a^2b+5ab^2-6b^3 \\
 & \quad \underline{-a^2b+5ab^2-6b^3}
\end{array}
$$

$\therefore$ the required H. C. D. $=4ab^2(a^2-5ab+6b^2)=4a^3b^2-20a^2b^3+24ab^4$.

Find the Highest Common Divisor of

36. $18x^3+45x^2+63x+54$ and $24x^3+24x^2+6x+36$

37. $4x^3-8ax^2-20a^2x+24a^3$ and $6x^3+24ax^2+6a^2x-36a^3$.

38. $3x^4-8x^3+8x$ and $4x^4-19x^2+6x$

39. $12x^4-14x^3+2x$ and $8x^4+10x^2-6x$

40. $3x^5+5x^4+2x^3+8x^2$ and $x^5-13x^3-14x^2+8x$

41. $8a^4b^2-34a^2b^4+24ab^5$ and $12a^5b-26a^4b^2+18a^2b^4$

Ex. 42. Find the H. C. D. of $6x^3+x^2-15x$ (i),

and $2x^3+x^2-10x+6$ (ii)

Here x is a factor of (i) but not of (ii), hence it will not be a factor of the required H. C. D., therefore remove it, [see Remark, Ex 11] We have thus the expressions $6x^2+x-15$ and $2x^3+x^2-10x+6$, whose H. C. D. will be found to be $2x-3$. This therefore will be the required H. C. D. of (i) and (ii).

Ex. 43. Find the H. C. D. of $x^4-9a^2x^2+10a^3x$ (i),

and $ax^3-a^2x^2-4a^4$ [*Cal.*, 1873] (ii)

First expression $=x(x^3-9a^2x+10a^3)$

Second expression $=a(x^3-ax^2-4a^3)$

Thus x is a factor of (i) but not of (ii), and a is a factor of (ii) but not of (i). Hence x and a cannot enter the required H. C. D. and must

therefore be rejected [see Remark, Ex. 11]. The required H. C. D. will thus be the H. C. D. of $x^3-9a^2x+10a^3$ and $x^3-ax^2-4a^3$.

$$x^3-ax^2-4a^3 \;)\; x^3-9a^2x+10a^3 \;(\; 1$$
$$\underline{x^3-ax^2-4a^3}$$
$$a \,|\, \underline{ax^2-9a^2x+14a^3}$$
$$x^2-9ax+14a^2 \;(\; x^3-ax^2-4a^3 \quad (\; x+8a$$
$$\underline{x^3-9ax^2+14a^2x}$$
$$8ax^2-14a^2x-4a^3$$
$$\underline{8ax^2-72a^2x+112a^3}$$
$$58a^2 \,|\, \underline{58a^2x-116a^3}$$
$$x-2a$$

$$x-2a \;)\; x^2-9ax+14a^2 \;(\; x-7a$$
$$\underline{x^2-2ax}$$
$$-7ax+14a^2$$
$$\underline{-7ax+14a^2}$$

$\therefore$ H. C. D. required $=x-2a$.

Find the Highest Common Divisor of

44. $2x^3-10x^2+20x-16$ and $3x^3-12x^2+21x-18$.
45. $8x^3-10x+4$ and $6x^3+21x^2-6$.
46. $2x^4+8x^3-7x^2+15x$ and $3x^4+17x^3+12x^2+13x+15$.
47. $6x^5-21x^4y+6x^3y^2-6x^2y^3-3xy^4$
 and $5y^5-10xy^4+20x^2y^3-15x^3y^2+10x^4y$.
48. $a^5-4a^4b+27ab^4$ and $a^4b-4ab^4+3b^5$.

Ex. 49. Find the H. C. D. of

$$x^3-(2a+b)x^2+(a^2+ab+b^2)x-(a+b)b^2$$

and

$$x^3-(a+2b)x^2+(a^2+ab+b^2)x-a^2(a+b).$$

$$x^3-(2a+b)x^2+(a^2+ab+b^2)x-(a+b)b^2 \;)$$
$$x^3-(a+2b)x^2+(a^2+ab+b^2)x-a^2(a+b) \;(\; 1$$
$$\underline{x^3-(2a+b)x^2+(a^2+ab+b^2)x-(a+b)b^2}$$
$$(a-b) \,|\, \underline{(a-b)x^2-(a+b)(a^2-b^2)}$$
$$x^2-(a+b)^2$$

$$x^2-(a+b)^2 \;)\; x^3-(2a+b)x^2+(a^2+ab+b^2)x-(a+b)b^2 \;(\; x-(2a+b)$$
$$\underline{x^3-(a+b)^2x}$$
$$-(2a+b)x^2+(2a^2+3ab+2b^2)x-(a+b)b^2$$
$$\underline{-(2a+b)x^2+(a+b)^2(2a+b)}$$
$$(2a^2+3ab+2b^2) \,|\, \underline{(2a^2+3ab+2b^2)x-(a+b)(2a^2+3ab+2b^2)}$$
$$x-(a+b)$$

$$x-(a+b) \;)\; x^2-(a+b)^2 \;(\; x+(a+b)$$
$$\underline{x^2-(a+b)x}$$
$$(a+b)x-(a+b)^2$$
$$\underline{(a+b)x-(a+b)^2}.$$

Find the Highest Common Divisor of

50. $x^3-6a^2x-9a^3$ and $2x^3-10ax^2+9a^2x+9a^3$.

51. $3x^3-15x^2y+xy^2-5y^3$ and $6x^4-25x^2y^2-9y^4$

52. $7a^3-6a^2b-18ab^2+4b^3$ and $14a^3-19a^2b-32ab^2+28b^3$

53. $3x^3-22x-15$ and $5x^4-17x^3+18x$

54. $x^4-9x^2-30x-25$ and $x^5+x^4-7x^3+5x$

55. $4x^6+8x^5-56x^4-12x^3$ and $6x^3-6x^2-36x$

56. $6x^4y\pm x^3y^2-xy^4$ and $4x^3-6x^2y-4xy^2+3y^3$

57. $6a^4x^3-10a^2x^4y-9a^3x^2y^2+15ax^3y^3$
and $10a^4xy^2-15a^3y^4+8a^2x^2y^3-12axy^5$.

58. $27a^5b^2-18a^4b^3-9a^3b^2$ and $36a^6b^3-18a^5b^2-27a^4b^3+9a^3b^2$.

59. $2x^5-4x^4+8x^3-12x^2+6x$ and $3x^5-3x^4-6x^3+9x^2-3x$

60. $x^4+4x^3y-21x^2y^2+10xy^3-y^4$ and $x^4+12x^3y+33x^2y^2-12xy^3+y^4$.

61. $2x^3+(2a-9)x^2-(9a+6)x+27$ and $2x^2-13x+18$

62. $3x^2-(4a+2b)x+2ab+a^2$ and $x^3-(2a+b)x^2+(2ab+a^2)x-a^2b$.

63. $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$

64. $x^4+px^3-(a-1)x^2-apx-a$ and $x^4-px^3-(a+1)x^2+apx+a$.

65. $x^4-2a(a-b)x^2+(a^2+b^2)(a-b)x-a^2b^2$
and $x^4-(a-b)x^3+(a-b)b^2x-b^4$

66. $x^4+(p-a)x^3-(ap+q+1)x^2-(p-aq)x+q$
and $x^4+(q-a)x^3-(aq+p+1)x^2-(q-ap)x+p$.

67. $6x^4-x^3y-3x^2y^2+3xy^3-y^4$ and $9x^4-3x^3y-2x^2y^2+3xy^3-y^4$.

68. $3x^5-10x^3+15x+8$ and $x^5-2x^4-6x^3+4x^2+13x+6$

69. $11x^4-9ax^3-a^2x^2-a^4$ and $13x^4-10ax^3-2a^2x^2-a^4$

70. $2x^4+x^3+x^2+3x-2$ and $6x^4-7x^3+4x^2-3x+1$

71. $6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$

72. $x^5+x^4-8x^3+12x^2-x-21$ and $x^5-3x^3+9x^2-4x-3$

73. $2x^5-11x^2-9$ and $4x^5+11x^4+81$. [See Ex. 4, § 157]

74. $a^5-3a^3b^2-8b^5$ and $a^5-5a^2b^3-12b^5$

156. H. C. D. of several Polynomials. Let $A, B, C, D, \ldots$ be any number of Polynomials.

Let G be the H. C. D. of A and B. Now every common divisor of A and B is a divisor of G [Note 3, § 155]; therefore every common divisor of G and C is a common divisor of A, B and C; thus the H. C. D. of G and C (say, H) will be the H. C. D. of A, B and C. Similarly it may be proved that the H. C. D. of H and D will be

the H. C. D. of A, B, C and D. And so on. We have thus the following

RULE.—*First find the* H. C. D. *of any two of the polynomials; then find the* H. C. D. *of this result and the third polynomial, and so on. The last* H. C. D. *will be the* H. C. D. *required.*

Examples.

Ex. 1. Find the H. C. D. of $x^3-x^2-10x-8$, $x^3+6x^2+11x+6$ and $x^3+4x^2-11x-30$

$$\begin{array}{r} x^3-x^2-10x-8\,)\,x^3+6x^2+11x+6\,(\,1 \\ \underline{x^3-\;\,x^2-10x-8} \\ 7x^2+21x+14\;: \end{array}$$

Reject the factor 7; thus we have x^2+3x+2

$$\begin{array}{r} x^2+3x+2\,)\,x^3-\;\,x^2-10x-8\,(\,x-4 \\ \underline{x^3+3x^2+\;\,2x} \\ -4x^2-12x-8 \\ \underline{-4x^2-12x-8} \end{array}$$

Thus x^2+3x+2 is the H. C. D. of the first two expressions. The required H. C. D. will therefore be the H. C. D. of this and the third expression.

$$\begin{array}{r} x^2+3x+2\,)\,x^3+4x^2-11x-30\,(\,x+1 \\ \underline{x^3+3x^2+\;\,2x} \\ x^2-13x-30 \\ \underline{x^2+\;\,3x+\;\,2} \\ -16x-32 \end{array}$$

Changing the sign and rejecting the factor 16, we get $x+2$.

$$\begin{array}{r} x+2\,)\,x^2+3x+2\,(\,x+1 \\ \underline{x^2+2x} \\ x+2 \\ \underline{x+2} \end{array}$$

$\therefore$ required H. C. D. $=x+2$

Find the Highest Common Divisor of

2. x^2-x-2, x^2+x-6 and x^2-3x+2

3. $2x^2-x-1$, $6x^2+x-1$ and $8x^3+1$

4. $x^3+3x^2y+3xy^2+y^3$, $x^3+x^2y+xy^2+y^3$ and $x^3+x^2y-xy^2-y^3$.

Find the Highest Common Divisor of

5. a^3+b^3, $a^4+a^2b^2+b^4$ and $a^4-a^3b+ab^3-b^4$.

6. $x^3-3x-70$, $x^3-39x+70$ and $x^3-48x+7$

7. $a^3-9a^2+26a-24$, $a^3-10a^2+31a-30$ and $a^3-11a^2+38a-40$.

8. $a^3+4a^2b-5b^3$, $a^3-3ab^2+2b^3$ and $a^3+4a^2b-8ab^2+3b^3$

9. x^2-2a^2-ax, x^2-4a^2, x^2-6a^2+ax and x^2-8a^2+2ax

10. $1-16x^4$, $1-5x^2+4x^4$, $1+2x-3x^3-6x^4$ and $1+2x-4x^3-8x^4$.

11. $6x^3-23x^2+29x-12$, $10x^3-19x^2+9$ and $15x^3-26x^2-x+12$

12. $4x^3-28x^2+39x+27$, $6x^3-47x^2+96x-27$
and $12x^3-52x^2-11x+9$.

We shall conclude this Chapter by giving an *ingenious method of finding the Highest Common Divisor*, depending on the Theorem of the next article

*157 **Theorem** *If* A *and* B *be any two expressions in* x, *then the* H C D *of* $lA+mB$ *and* $pA+qB$ *will be the same as the* H C D *of* A *and* B, *where the quantities* l, m, p *and* q *do not involve* x, *and are such that* $lq-mp$ is not$=0$

Let the H C D of A and B be G and that of $lA+mB$ and $pA+qB$ be H

We know that *every common* divisor of A and B which involves x, is a divisor of $lA+mB$ and of $pA+qB$ [§ 153], therefore their *highest common* divisor G is a common divisor of $lA+mB$ and $pA+qB$ Therefore either $G=H$, or G is a divisor of H [§ 155, Note 3] and consequently of a *lower* degree than H.

Again *every common* divisor of $lA+mB$ and $pA+qB$ which involves x, is a divisor of

$$q(lA+mB)-m(pA+qB)$$

and of $$l(pA+qB)-p(lA+mB) \quad [\S\ 153],$$

and consequently their *highest common* divisor H is a common divisor of these expressions,

now $$q(lA+mB)-m(pA+qB)=(lq-mp)A,$$

and $$l(pA+qB)-p(lA+mB)=(lq-mp)B,$$

therefore H is a common divisor of $(lq-mp)A$ and $(lq-mp)B$, and since by supposition $lq-mp$ is not$=0$, and does not contain x, H cannot be a divisor of $lq-mp$, and must therefore be a common divisor of A and B Hence either $H=G$, or H is a divisor of G [§ 155, Note 3], and therefore G is of a *higher* degree than H.

Thus G is at once higher and lower than H, which is absurd.

Therefore $$H=G$$

Note In applying the above method to examples, we must so choose l, m and p, q that the *highest* and *lowest* terms shall disappear from $lA+mB$ and $pA+qB$ respectively

The following examples will illustrate the method.

Ex 1. Find the H C D of $2x^2+x-1$ and $6x^2+x-2$.

Let $A=2x^2+x-1$, and $B=6x^2+x-2$

Now $3A-B=3(2x^2+x-1)-(6x^2+x-2)=2x-1$,

and $B-2A=(6x^2+x-2)-2(2x^2+x-1)=2x^2-x=x(2x-1)$.

Hence the H C D of $3A-B$ and $B-2A$, which is evidently $2x-1$, is the H C D of A and B, and therefore the H C D required

Ex 2 Find the H C D of

$$A=4x^3+9x^2+3x-2,$$

and
$$B=4x^3-3x^2-5x+2$$

We have
$$A+B=8x^3+6x^2-2x=2x(4x^2+3x-1);$$

and
$$A-B=12x^2+8x-4=4(3x^2+2x-1).$$

Reject $2x$ and 4 which do not form part of the required H. C. D. and put

$$A_1=4x^2+3x-1 \text{ and } B_1=3x^2+2x-1.$$

Thus the H C D of A_1 and B_1 will be the H C D. of $A+B$ and $A-B$, and therefore of A and B Now

$$A_1-B_1=x^2+x=x(x+1),$$

and
$$3A_1-4B_1=3(4x^2+3x-1)-4(3x^2+2x-1)=x+1$$

Therefore the H C D of A_1-B_1 and $3A_1-4B_1$ which is $x+1$, is the H. C. D of A_1, and B_1, and therefore of A and B

Ex. 3. Find the H C D of

$$A=2x^3+7x^2+2x-3,$$

and
$$B=3x^3+8x^2-2x+3 \text{ [§ 155, Ex. 11)}$$

Now
$$A+B=5x^3+15x^2=5x^2(x+3),$$

and
$$3A-2B=3(2x^3+7x^2+2x-3)-2(3x^3+8x^2-2x+3)$$
$$=5(x^2+2x-3)$$

As $5x^2$ and 5 do not form part of the required H. C. D, we reject them, thus the H C D of $A+B$ and $3A-2B$ will be the same as the H C D. of $x+3$ and x^2+2x-3

Let
$$A_1=x+3 \text{ and } B_1=x^2+2x-3:$$

thus
$$A_1+B_1=x^2+3x=x(x+3)$$

Hence the H C D of A_1 and A_1+B_1, which is evidently $x+3$, is the H C. D of A_1 and B_1, and therefore of $A+B$ and $3A-2B$ Therefore $x+3$ is the H C D required

Ex. 4 Find the H. C. D. of

$$A=2x^6-11x^3-9,$$

and
$$B=4x^5+11x^4+81 \text{ [§ 155, Ex. 73.]}$$

Now
$$2A-B=-11x^4-22x^2-99=-11(x^4+2x^2+9),$$

and
$$9A+B=22x^6+11x^4-99x^2=11x^2(2x^3+x^2-9).$$

Hence the H. C. D. of $2A-B$ and $9A+B$, that is, of x^4+2x^2+9 and $2x^3+x^2-9$, will be the H. C. D. of A and B. Let

$$A_1=x^4+2x^2+9,$$

and
$$B_1=2x^3+x^2-9,$$

thus
$$A_1+B_1=x^4+2x^3+3x^2=x^2(x^2+2x+3)$$

Therefore the H. C. D. of B_1 and A_1+B_1, that is, of B_1 and x^2+2x+3 [$=A_2$ say] will be the H. C. D. of A_1 and B_1 and therefore of A and B.

Now
$$3A_2+B_1=2x^3+4x^2+6x=2x(x^2+2x+3),$$

therefore the H. C. D. of A_2 and $3A_2+B_1$, which is obviously x^2+2x+3, is the H. C. D. of A_2 and B_1, and therefore of A and B. Thus x^2+2x+3 is the H. C. D. required.

[The student may try by this method other examples given under § 155.]

CHAPTER XIII

Lowest Common Multiple

158. Definitions. A Quantity is said to be a MULTIPLE of another, when the latter is contained in the former an exact number of times. Thus 16 is a multiple of 2, of 4, or of 8. $5ab$ is a multiple of 5, of a, or of b.

A COMMON MULTIPLE of two or more quantities is that quantity which is divisible by each of the latter without remainder. Thus 16 is a Common Multiple of 2, 4 and 8. $5ab$ is Common Multiple of 5, a and b.

The LOWEST COMMON MULTIPLE of two or more quantities is the quantity of *lowest dimensions* which is divisible by each of the latter without remainder. Thus $5ab$ is the Lowest Common Multiple of 5, a and b, but not so are $5a^2b$, $10a^2b^2$, $20a^3b^4$, &c.

The term Lowest Common Multiple is often shortened into L. C. M.

159. L. C. M. of Monomials. Since the required L. C. M. must be that multiple of *lowest dimensions*, which is common to all the given expressions, we have the following

RULE.—*Take the several factors occurring in the given expressions, each raised to the highest power which it has in any one of them: the product of these powers will be the* L. C. M. *required.*

If there be numerical co-efficients, find their L. C. M. as in Arithmetic, after which put the L. C. M. of the literal factors.

Examples.

Ex 1 Find the L. C. M. of a^3x and ax^2.

Here the factors are a and x, of which the highest indices are 3 and 2 respectively;

$$\therefore \text{L. C. M. required} = a^3x^2$$

Ex 2 Find the L. C. M. of $2ax$, $4by$ and $6xy$

Here the factors are a, b, x and y, each of the first degree, and the L. C. M. of co efficients is 12;

$$\therefore \text{L. C. M. required} = 12abxy.$$

Ex. 3. Find the L. C. M. of $8x^2y$, $12y^2z$ and $16xz^3$

Here the factors are x, y and z of which the indices are 2, 2 and 3 respectively, and L. C. M. of co-efficients is 48,

$$\therefore \text{L. C. M. required} = 48x^2y^2z^3.$$

Find the Lowest Common Multiple of

4. $5a^2b$ and $10ab^3$ 5. $6xy^2$ and $8xy$ 6. $12a^3$ and $16a^2b$
7. $9ax^3$ and $21a^2x^4$ 8. $24x^3y^4$ and $28x^2y^5$ 9. xy, yz and xz
10. $2a^2x^2$, ax^3 and a^3x. 11. ab^3, 20 and $15a^2x^2$.
12. $5m^3$, $10m^2n$ and $25n^3$ 13. $38a^2xy$, $57ax^2y^3$ and $95bxy^3$.
14. 12, $3a^2$, $6ax$ and $8x^2$ 15. $2a^3x$, $3a^2x^2$, $4ax^3$ and $5x^4$
16. $32a^2xy^3$, $48x^2yz$, $64ay^2z$ and $80a^2x^3z^2$.

160 L. C. M. of expressions whose factors can be found by inspection The method is the same as given in the last article; we therefore follow the same Rule

Examples.

Ex. 1. Find the L. C. M. of a^2 and $ax - a^2$

The second expression $= a(x - a)$; therefore the factors are a and $x - a$ whose highest indices are 2 and 1 respectively,

$$\therefore \text{L. C. M. required} = a^2(x - a) = a^2x - a^3$$

Ex. 2 Find the L. C. M. of $a+x$ and a^2-x^2.

Here the second expression itself contains the first, for it $=(a+x)(a-x)$;

$\therefore$ L. C. M. required $=a^2-x^2$.

Ex. 3 Find the L. C. M. of $4a^2x^2$, $6(a^2+ax)$ and $8(ax-x^2)$.

Second expression $=6a(a+x)$, third expression $=8x(a-x)$.

Thus the factors are a, x, $a+x$ and $a-x$ whose highest indices are 2, 2, 1 and 1, and L. C. M. of coefficients is 24,

$\therefore$ L. C. M. required $=24a^2x^2(a+x)(a-x)=24a^2x^2(a^2-x^2)$.

Ex. 4 Find the L. C. M. of

$(a-b)(a-c)$, $(b-c)(b-a)$ and $(c-a)(c-b)$.

First expression $=(a-b)\times-(c-a)=-(a-b)(c-a)$.

Similarly second expression $=-(a-b)(b-c)$,

and third expression $=-(b-c)(c-a)$.

The factors of the highest degree are $a-b$, $b-c$ and $c-a$,

$\therefore$ L. C. M. required $=-(a-b)(b-c)(c-a)$.

Find the Lowest Common Multiple of

5. $3ax^2$ and $2x+x^2$. 6. axy and $a(xy-y^2)$. 7. x^2+xy and y^2+xy.

8. $2(ax+ay)$ and $3(ax-ay)$. 9. $4(x^2+x)$ and $6(x^2-1)$.

10. $15y(a-x)$ and $20(a^2x-x^3)$. 11. $30(a^2-b^2)$ and $42(a^3-b^3)$.

12. $4x^2-1$ and $8x^3+1$. 13. $2(x^2-y^2)$ and $10(x+y)^2$.

14. a^3+x^3 and $a^4+a^2x^2+x^4$. 15. $14(x^3-y^3)$ and $21(x^3+x^2y+xy^2)$.

16. $2ax$, $a+x$ and $a-x$. 17. $x+y$, x^2-y^2 and x^3+y^3.

18. $x+1$, x^2+1 and x^3+1. 19. $1+a$, $1-a^2$ and $1-2a+a^2$.

20. $2(x^3-x^2y)$, $3(xy^2+y^3)$ and $4(x^3y-xy^3)$.

21. $(x-y)(x-z)$ and $(y-x)(y-z)$. 22. ax^2-x^3 and ax^2-a^3.

23. x^2+x, $(x+1)(x+2)$ and x^3+2x^2.

24. $6(x^2+xy)$, $8(xy-y^2)$ and $10(x^2-y^2)$.

25. x^2-4, x^2+4x+4 and $(x-2)^3$.

26. $3(x+y)(x+2y)$, $15(x+2y)(x+3y)$ and $25(x+y)(x+3y)$.

27. $6(x^2y+xy^2)$, $9(x^3-xy^2)$ and $4(y^3+xy^2)$.

28. $2x$, $3y$, $1+a$ and $4(1-a^2)$. 29. a^3x^2, a^2x^3, $ax-xy$ and a^2+ay.

30. $x+1$, $x-1$, x^2+x+1 and x^2-x+1.

31. x^2-y^2, $4(x+y)^2$, $6(x-y)^2$ and $15(x^3+y^3)$.

32. x, $x-1$, x^2-1, x^3-1 and x^4-1.

33. $x^3(x-y)^2$, $y^2(x+y)^2$, x^4-xy^3, x^3y+x^4 and $x^2y^3-y^5$.

161. L. C. M. of two Polynomials. Let A and B be any two expressions whose H. C. D. is H and whose L. C. M. is M.

Divide A and B respectively by H, and let the quotients be a and b, so that $A=aH$ and $B=bH$.

Then since H is the H. C. D. of A and B, a and b cannot have a *common* factor, and therefore their L. C. M. must be ab.

Hence the L. C. M. of A and B, that is, of aH and bH is abH.

Thus $M=abH$.

Now $\qquad M=abH=aH\cdot b=Ab$, or $=a\cdot bH=aB \qquad$ (i)

Also $\qquad M=abH=aH\,bH\div H=AB\div H \qquad$ (ii).

Thus from (i) and (ii), we have the following

RULE.—*Divide either polynomial by the* H. C. D. *of the two, and multiply the quotient by the other* or in other words.—*Multiply the polynomials together and divide the product by their* H. C. D.

COROLLARY. From (ii), it is evident that

$$M\times H=AB\div H\times H=A\times B.$$

Thus *the product of two expressions is equal to the product of their* L. C. M. *and* H. C. D.

Examples

Ex. 1. Find the L. C. M. of x^2+5x+6 and x^2-x-6.

First expression $=(x+2)(x+3)$, second expression $=(x+2)(x-3)$; and their H. C. D. is $x+2$,

$$\therefore \text{ L. C. M. required } =\frac{(x+2)(x+3)\times(x+2)(x-3)}{x+2}$$

$$=(x+2)(x^2-9)=x^3+2x^2-9x-18.$$

Ex. 2. Find the L. C. M. of $a^3+2a^2x-3x^3$ and $a^3+a^2x-3ax^2-6x^3$.

First expression $=a^3-x^3+2a^2x-2x^3$

$$=(a^3-x^3)+2x(a^2-x^2)$$

$$=(a-x)(a^2+ax+x^2+2ax+2x^2)$$

$$=(a-x)(a^2+3ax+3x^2).$$

Second expression $=a^3-8x^3+a^2x-3ax^2+2x^3$

$$=(a^3-8x^3)+x(a^2-3ax+2x^2)$$

$$=(a-2x)(a^2+2ax+4x^2)+x(a-2x)(a-x)$$

$$=(a-2x)(a^2+3ax+3x^2).$$

Hence removing $a^2+3ax+3x^2$, the H. C. D. from *one* of the expressions which is the same as dividing it by the H. C. D., we have

$$\text{L. C. M. required}=(a-x)(a-2x)(a^2+3ax+3x^2).$$

Ex. 3. Find the L. C. M. of $a^2b+abx-2bx^2$
and $a^2b-(2a^2+ab)x+2ax^2$.

First expression $=b(a^2+ax-2x^2)=b(a-x)(a+2x)$

Second expression $=a\{ab-(2a+b)x+2x^2\}=a(a-x)(b-2x)$

removing $a-x$, the H. C. D., from *one* of the expressions, we have

L. C. M. required $=ab(a-x)(a+2x)(b-2x)$

Find the Lowest Common Multiple of

4. $3x^2-10x+3$ and $3x^2-19x+6$
5. $2x^2-13x+21$ and $3x^2-23x+42$
6. x^3-3x^2+3x-1 and x^3-x^2-x+1
7. $x^2-(a+b)x+ab$ and $x^2-(a+c)x+ac$
8. $3x^2-5xy+2y^2$ and $4x^3-4x^2y-xy^2+y^3$
9. $x^3-6x^2+11x-6$ and x^3+4x^2+x-6
10. x^3-6x^2+8x and x^2+x-6
11. $mx^2-6mx+5m$ and $nx^2+5nx-6n$
12. $2a^2+ax-3x^2$ and $3a^3-a^2x-ax^2-x^3$
13. $(a^3+a^2b)x^2+a(a^2-b^2)xy-(a^2b+ab^2)y^2$
    and $(a^2b-ab^2)x^2-b(a^2-b^2)xy+(ab^2-b^3)y^2$.
14. $x^3+3x^2-4x-12$ and x^3+2x^2-x-2
15. $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$
    and $x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5$
16. $2a^4+3a^3x-9a^2x^2$ and $6a^4x-17a^3x^2+14a^2x^3-3ax^4$
17. $x^3+(5a-3)x^2+(6a^2-15a)x-18a^2$
    and $x^3+(a-3)x^2-(2a^2+3a)x+6a^2$
18. $x^4-4x^3+2x^2+4x-15$ and $x^4-4x^3+3x^2+2x-12$
19. $2x^4+4x^3-97x^2-2x+48$ and $6x^4+118x^3-43x^2-59x+20$

162 Theorem *Every Common Multiple of two expressions is a multiple of their L C M*

Let A and B be two expressions whose L C M is M.

Let μ be any other multiple of A and B Then μ is divisible by M without remainder

If not, if possible, let M be contained q times in μ with a remainder R; then $R=\mu-qM$ [§ 96].

Now A and B divide M and also μ, therefore they divide qM, and $\mu-qM$ or R [§ 153]

But R is of a lower degree than M, the divisor [§ 96].

Hence A and B divide an expression which is of lower dimensions than M, their L. C. M., which is absurd.

Therefore there can be no remainder, *i. e.*, μ is a multiple of M

163. L. C. M. of Several Polynomials. Let A, B and C be any three expressions, and let M be the L. C. M. of A and B

Now every multiple of M is a common multiple of A and B; therefore every common multiple of M and C is a common multiple of A, B and C.

Also every common multiple of A, B and C is a common multiple of M and C [§ 162]

Therefore the L. C. M. of M and C is the L. C. M. of A, B and C.

Similarly the reasoning may be extended to the case of any number of polynomials

We have thus the following

RULE —*Find the* L. C. M. *of any two expressions; then find the* L. C. M. *of this* L. C. M. *and a third expression; next find the* L. C. M. *of the second* L. C. M. *and a fourth expression; and so on; the last* L. C. M. *will be the* L. C. M. *required*

Examples

Ex 1 Find the L. C. M. of $6x^2+5x-6$, $12x^2+7x-10$ and $4x^2+x-5$

The L. C. M. of $6x^2+5x-6$ and $12x^2+7x-10$ will be found to be $(3x-2)(2x+3)(4x+5)$ Therefore the L. C. M. required will be the L. C. M. of this expression and $4x^2+x-5$, which will therefore be

$$(3x-2)(2x+3)(4x+5)(x-1)$$
$$=24x^4+26x^3-49x^2-31x+30.$$

Find the Lowest Common Multiple of

2. $6x^2-x-1$, $3x^2+7x+2$ and $2x^2+3x-2$.
3. $2x^2-7x+3$, $4x^2-7x-15$ and $8x^2+6x-5$.
4. $3x^2-14x-80$, $3x^2+17x-90$ and $x^3-7x^2-80x+576$.
5. $1+4x+8x^2+8x^3$, $1+4x+4x^2-16x^4$ and $1+2x-8x^3-16x^4$.
6. $9x^4-28x^2+3$, $27x^4-12x^2+1$, $27x^4+6x^2-1$ and x^4-6x^2+9.

164. Examination upon Chapters XII and XIII.

1. When is one quantity said to be a *Divisor* of another, and when a *Multiple*?

2 What is a *common divisor*, and what a *common multiple*, of two or more expressions? Shew by examples that the number of common divisors is limited, but that of common multiples is unlimited.

3. Define the H. C D and the L C M of two or more expressions. Shew that the L C M is a multiple of the H C D

4 State and prove the Rules for finding the H. C D and the L C M of two polynomials

5 Shew that if a quantity divide two others, it will also divide the sum or difference of any multiples whatever of them.

6 Prove that every Common Divisor of A and B is a divisor of their H C D

7 Prove that every Common Multiple of two or more quantities is a multiple of their L C M

8 If h be the H C D of two expressions a and b whose L C M. is l, prove that $hl=ab$

9 If m and n be any multiples of a and b whose L C M is l, prove that l divides

(1) $mx+ny$, (2) $mx-ny$, (3) m^2+n^2, (4) m^2-n^2

***165 Some Examples Worked out** We shall conclude this Chapter by working out a few examples

Ex 1. If x^2-3x+a and $x^2+2x+6a$ have a common factor, find it

Let F=their common factor, thus by § 153, Cor 1, F must be a factor of their difference $5x+5a=5(x+a)$, and as it is of the *first* degree and the given expressions are of the *second* degree, F must $=x+a$, for 5 cannot evidently be a factor

Ex 2 Shew that the condition that x^2+px+q and $x^2+p'x+q'$ may have a common divisor is

$$(q-q')^2=(p-p')(p'q-pq')$$

The common divisor must be *linear* and of the form $x+a$

Divide x^2+px+q by $x+a$ The remainder after division will be seen to be a^2-pa+q, which must vanish, that is, be$=0$, as $x+a$ is a factor of x^2+px+q Hence

$$a^2-pa+q=0 \qquad \ldots\ldots\ldots\ldots(\text{i}).$$

Similarly we see that

$$a^2-p'a+q'=0 \qquad \ldots\ldots\ldots\ldots(\text{ii}).$$

Subtract (i) from (ii); thus

$$(p-p')a-(q-q')=0,$$

whence
$$a=\frac{q-q'}{p-p'} \qquad \ldots\ldots\ldots\ldots(\text{iii})$$

Substitute a in (1), thus

$$\left(\frac{q-q'}{p-p'}\right)^2-p\left(\frac{q-q'}{p-p'}\right)+q=0\,;$$

Transpose and multiply by $(p-p')^2$; thus

$$\begin{aligned}(q-q')^2&=p(p-p')(q-q')-q(p-p')^2\\&=(p-p')\{p(q-q')-q(p-p')\}.\\&=(p-p')(p'q-pq').\end{aligned}$$

Ex 3. For what value of m will

$$x^3-(m-6)x^2-2mx+24 \text{ and } x^3-(m+3)x^2-(2m-45)x-30$$

have a common factor, and what is that factor?

Let L and M denote the given expressions and F their common factor, thus F is a factor of $L-M$ or $x^2-5x+6=(x-2)(x-3)$, hence either $F=x^2-5x+6$ (in which case $x-2$ and $x-3$ are each factors of L and M), or $F=x-2$ or $x-3$ First suppose $F=x-2$; then by actual division, the remainder is found to be $-(8m-32)+24$ which must $=0$, whence $m=7$ Next suppose $F=x-3$, thus the remainder is $-(15m-81)+24$ which must$=0$, whence, as before, $m=7$.

Thus when $m=7$, both $x-2$ and $x-3$ are factors of L and M, i.e., $(x-2)(x-3)$ or x^2-5x+6 is the required factor.

Ex 4 If $x+f$ be a common factor of x^2+ax+b and $x^2+a'x+b'$ shew that

$$\text{(i)}\quad f=\frac{b-b'}{a-a'}\,;\qquad\qquad \text{(ii)}\quad f=\frac{a'b-ab'}{b-b'}.$$

If $x+f$ be a factor of the first expression, the remainder after division must be 0

$$\begin{array}{l}x+f\;)\;x^2+ax+b\;(x+(a-f)\\ \qquad\quad \underline{x^2+fx\qquad\quad}\\ \qquad\quad (a-f)x+b\\ \qquad\quad \underline{(a-f)x+(a-f)f}\\ \qquad\qquad -(a-f)f+b\end{array}$$

$\therefore\quad -(a-f)f+b=0\quad (1),\qquad\qquad \text{or } f^2-af+b=0\quad (2).$

Similarly by dividing the second expression by $x+f$, we get

$-(a'-f)f+b'=0\quad (3),\qquad\qquad \text{or } f^2-a'f+b'=0\quad (4).$

Subtracting (2) from (4) and transposing, $(a-a')f=b-b'$,

$$\therefore\quad f=\frac{b-b'}{a-a'}\qquad\qquad (1)$$

Again from (1) and (3), we have $(a-f)f=b$ and $(a'-f)f=b'$,

by division, $\frac{(a-f)f}{(a'-f)f}=\frac{b}{b'}$, or $b'(a-f)=b(a'-f)$,

or $$(b-b')f=a'b-ab', \text{ i.e., } f=\frac{a'b-ab'}{b-b'} \qquad \text{(ii)}.$$

Ex. 5. If $x+c$ be the H. C. D. of x^2+ax+b and $x^2+a'x+b'$, prove that their L. C. M. will be

$$x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c \qquad \text{(A)}$$

Dividing x^2+ax+b by $x+c$, we get $x+a-c$ for the quotient, also the remainder, when $x^2+a'x+b'$ is divided by $x+c$, is $b'-c(a'-c)$, and since this must $=0$,

$$b'=c(a'-c) \quad (1)$$

By the Rule [§ 161], L. C. M. of the given expressions

$$=\frac{(x^2+ax+b)(x^2+a'x+b')}{x+c}$$
$$=(x+a-c)(x^2+a'x+b')$$
$$=(x+a-c)\{x^2+a'x+(a'-c)c\} \text{ from (1)}$$
$$=x^3+(a+a'-c)x^2+\{(a'-c)c+(a-c)a'\}x+(a-c)(a'-c)c$$
$$=x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c$$

Otherwise.—Divide the given expressions by $x+c$, thus the quotients are $x+a-c$ and $x+a'-c$ respectively. Hence, since $M=abH$ [§ 161], the required L. C. M.

$$=(x+c)(x+a-c)(x+a'-c)$$

which when reduced gives (A).

Miscellaneous Examples V.

Find the H. C. D. of

1. $6a^4-5a^2x^2-6x^4$ and $4a^5-6a^3x^2-2a^2x^3+3x^5$.
2. $2a^3-3a^2b-2ab^2+3b^3$ and $3a^4+2a^3b-2a^2b^2-2ab^3-b^4$.
3. $9x^2-3xy-6x+2y$ and $6x^3-4x^2-3xy^2+2y^2$.
4. $x(6x^3-8y^3)-y(3x^2-4y^2)$ and $2xy(2y-x)+4x^3-2y^3$.
5. $a^2-acx+(ac-b^2+bc)x^2-bcx^3$ and $a^2+abx+(ac-c^2+bc)x^2+c^2x^3$
6. $x^3+ax^2-axy-y^3$ and $x^4+2x^3y-a^2x^2+x^2y^2-2axy^2-y^4$
7. $(a^2-b^2)x^2+2b^2x-a(a-2b)$ and $(a^2+ab-2b^2)x^2+3b^2x-(a^2-ab-2b^2)$.
8. $x^4-2ax^3-4a^2x^2+16a^3x+16a^4$ and $x^4-6ax^3-4a^2x^2+16a^3x-16a^4$

Find the H. C. D.

9. $2x^4-7x^3+16x^2-17x+12$ and $3x^4-7x^3+13x^2-7x+6$

10. $x^5-3x^4+x^3-3x^2+x-3$ and $7x^4-16x^3-21x^2+9x+27$.

11. x^5+11x^3-54 and $x^5+11x+12$

12. $20a^4-3a^3b+b^4$ and $64a^4-3ab^3+5b^4$.

Find the L. C. M. of

13. x^4+2x^3+6x-9 and $x^4+4x^3+4x^2-9$

14. $3x^3-27ax^2+78a^2x-72a^3$ and $2x^3+10ax^2-4a^2x-48a^3$.

15. $x^3-6x^2-37x+210$ and $x^3+4x^2-47x-210$

16. $ab^3-(ab-b^2)x+(a-b)x^2+x^3$ and $ab^2+(ab+b^2)x+(a+b)x^2+x^3$.

17. $3x^3-2x^2-x$ and $4x^3-2x^2-3x+1$.

18. $7x^3-19x^2+17x-5$ and $2x^4-x^3-9x^2+13x-5$

Find the H. C. D. and the L. C. M. of

19. $ab(x^2+1)+x(a^2+b^2)$ and $ab(x^2-1)+x(a^2-b^2)$

20. $x^2+y^2+2xy-1$ and $x^3+y^3+3xy-1$

21. $24(x^3+x^2y+xy^2+y^3)$ and $16(x^3-x^2y+xy^2-y^3)$.

22. $3x^2-10ax+7a^2$ and $x^3-5ax^2+7a^2x-3a^3$.

23. $6a^3+a^2x-11ax^2-6x^3$ and $6a^3+11a^2x-ax^2-6x^3$

24. $4x^4+2x^3-18x^2+3x-5$ and $6x^5-4x^4-11x^3-3x^2-3x-1$

25. $(a^2+a-2)x^2+(2a^2+a+3)x+a^2-1$
and $(a^2+4a+4)x^2+(2a^2+a-6)x+a^2-3a+2$

26. $2ab+a^2+b^2-c^2$, $3abc-a^3-b^3-c^3$ and $(b+c)(c+a)(a+b)+abc$.

27. Find the H. C. D. of $x^3-7x^2-80x+576$ and $3x^2-14x-80$, and the L. C. M. of these two expressions and $3x^2+17x-90$

28. Find the H. C. D. of $2x^3+x^2y-xy^2-2y^3$ and $x^5-x^3y^2-2x^2y^3+2xy^4$, and shew that its square is a factor of the latter expression

29. Find the H. C. D. of $2x^5-5x^2+3$ and $3x^5-5x^3+2$, and shew that if $x=1$, each of these expressions vanishes

30. Find the H. C. D. of $x^3-7a^2x+6a^3$ and $x^4-3ax^3-2a^2x^2+12a^3x-8a^4$, and shew that each of these expressions vanishes, when $x=a$ or $2a$

31. Find the H. C. D. of $6x^4-2x^3+9x^2+9x-4$ and $9x^4+80x^2-9$. What value of x makes these expressions vanish?

32. For what value of x will the expression $3x^4-5x^3+2x^2+48x+7$ be divisible by x^2+2x-1?

33. What value must be given to a in order that
$$x^3-ax^2+19x-a-4 \text{ and } x^3-(a+1)x^2+23x-a-7$$
may have a common divisor?

34 If x^3+px+q and x^3+qx+p have a common divisor, then

$$p+q+1=0$$

35 Shew that if x^3+px+q and $x^3+p'x+q'$ have a common divisor, then $$(q'-q)^3+(p'-p)^2(pq'-p'q)=0$$

36 If $x+a$ be a common divisor of ax^2+bx+c and $a'x^2+b'x+c'$, shew that

$$\text{(i)}\quad a=\frac{ac'-a'c}{ab'-a'b},\qquad \text{(ii)}\quad a=\frac{bc'-b'c}{ac'-a'c},\qquad \text{(iii)}\quad a^2=\frac{bc'-b'c}{ab'-a'b}$$

37 If x^2+ax+b and $x^2+\alpha x+\beta$ have a common divisor of the form $x+c$, prove that their L C M is

$$x^3+\frac{ab-\alpha\beta}{b-\beta}x^2+\frac{a\beta^2-\alpha b^2}{a\beta-\alpha b}x+\frac{b\beta(a-\alpha)}{b-\beta}$$

CHAPTER XV

FRACTIONS

166 Definitions We have seen [§ 2] that to ascertain any magnitude, we refer it to a known standard, called its Unit, and find how often this unit is contained in the magnitude But sometimes a magnitude may be seen to contain its unit a certain number of times, together with *a part remaining over* less than the unit, or the magnitude itself may be *less* than its unit Thus, when the unit is *one foot*, a stick 4 feet 5 inches long will be seen to contain 4 units together with a part less than the unit, so a length of 7 inches will be seen to contain not even one unit In such cases, therefore, we have to divide the original unit into a number of *equal* parts, and take *one* of these parts as our *new* unit, to measure the length which is less than the original unit Thus in the above examples, the original unit is divided into 12 equal parts, one of which called an *inch* and represented by $\frac{1}{12}$, is taken as our unit, which is contained 5 and 7 times respectively, in the examples considered

The original unit (in this case one foot) is termed the Primary Unit, or briefly, the UNIT, and the new unit obtained by subdividing the unit (in this case $\frac{1}{12}$ foot), is termed the SUB-UNIT. The sub unit is thus related to the unit by means of the divisor, which we call the *Denominator*, because it "*denominates*" or determines the sub-unit Thus, in the example given, $\frac{1}{12}$ represents one sub unit, $\frac{5}{12}$ represents 5 sub-units, or 5 times the sub unit, $\frac{12}{12}$ represents 12 sub-units, and therefore the unit, $\frac{17}{12}$ represents 17 sub-

units, and so on. Hence, generally, if b represent the Denominator, $\frac{1}{b}$ will represent the sub-unit, and a of them will be represented by $\frac{a}{b}$, and b of them by $\frac{b}{b}$ or 1, therefore b times $\frac{a}{b}=a$ units, that is,

$$\frac{a}{b}\times b=a.$$

As a "*numerates*" or numbers what *multiple* of the sub-unit is to be taken, it is called the *Numerator.*

Thus FRACTION (lit a *broken* part) denotes a part or parts of a Unit, and is represented by writing the numerator over the denominator with a line between them The DENOMINATOR indicates into how many equal parts the unit is to be divided, and the NUMERATOR indicates how many of such parts are to be taken The numerator and denominator are called the TERMS of a fraction.

A fraction is said to be PROPER or IMPROPER, according as the numerator is *less* or *greater* than the denominator

Corollary 1 It is clear from this article that

$$\frac{m}{a}+\frac{n}{a}=\frac{m+n}{a};$$

for here the sub unit is $\frac{1}{a}$, and $\frac{m}{a}+\frac{n}{a}$ denotes that m of such sub-units together with n of such sub units are taken, which is evidently the same thing as to take at once $m+n$ of the sub-units. Reasoning similarly we see that

$$\frac{m}{a}+\frac{n}{a}+\frac{p}{a}+\ldots\ldots=\frac{m+n+p+\ldots}{a}.$$

Hence we arrive at the following important conclusion .—*If several fractions have the same denominator, their sum is a fraction, having for its denominator the common denominator, and for its numerator the sum of the numerators of the given fractions*

Corollary 2 By a reasoning exactly similar to the above, we may shew that

$$\frac{m}{a}-\frac{n}{a}=\frac{m-n}{a}$$

Corollary-3. Hence we may easily see that

$$\frac{m}{a}-\frac{n}{a}-\frac{p}{a}+\frac{q}{a}+\ldots\ldots=\frac{m-n-p+q+\ldots}{a}.$$

Examples.

1 Shew that $\frac{2a}{b}+\frac{a}{b}=\frac{3a}{b}$ 2 Shew that $\frac{5a}{m}+\frac{3a}{m}-\frac{6a}{m}=\frac{2a}{m}$

3 Shew that $\frac{a}{a+b}+\frac{b}{a+b}=1$. 4 Shew that $\frac{x}{x-a}-\frac{a}{x-a}=1$

5. Shew that $\frac{2a+b}{a-b}-\frac{a}{a-b}=\frac{a+b}{a-b}$.

6 Shew that $\frac{2x}{x+y}+\frac{y}{x+y}-\frac{x}{x+y}=1$

7. Shew that $\frac{a}{a+b+c}+\frac{b}{a+b+c}+\frac{c}{a+b+c}=1$

167 Fraction expresses a Quotient We have seen from the definition of fraction that $\frac{a}{b}\times b=a$ [§ 166] Also by the definition of division [§ 69], we get $a \div b \times b = a$ Thus $\frac{a}{b}\times b=a\div b\times b$, whence dividing both sides by b, we get $\frac{a}{b}=a\div b$

REMARK The line between the numerator and denominator is supposed, not without reason, to be the sign of division $\div$, in the place of whose dots are written the terms of a fraction The fraction $\frac{a}{b}$ is read "a by b"

Corollary 1 From this article, we see that $\frac{a}{b}$ expresses either a times b sub units, or $\frac{1}{b}$th part of a units Ex $\frac{3}{5}$ is either 3 times the fifth part of unity or $\frac{1}{5}$ of 3 units

Corollary 2 An *integer* may be considered as a *fraction* whose denominator is 1 Thus $3=\frac{3}{1}$, $a=\frac{a}{1}$.

168 Important Proposition *If the numerator and denominator of a fraction be both multiplied, or both divided, by the same number, its value is not altered*

Let $\frac{a}{b}$ be a fraction, we shall prove that for *all values* of m,

$$(1)\ \frac{a}{b}=\frac{am}{bm},\ \text{and}\ (2)\ \frac{a}{b}=\frac{a\div m}{b\div m}$$

We have $\frac{a}{b}=a\div b$ [§ 167] $=a\div b\times m\div m$

$=a\times m\div b\times m$ [§ 73] $=(am)\div b\div m$

$=(am)\div(bm)$ [§ 70] $=\frac{am}{bm}$ [§ 167].

Again, as above, $\frac{a}{b}=a\div b\times m\div m=a\div m\div b\times m$ [§ 73]

$=(a\div m)\div(b\div m)$ [§ 70] $=\frac{a\div m}{b\div m}$.

Otherwise thus.—Let $k=\frac{a}{b}$; thus $k\times b=\frac{a}{b}\times b=a$ [§ 166];

multiply by m, thus $k\times b\times m=a\times m$ or $k\times(bm)=(am)$ [§ 65];

whence $k=(am)\div(bm)$ [§ 69] $=\frac{am}{bm}$ [§ 167];

$$\therefore \frac{a}{b}=\frac{am}{bm}$$

Again, we have $k\times b=\frac{a}{b}\times b=a$,

thus $k\times b\div m=a\div m$ or $k\times(b\div m)=(a\div m)$,

whence $k=(a\div m)\div(b\div m)=\frac{a\div m}{b\div m}$ [§ 167],

$$\frac{a}{b}=\frac{a\div m}{b\div m}.$$

Examples

1 Shew that $\frac{x}{a}=\frac{x^2}{ax}=\frac{x^2+ax}{ax+a^2}=\frac{ax+bx}{a^2+ab}$.

2. Shew that $\frac{1}{a+b}=\frac{4}{4a+4b}=\frac{a-b}{a^2-b^2}=\frac{a+b}{a^2+2ab+b^2}$.

3 Shew that $y=\frac{ay}{a}=\frac{y^2-y}{y-1}=\frac{ay+xy}{a+x}=\frac{ay-xy}{a-x}$.

4 Shew that $\frac{1+\frac{1}{x}}{x}=\frac{x+1}{x^2}=\frac{x^2+x}{x^3}=\frac{x^2-1}{x^3-x^2}$

Corollary. Hence *if the signs of both the numerator and denominator of a fraction be changed, its value is not altered* For this is equivalent to multiplying the numerator and denominator by -1 [§ 168]

Examples

1 Shew that $1-\dfrac{a}{a-b}=1-\dfrac{-a}{-a+b}=1+\dfrac{a}{b-a}=1-\dfrac{-a}{b-a}$

2 Shew that $\dfrac{x}{x-a}+\dfrac{a}{a-x}=\dfrac{x}{x-a}+\dfrac{-a}{x-a}=\dfrac{x-a}{x-a}=1.$

3 Shew that $\dfrac{x}{x-a}+\dfrac{a}{a-x}=1$, by changing the sign of $\dfrac{x}{x-a}$.

169 Omitted.

170 Reduction to a Mixed Quantity Since a fraction repressnts the quotient in the division of the numerator by the denominator [§ 167], an Improper Fraction may be reduced either to an integer, or to an integer and a fraction, by the Rules of §§ 95 and 96.

DEFINITION When a quantity contains an integral part together with a fraction, it is termed a *Mixed Quantity*, thus

$m+\dfrac{a}{b}$, $a+b-\dfrac{c}{d}$, &c are mixed quantities

Examples.

Ex 1 Reduce $\dfrac{ab+c}{b}$ to a mixed quantity

By actual division $(ab+c)\div b=a+\dfrac{c}{b}$ [§ 96], or $=\dfrac{ab}{b}+\dfrac{c}{b}=a+\dfrac{c}{b}$.

Ex 2 Reduce $\dfrac{1+x}{1-x}$ to a mixed quantity.

$$\begin{array}{l} 1-x\,)\,1+x\,(\,1 \\ \qquad\;\; \underline{1-x} \\ \qquad\quad 2x \end{array} \qquad \therefore \textit{Ans.} =1+\frac{2x}{1-x}$$

Reduce to mixed quantities or whole numbers

3 $\dfrac{30ab}{5a}$ 4 $\dfrac{51x}{5}$ 5 $\dfrac{20a+3b}{4}$. 6 $\dfrac{a^2-x^2}{a}$ 7. $\dfrac{a+x}{a-x}$.

8. $\dfrac{x^2+x}{x-1}$. 9 $\dfrac{a^3-b^3}{(a-b)^2}$ 10 $\dfrac{ax+b}{x+a}$. 11 $\dfrac{a^3-x^3}{a+x}$

12 $\dfrac{2a^3-ax^2+3x^3}{2a^2-x^2}$. 13. $\dfrac{a^4-x^4-a^2+x^2-ax+2}{a^2+x^2-1}$.

14 Shew that

$$\frac{60x^3-17x^2-4x+1}{5x^2+9x-2}=12x-25+\frac{49}{x+2}$$

171 Reduction of a Mixed Quantity. Let $a+\frac{b}{c}$ be a mixed quantity Then $a+\frac{b}{c}=\frac{a}{1}+\frac{b}{c}$ [§ 167, Cor 2] $=\frac{ac}{c}+\frac{b}{c}$ [§ 168] $=\frac{ac+b}{c}$ [§ 166, Cor 1]

Hence the Rule :—Multiply the integral part, by the denominator of the fractional part and add this product to the numerator.

Examples.

Reduce the mixed quantities to improper fractions

1. $a+\frac{2x}{a}$ 2 $10+\frac{13x-y}{2x}$. 3 $x-\frac{x^2-y^2}{x}$. 4. $\frac{(a+b)^2}{4ab}-1$

5. $1+\frac{a^2-b^2-c^2}{2bc}$. 6. $2x+\frac{2a^2}{x-2a}$ 7. $a+x-\frac{a^2+x^2}{a-x}$.

8 $a^2-a+1-\frac{a+2}{a+1}$.

172. Reduction to Lowest Terms A fraction is said to be in its *lowest terms* when its numerator and denominator have no *common* factor A fraction is therefore reduced to its lowest terms by dividing its numerator and denominator by their H C D Hence to reduce a fraction to its lowest terms, we *strike out their* H. C. D. *from the numerator and denominator.*

Examples.

Ex. 1 Reduce $\frac{54a^3bc}{36ab^3c}$ to its lowest terms

Given fraction $=\frac{18abc\ 3a^2}{18abc\ 2b^2}=\frac{3a^2}{2b^2}$.

Note When the factors of numerator and denominator are found by inspection as here, we simply stike out the *common factors*

Ex. 2 Reduce $\frac{6ab^3(a^2+ab)}{8a^2b(a^2b-b^3)}$ to its lowest terms.

Given fraction $=\frac{6a^2b^3(a+b)}{8a^2b^2(a^2-b^2)}=\frac{6a^2b^3(a+b)}{8a^2b^2(a+b)(a-b)}=\frac{3b}{4(a-b)}$.

Ex. 3. Reduce $\frac{2x^2-x-6}{3x^2-8x+4}$ to its lowest terms

Given fraction $=\frac{(x-2)(2x+3)}{(x-2)(3x-2)}=\frac{2x+3}{3x-2}$

Ex. 4. Reduce $\dfrac{x^3-39x+70}{x^3+14x^2+39x-70}$ to its lowest terms

As in § 155, find the H. C. D. of the numerator and denominator, which will be seen to be $x+7$

And
$$(x^3-39x+70)\div(x+7)=x^2-7x+10,$$
$$(x^3+14x^2+39x-70)\div(x+7)=x^2+7x-10\ ,$$
$$\therefore\ \text{reduced fraction}=\frac{x^2-7x+10}{x^2+7x-10}$$

Ex. 5. Reduce $\dfrac{2a^4-5a^3b+3a^2b^2-5ab^3-3b^4}{3a^4-7a^3b+a^2b^2-3ab^3-2b^4}$ to its lowest terms.

Find the H. C. D. of numerator and denominator as in § 155, which will thus be $a^2-2ab-b^2$

Now
$$\text{numerator}\div(a^2-2ab-b^2)=2a^2-ab+3b^2,$$
$$\text{denominator}\div(a^2-2ab-b^2)=3a^2-ab+2b^2\ ,$$
$$\text{reduced fraction}=\frac{2a^2-ab+3b^2}{3a^2-ab+2b^2}$$

Reduce to lowest terms

6. $\dfrac{12a^2b}{4ab^2c}$ 7. $\dfrac{16axy^2}{20ax^2y}$ 8. $\dfrac{28a^3b^2x^4}{42abx^3y}$ 9. $\dfrac{63m^4n^5}{405m^6n^8}$

10. $\dfrac{a^2-ax}{a^2+ax}$ 11. $\dfrac{8x^2-12xy}{16x^2y^2+20xy^3}$ 12. $\dfrac{a^2xy^2}{a^2xy-axy^2}$

13. $\dfrac{8m^2-2n^2}{8am+4an}$ 14. $\dfrac{3mp+3mq}{p^2-q^2}$ 15. $\dfrac{a^2bc-ab^2c}{a^2bd-ab^2d}$

16. $\dfrac{cx+x^2}{a^2c^2-x^2a^2}$. 17. $\dfrac{x^2-xy}{y^2-xy}$ 18. $\dfrac{24a^2x^2-16ax^3}{24a^2x^2-54a^4}$.

19. $\dfrac{3ab^2(a^2-b^2)}{12a^2b(a^3+b^3)}$ 20. $\dfrac{4a^3(x^3-a^3)}{6ax^2(a^4+a^2x^2+x^4)}$ 21. $\dfrac{m^3a^2+n^3a^2}{a(m^2+n^2)-man}$.

22. $\dfrac{x^2-(a+b)x+ab}{x^2-(a-c)x-ac}$. 23. $\dfrac{1-2a+a^2-b^2}{1-a+ab-b^2}$. 24. $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$

25. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}$ 26. $\dfrac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$ 27. $\dfrac{1+a+a^2}{1+a^2+a^4}$.

28. $\dfrac{a^2+b^2+c^2+2ab-2ac-2bc}{a^2+b^2-c^2+2ab}$ 29. $\dfrac{1-a^2b+b-a^2}{1-ab^2+a-b^2}$

30. $\dfrac{a^2+b^2+c^2-bc-ca-ab}{a^3+b^3+c^3-3abc}$ 31. $\dfrac{x^2-4x+3}{x^2-2x-3}$ 32. $\dfrac{x^2-x-20}{x^2-9x+20}$

33. $\dfrac{x^2+2x-3}{x^2+5x+6}$ 34. $\dfrac{6a^2+5ax-6x^2}{6a^2+13ax+6x^2}$ 35. $\dfrac{6a^2-7ax-3x^2}{6a^2+11ax+3x^2}$.

Reduce to lowest terms

36. $\dfrac{4+12x+9x^2}{2+13x+15x^2}$. 37. $\dfrac{1+x-12x^2}{3-7x-6x^2}$. 38. $\dfrac{6xy+8x+9y+12}{10xy-8x+15y-12}$.

39. $\dfrac{x^2+2axy+(a^2-b^2)y^2}{x^2+2bxy-(a^2-b^2)y^2}$. 40. $\dfrac{a^2x^2-2acxz-b^2y^2+c^2z^2}{a^2x^2+2abxy+b^2y^2-c^2z^2}$.

41. $\dfrac{(a+b)x^2-(2a+b)bx+ab^2}{(a-b)x^2-(2a-b)bx+ab^2}$. 42. $\dfrac{x^3+2x^2+2x}{x^5+4x}$.

43. $\dfrac{a^3+2a^2b-ab^2-2b^3}{a^3-3ab^2+2b^3}$. 44. $\dfrac{2y^3+y^2-8y+5}{7y^2-12y+5}$.

45. $\dfrac{x^3+2x^2-2x+3}{x^3-8x+3}$. 46. $\dfrac{2x^3-x^2+x+1}{2x^3+3x^2+3x+1}$. 47. $\dfrac{x^4+3x^3+x+3}{x^3-13x-12}$.

48. $\dfrac{9x^3+6x^2-2x-4}{12x^3-5x^2+4x-4}$. 49. $\dfrac{2x^3-13x+15}{3x^3+9x^2-5x-15}$.

50. $\dfrac{x^3+11x^2+30x}{9x^3+53x^2-9x-18}$. 51. $\dfrac{18x^3-11a^2x-2a^3}{18x^2-6ax-12a^2}$.

52. $\dfrac{3(x^3-y^3)-5xy(x-y)+y^3}{3(x^3+y^3)+xy(x+y)-5y^3}$. 53. $\dfrac{a^2-acx+(ac-b^2+bc)x^2-bcx^3}{a^2+abx+(ac-c^2+bc)x^2+c^2x^3}$.

54. $\dfrac{a^4+a^3b+ab^3+b^4}{a^4+3a^3b+4a^2b^2+3ab^3+b^4}$. 55. $\dfrac{3a^4-a^2b^2-2b^4}{10a^4+15a^3b-10a^2b^2-15ab^3}$.

56. $\dfrac{4x^4+11x^2+25}{4x^4-9x^2+30x-25}$. 57. $\dfrac{x^4-2x^3-25x^2+26x+120}{x^4-4x^3-19x^2+46x+120}$.

58. $\dfrac{3a^4-14a^3x-9ax^3+2x^4}{2a^4-9a^3x-14ax^3+3x^4}$. 59. $\dfrac{3a^5b-27a^4b^2+78a^3b^3-72a^2b^4}{2a^4b^2+10a^3b^3-4a^2b^4-48ab^5}$.

60. $\dfrac{16a^2x^6-26a^3x^5+46a^4x^4-42a^5x^3}{18a^3x^6+3a^4x^5-132a^5x^4+63a^6x^3}$.

173. Reduction to Lowest Common Denominator. Let it be required to reduce the fractions

$$\frac{a}{b},\quad \frac{c}{d},\quad \frac{e}{f},$$

to a common denominator. By § 168, we have

$$\frac{a}{b}=\frac{a\times df}{b\times df}=\frac{adf}{bdf},$$

$$\frac{c}{d}=\frac{c\times bf}{d\times bf}=\frac{bcf}{bdf},$$

$$\frac{e}{f}=\frac{e\times bd}{f\times bd}=\frac{bde}{bdf}.$$

Hence to reduce fractions to a *common denominator*, we *multiply the terms of each fraction by the product of the denominators of all the others.*

If the denominators b, d, f, have no common factors, it is clear that the reduced fractions are in their *lowest terms* If, however, they have one or more factors in common, it is evident that a common denominator can be found of dimensions *lower* than bdf that is, that common denominator in this case is the L C M. of b, d and f For let m be the L C M of b, d, f, so that $m=bx=dy=fz$ Then the quotients obtained from the division of m successively by b, d and f, are x, y and z Therefore the reduced fractions are

$$\frac{ax}{bx}, \quad \frac{cy}{dy}, \quad \frac{ez}{fz}, \text{ or } \frac{ax}{m}, \quad \frac{cy}{m}, \quad \frac{ez}{m}.$$

But m is the *least* number which is divisible by b, d, and f, therefore these fractions must be in their *lowest possible terms* Hence we get the following

RULE —*Find the* L C M, *of denominators, divide it by the denominator of each fraction, and multiply its terms by the quotient thus obtained.*

Examples.

Ex. 1. Reduce $\frac{3}{2a}$, $\frac{5}{3a^2}$ and $\frac{4}{ab}$ to the Lowest Common Denominator.

The L C M. of the denominators $=6a^2b$

$$\frac{3}{2a}=\frac{3\times 3ab}{2a\times 3ab}=\frac{9ab}{6a^2b}.$$

$$\frac{5}{3a^2}=\frac{5\times 2b}{3a^2\times 2b}=\frac{10b}{6a^2b},$$

$$\frac{4}{ab}=\frac{4\times 6a}{ab\times 6a}=\frac{24a}{6a^2b}$$

Ex. 2. Reduce $\frac{x}{a}$, $\frac{x-a}{a(x+a)}$ and $\frac{3a}{x^2-a^2}$ to the L C D

The L. C M of denominators$=a(x^2-a^2)$

$$\frac{x}{a}=\frac{x}{a}\times\frac{x^2-a^2}{x^2-a^2}=\frac{x^3-a^2x}{a(x^2-a^2)},$$

$$\frac{x-a}{a(x+a)}=\frac{x-a}{a(x+a)}\times\frac{x-a}{x-a}=\frac{(x-a)^2}{a(x^2-a^2)},$$

$$\frac{3a}{x^2-a^2}=\frac{3a}{x^2-a^2}\times\frac{a}{a}=\frac{3a^2}{a(x^2-a^2)}$$

Ex. 3. Reduce $\frac{x}{a-1}$, $\frac{2x}{a+1}$ and $\frac{ax-1}{1-a^2}$ to the L. C. D.

The L. C. M. of the denominators of the first and second fractions is a^2-1, which differs from $1-a^2$, the denominator of the third fraction, *only in sign*. Changing therefore the sign of the third fraction [§ 168, *Cor*.], the given fractions become

$$\frac{x}{a-1}, \frac{2x}{a+1}, \frac{1-ax}{a^2-1};$$

of which the equivalent forms are

$$\frac{x(a+1)}{a^2-1}, \frac{2x(a-1)}{a^2-1}, \frac{1-ax}{a^2-1}.$$

Reduce to Lowest Common Denominator

4. $\frac{a}{9}, \frac{5a}{12}, \frac{7a}{15}$.

5. $\frac{2x}{3}, \frac{3y}{4}, \frac{5z}{18}$.

6. $\frac{a^2}{bc}, \frac{b^2}{ca}, \frac{c^2}{ab}$.

7. $\frac{5x}{yz}, \frac{4y}{zx}, \frac{6z}{xy}, \frac{x^2+2y^2}{xyz}$.

8. $\frac{a+x}{9a}, \frac{a-x}{12x}$.

9. $\frac{4x-5}{10}, \frac{2x}{5}, \frac{7x+6}{25}$.

10. $\frac{1+a}{5}, \frac{3-a}{6}, \frac{a-8}{10}$.

11. $\frac{x^2-ab}{ab}, \frac{y^2-bc}{bc}, \frac{z^2-ac}{ac}$.

12. $\frac{a}{a+b}, \frac{b}{a-b}, \frac{c}{a+b}$.

13. $\frac{x}{3}, \frac{5a}{4}, \frac{1+x}{2(1-x)}$.

14. $\frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{1-x^2}$.

15. $\frac{2}{x-1}, \frac{3}{2-x}, \frac{4}{x^2-4}$.

16. $\frac{x}{2y}, \frac{x}{x-y}, \frac{y-1}{y^2-xy}$.

17. $\frac{a-x}{a^2(a+x)}, \frac{a+x}{x^2(a-x)}, \frac{x^2-1}{ax(x^2-a^2)}$.

18. $\frac{ab}{a-b}, \frac{bc}{b-c}, \frac{ca}{c-a}$.

19. $\frac{a}{ab-b^2}, \frac{b}{a^2+ab}, \frac{a+b}{a^2-b^2}$.

20. $\frac{3}{2x-1}, \frac{5}{2x+1}, \frac{3x}{4x^2-1}, \frac{4x}{(2x-1)^2}$.

174. Addition and Subtraction of Fractions. If two or more fractional expressions be connected by the signs + or −, or both, the whole may be reduced to a simple form by first reducing the given fractions to a common denominator [§ 173], and then finding the *algebraic sum* of the numerators of the reduced fractions [§ 166, Cor. 3]. We have thus the following

Rule.—*Reduce the fractions, if necessary, to a common denominator, then find the algebraic sum of the numerators and retain the common denominator.*

Examples.

Ex. 1. Add together $\frac{a}{9}$, $\frac{5a}{12}$ and $\frac{7a}{15}$

The L. C. M. of denominators $=180$

$$\text{Required sum } = \frac{a}{9}+\frac{5a}{12}+\frac{7a}{15}=\frac{20a}{180}+\frac{75a}{180}+\frac{84a}{180}$$

$$=\frac{20a+75a+84a}{180}=\frac{179a}{180}.$$

Ex. 2. Find the sum of $\frac{5x+20}{6}$ and $\frac{10x-81}{18}$

The L. C. M. of denominators $=18$.

$$\text{Required sum } = \frac{5x+20}{6}+\frac{10x-81}{18}=\frac{3(5x+20)}{18}+\frac{10x-81}{18}$$

$$=\frac{15x+60+10x-81}{18}=\frac{25x-21}{18}$$

Ex. 3. Find the value of $\frac{a}{b}+\frac{a+b}{a-b}$

The L. C. M. of denominators $=b(a-b)$.

$$\therefore \text{ Required value}=\frac{a(a-b)}{b(a-b)}+\frac{b(a+b)}{b(a-b)}=\frac{a^2-ab+ab+b^2}{ab-b^2}=\frac{a^2+b^2}{ab-b^2}.$$

Ex. 4. Subtract $\frac{5x}{21}$ from $\frac{7x}{18}$

The L. C. M. of denominators $=126$

$$\therefore \text{ Required difference } = \frac{49x}{126}-\frac{30x}{126}=\frac{49x-30x}{126}=\frac{19x}{126}.$$

Ex. 5. From $\frac{4x+2}{3}$ take $\frac{2x-3}{3x}$

The L. C. M. of denominators $=3x$

$$\text{Required difference } = \frac{x(4x+2)}{3x}-\frac{2x-3}{3x}$$

$$=\frac{4x^2+2x-2x+3}{3x}=\frac{4x^2+3}{3x}.$$

Ex. 6. Simplify $\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}$

The L. C. M. of denominators $=a^2-x^2$

$$\text{Required value} = \frac{a(a+x)}{a^2-x^2} + \frac{3a(a-x)}{a^2-x^2} - \frac{2ax}{a^2-x^2}$$
$$= \frac{a^2+ax+3a^2-3ax-2ax}{a^2-x^2}$$
$$= \frac{4a^2-4ax}{a^2-x^2} = \frac{4a(a-x)}{a^2-x^2} = \frac{4a}{a+x}.$$

Ex 7. Simplify $\dfrac{x}{3(a-1)} + \dfrac{5x}{6(a+1)} + \dfrac{ax-1}{4(1-a^2)}$.

Changing the sign of the third fraction [§ 168, *Cor*], we have

$$\frac{x}{3(a-1)} + \frac{5x}{6(a+1)} + \frac{1-ax}{4(a^2-1)}$$

The L. C. M. of denominators $= 12(a^2-1)$.

$\therefore$ Required sum $= \dfrac{4x(a+1)}{12(a^2-1)} + \dfrac{10x(a-1)}{12(a^2-1)} + \dfrac{3(1-ax)}{12(a^2-1)}$

$$= \frac{11ax-6x+3}{12(a^2-1)}.$$

Ex 8 Simplify $\dfrac{1}{(b-c)(c-a)} + \dfrac{1}{(c-a)(a-b)} + \dfrac{1}{(a-b)(b-c)}$.

The L. C. M. of donominators $= (b-c)(c-a)(a-b)$;

$\therefore$ given fraction $= \dfrac{(a-b)+(b-c)+(c-a)}{(b-c)(c-a)(a-b)} = \dfrac{0}{\text{Denominator}} = 0.$

Simplify the following expressions

9. $\dfrac{x}{5} + \dfrac{5x}{12} + \dfrac{7x}{20}$ 10 $\dfrac{ax}{18} - \dfrac{2ax}{7} + \dfrac{7ax}{30}$ 11 $\dfrac{a+x}{15} + \dfrac{2a}{5} - \dfrac{4x-3a}{25}$.

12. $\dfrac{x}{2y} + \dfrac{5x}{6y} + \dfrac{3x}{8y}$. 13. $\dfrac{2a}{x} + \dfrac{3b}{y^2} - \dfrac{ab}{x^2y}$ 14 $\dfrac{4x^2}{y} - \dfrac{3x}{8} + \dfrac{5y}{12x}$.

15 $\dfrac{a}{bc} + \dfrac{b}{ca} + \dfrac{c}{ab}$ 16. $\dfrac{x-y}{xy} + \dfrac{y-z}{yz} + \dfrac{z-x}{zx}$.

17 $\dfrac{a+x}{5ax} - \dfrac{a-x}{3a^2} + \dfrac{a}{10x^2} + \dfrac{a^2+x^2}{6ax^2}$ 18. $\dfrac{x}{y} - \dfrac{y}{x}$ 19. $\dfrac{ax}{b} + \dfrac{by}{c}$.

20. $\dfrac{x}{y} - \dfrac{x+y}{x-y}$. 21. $\dfrac{1}{1+x} + \dfrac{1}{1-x}$. 22 $\dfrac{1}{1-x} - \dfrac{2}{1-x^2}$.

23 $\dfrac{a}{a+b} - \dfrac{b}{a-b}$. 24. $\dfrac{m}{p+q} - \dfrac{m}{p-q}$ 25. $\dfrac{x}{a^2} + \dfrac{a-x}{a(a+x)}$.

26 $\dfrac{a}{c} - \dfrac{(ad-bc)c}{c(c+dx)}$. 27. $\dfrac{3ax}{7} + \dfrac{13by}{14} - \dfrac{2ax+9by}{21}$.

Simplify the following expressions

28. $\frac{15x-11}{18}-\frac{5x-3}{6}+\frac{5}{9}$ 29. $\frac{x}{2a}-\frac{x-a}{2(x+a)}$ 30. $\frac{x}{2x-2y}+\frac{y}{2y-2x}$

31. $\frac{5x-18}{10}+\frac{2x+7}{25}+1\frac{13}{25}$ 32. $\frac{1}{x}+\frac{m-n}{2x^2}+\frac{4a^2}{x^3}$ 33. $\frac{2}{x}+\frac{1+x}{1-x}-\frac{1-x}{1+x}$.

34. $\frac{a(a+x)}{a-x}+\frac{x(3a-x)}{x-a}$. 35. $\frac{m}{m-n}+\frac{n}{m+n}+\frac{2mn}{n^2-m^2}$.

36. $\frac{1}{a^m-1}+\frac{1}{a^m+1}$ 37. $\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}$ 38. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$

39. $\frac{a-b}{a}+\frac{a}{a-b}$. 40. $\frac{1}{x-1}-\frac{1}{2x+2}-\frac{x+3}{2x^2+2}$.

41. $\frac{1}{3(3x+2)}+\frac{1}{3x-2}-\frac{4}{3(3x-1)}$. 42. $\frac{1}{x-1}-\frac{3}{x+1}+\frac{2(x-2)}{x^2+1}$.

43. $\frac{x+y}{y}-\frac{2x}{x+y}+\frac{x^3-x^2y}{y^3-x^2y}$ 44. $\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{2(1+x^2)}$.

45. $\frac{x}{x-3}-\frac{x-3}{x}+\frac{x}{x+3}-\frac{x+3}{x}$. 46. $\frac{1}{2(x-1)}-\frac{4}{x-2}+\frac{9}{2(x-3)}$.

47. $\frac{1}{6m-2n}+\frac{1}{3m+2n}-\frac{3}{6m+2n}$. 48. $\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}$.

49. $\frac{4}{x(x-2)}+\frac{1}{x^2-5x+6}-\frac{3}{x(x-3)}$. 50. $\frac{1}{8(x-1)}-\frac{1}{4(x-3)}+\frac{1}{8(x-5)}$.

51. $\frac{1}{x^2-2}-\frac{2}{x^2-1}+\frac{2}{x^2+1}-\frac{1}{x^2+2}$. 52. $\frac{x+a}{x-a}+\frac{x-a}{x+a}+2\frac{x^2-a^2}{x^2+a^2}$

53. $\frac{10x-11}{3(x^2-1)}-\frac{10x-1}{3(x^2+x+1)}+\frac{x^2-2x+5}{(x^3-1)(x+1)}$

54. $\frac{7}{2(x+1)}-\frac{1}{6(x-1)}-\frac{10x-1}{3(x^2+x+1)}$ 55. $\frac{a}{b}-\frac{(a^2-b^2)x}{b^2}+\frac{a(a^2-b^2)x^2}{b^2(b+ax)}$.

56. $\frac{3}{1-x}-\frac{17}{1-2x}+\frac{17}{1-3x}$ 57. $\frac{1}{1-x}-\frac{1}{(1-x)^2}+\frac{1}{(1-x)^3}-\frac{1}{(1-x)^4}$

58. $\frac{1}{2(a-b)}+\frac{1}{2(a+b)}+\frac{a}{a^2+b^2}-\frac{a^3}{a^4-b^4}$

59. $\frac{1}{(x+1)^3}-\frac{3}{2(x+1)^2}+\frac{5}{4(x+1)}-\frac{5}{4(x+3)}$

60. $\frac{a+c}{(a-b)(x-a)}-\frac{b+c}{(a-b)(x-b)}$.

61. $\frac{1+2x}{(3-x)(1+x)}+\frac{7}{(2+x)(1-3x)}+\frac{x}{(1+x)(2+x)}$.

Simplify the following expressions

62 $\quad \dfrac{1}{(x+1)(x+2)}-\dfrac{1}{(x+1)(x+2)(x+3)}$.

63 $\quad \dfrac{x^{2n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$.

175 Multiplication by an Integer. Let $k=\dfrac{a}{b}$,

thus $$k=a\div b \text{ [§ 167]};$$

$$\therefore k\times m=a\div b\times m=a\times m\div b \text{ [§ 73]}=am\div b\ ;$$

whence $$\frac{a}{b}\times m=\frac{am}{b}, \text{ where } m \text{ is any integer.}$$

Hence *to multiply a fraction by an integer, we multiply the numerator by that integer*

176 Multiplication of fractions Let $k=\dfrac{a}{b}\times\dfrac{c}{d}$;

$$\therefore k\times b\times d=\frac{a}{b}\times\frac{c}{d}\times b\times d=\frac{a}{b}\times b\times\frac{c}{d}\times d \text{ [§ 67]};$$

but by § 166, $\dfrac{a}{b}\times b=a$ and $\dfrac{c}{d}\times d=c$; therefore

$$k\times b\times d=a\times c, \text{ i e, } k\times(bd)=ac,$$

whence $$k=ac\div bd \text{ [§ 69]}=\frac{ac}{bd} \text{ [§ 167]},$$

$$\therefore \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

Similarly it may be shewn that

$$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}\times\ldots=\frac{ace\ldots}{bdf\ldots}.$$

Hence to multiply several fractions together, we *multiply all the numerators for the new numerator, and all the denominators for the new denominator.*

Examples

Ex 1. $\dfrac{ax}{cd}\times\dfrac{cx}{ab}=\dfrac{acx^2}{abcd}=\dfrac{x^2}{bd}$

REMARK It is always advisable, before we multiply out, according to Rule, first to *strike out any factor or factors common to both the numerator and denominator* The product will thus be found at once in

its *lowest terms* Thus striking out a and c from the numerator and denominator first, we have

$$\frac{x}{d}\times\frac{x}{b}=\frac{x^2}{bd}.$$

Ex. 2 $\dfrac{8mn}{5cd}\times\dfrac{5c+5d}{m^2n-mn^2}=\dfrac{8mn}{5cd}\times\dfrac{5(c+d)}{mn(m-n)}=\dfrac{8(c+d)}{cd(m-n)}=\dfrac{8c+8d}{cdm-cdn}.$

Ex 3 $\dfrac{3(ax+x^2)}{a^2-x^2}\times\dfrac{bc+cd}{a^2-ax}\times\dfrac{(a-x)^2}{3b^2-3d^2}$

$$=\frac{3x(a+x)}{(a+x)(a-x)}\times\frac{c(b+d)}{a(a-x)}\times\frac{(a-x)(a-x)}{3(b+d)(b-d)}=\frac{cx}{a(b-d)}.$$

Ex. 4. $\left(\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}\right)\left(\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}\right)$

$$=\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\frac{a}{x}+\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\frac{b}{y}+\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\frac{c}{z}$$

$$=1+\frac{ay}{bx}+\frac{az}{cx}+\frac{bx}{ay}+1+\frac{bz}{cy}+\frac{cx}{az}+\frac{cy}{bz}+1$$

$$=3+\frac{ay}{bx}+\frac{az}{cx}+\frac{bx}{ay}+\frac{bz}{cy}+\frac{cx}{az}+\frac{cy}{bz}$$

Simplify

5 $\dfrac{2ax}{5by}\times\dfrac{3y^2}{4x^2}$ 6 $\dfrac{10a^2m}{7m^2z}\times\dfrac{4mx}{5az}\times\dfrac{21z^2}{16ax}.$ 7 $\dfrac{ab}{xy}\times\dfrac{a+x}{a-x}\times\dfrac{ax}{by}.$

8 $\dfrac{ax}{(a-x)^2}\times\dfrac{a^2-x^2}{ab}$ 9 $\left(a-\dfrac{x^2}{a}\right)\left(\dfrac{a}{x}+\dfrac{x}{a}\right)$

10 $\left(1+\dfrac{3x}{a-x}\right)\left(\dfrac{a-x}{a+2x}\right)^2$ 11 $\dfrac{a^2-b^2}{5b}\times\dfrac{15a^2}{a+b}$ 12 $\left(\dfrac{a}{x}+\dfrac{b}{y}\right)\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right).$

13 $\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)\left(\dfrac{x}{y}+\dfrac{y}{z}\right).$ 14 $\left(\dfrac{a}{x}+\dfrac{b}{y}\right)\left(\dfrac{x}{a}-\dfrac{y}{b}\right)$

15. $\left(\dfrac{4a}{3x}+\dfrac{3x}{2b}\right)\left(\dfrac{2b}{3x}+\dfrac{3x}{4a}\right)$ 16 $\left(1-\dfrac{a}{x}\right)\left(1-\dfrac{a^2}{a^2-x^2}\right)$

17 $\left(a+\dfrac{ax}{a-x}\right)\left(a-\dfrac{ax}{a+x}\right)\dfrac{a^2-x^2}{a^2+x^2}$ 18 $\dfrac{2ax}{3by}\times\dfrac{x^2-y^2}{2c^2-2y^2}\times\dfrac{3bc+3by}{x^2-xy}$

19. $\dfrac{x+y}{(x-y)^3}\times\dfrac{x^3-y^3}{x^3+y^3}\times\dfrac{(x-y)^2+xy}{(x+y)^2-xy}.$ 20 $\left(a^4-\dfrac{a^2}{x^2}\right)\left(\dfrac{a^2x^2+abx^2}{ax+1}\right)\dfrac{ax}{a^2-b^2}$

21 $\dfrac{pr+(pq+qr)x+q^2x^2}{p-qx}\times\dfrac{ps+(pt-qs)x-qtx^2}{p+qx}.$

Multiply

22. x^2+x+1 by $\frac{1}{x^2}-\frac{1}{x}+1$. 23. $1-x+x^2-\frac{x^3}{1+x}$ by $1-x^2$.

24. $ax+1+\frac{1}{ax}$ by $ax-1+\frac{1}{ax}$. 25. $\frac{a^2}{x^2}-\frac{ab}{2xy}+\frac{b^2}{y^2}$ by $\frac{3a^2}{x^2}-\frac{ab}{5xy}+\frac{b^2}{y^2}$

177. Division by an Integer Let $\lambda=\frac{a}{b}$; thus

$$\lambda=a\div b \text{ [§ 167]};$$

$$\lambda\div m=a\div b\div m=a\div(bm) \text{ [§ 70]} =\frac{a}{mb} \text{ [§ 167]};$$

whence $\frac{a}{b}\div m=\frac{a}{bm}$, where m is any integer.

Hence *to divide a fraction by an integer*, we *multiply the denominator by that integer.*

178 Division of fractions Let $q=\frac{a}{b}\div\frac{c}{d}$;

then, since *Divisor* × *Quotient* = *Dividend*, we have $\frac{c}{d}\times q=\frac{a}{b}$;

$$q\times\frac{c}{d}\times b\times d=\frac{a}{b}\times b\times d\text{ ; or }q\times b\times\frac{c}{d}\times d=\frac{a}{b}\times b\times d \text{ [§ 67]},$$

but $\frac{c}{d}\times d=c$ and $\frac{a}{b}\times b=a$ [§ 166]; therefore

$$q\times b\times c=a\times d,\ i\,e,\ q\times(b\times c)=(a\times d) \text{ [§ 65]};$$

whence $q=\frac{a\times d}{b\times c}$ [§ 69], $\therefore\ \frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$ [§ 176].

Hence to divide one fraction by another, we *invert the divisor, and proceed as in multiplication.*

Examples.

Ex. 1 $\frac{25x^2}{24y^2}\div\frac{5x}{12y}=\frac{25x^2}{24y^2}\times\frac{12y}{5x}=\frac{25\times12\times x^2y}{24\times5\times xy^2}=\frac{5x}{2y}$,

or it may be worked thus—

$\frac{25x^2}{24y^2}\times\frac{12y}{5x}=\frac{5x\times5x}{12y\times2y}\times\frac{12y}{5x}=\frac{5x}{2y}$ (striking out the common factors; *Vide* REMARK, Ex 1, § 176).

Ex 2 $\dfrac{4x^2-9}{3xy}\div\dfrac{2x^2+3x}{6y}=\dfrac{(2x+3)(2x-3)}{3xy}\times\dfrac{6y}{x(2x+3)}$

$$=\frac{2(2x-3)}{x^2}=\frac{4x-6}{x^2}.$$

Ex. 3. $\left(1-\dfrac{5x^2}{x^2+y^2}\right)\div\left(1+\dfrac{2x}{y}\right)=\dfrac{y^2-4x^2}{x^2+y^2}\div\dfrac{y+2x}{y}=\dfrac{y^2-4x^2}{x^2+y^2}\times\dfrac{y}{y+2x}$

$$=\frac{y^2-2xy}{y^2+x^2}.$$

Ex. 4 Divide $\dfrac{a^3}{x^3}+a^2-x^2+\dfrac{x}{a}-\dfrac{a}{x}-\dfrac{x^3}{a^3}$ by $\dfrac{a}{x}-\dfrac{x}{a}$.

Arrange according to powers of a

$$\frac{a}{x}-\frac{x}{a}\Big)\frac{a^3}{x^3}+a^2-\frac{a}{x}-x^2+\frac{x}{a}-\frac{x^3}{a^3}\Big(\frac{a^2}{x^2}+ax+\frac{x^2}{a^2}$$

$$\begin{array}{l}
\dfrac{a^3}{x^3}-\dfrac{a}{x} \\
\hline
\qquad a^2-x^2 \\
\qquad a^2-x^2 \\
\hline
\qquad \dfrac{x}{a}-\dfrac{x^3}{a^3} \\
\qquad \dfrac{x}{a}-\dfrac{x^3}{a^3} \\
\hline
\end{array}$$

$$\text{required quotient}=\frac{a^2}{x^2}+ax+\frac{x^2}{a^2}$$

Simplify

5. $\dfrac{5ax}{8x^2y}\div\dfrac{30a^2}{12xy}$ 6. $\dfrac{4xyz}{5a^2b}\div\dfrac{3yz}{10ax}$ 7. $\dfrac{ma^3}{nbc}\div\dfrac{m^2a^2}{nac^2}$

8. $\left(1-\dfrac{x^2}{a^2}\right)\div\dfrac{a+x}{a^2}$. 9. $\dfrac{a^2+ab}{a-b}\div\dfrac{a^4-b^4}{(a-b)^2}$ 10. $\dfrac{2ax-x^2}{c^3-x^3}\div\dfrac{2a-x}{(c-x)^2}$

11. $\dfrac{2a(1-x^2)^2}{cy}\div\dfrac{(1-x)(1+x)^2}{y^3}$ 12. $\left(1+\dfrac{1}{x}\right)\div\left(x-\dfrac{1}{x}\right)\div\left(\dfrac{x}{x-1}\right)^2$.

13. $\dfrac{a-x}{3x-3}\div\left(1+\dfrac{2}{x^2-1}\right)\div\left(\dfrac{a}{x}-1\right)$

Divide

14. $\dfrac{a}{x}-\dfrac{y}{b}$ by $\dfrac{y}{a}-\dfrac{b}{x}$ 15. $1-\dfrac{2xy}{x^2+y^2}$ by $\dfrac{x^3-y^3}{x-y}-3xy$.

16. $\dfrac{a+2x}{a+x}+\dfrac{a}{x}$ by $\dfrac{x}{a+x}+\dfrac{a+x}{x}$ 17. $\dfrac{x}{y}-\dfrac{2x^2}{(x+y)^2}$ by $\dfrac{y}{x}-\dfrac{2y^2}{(x+y)^2}$.

Divide

18. $1+\dfrac{6}{x^2-5x}$ by $1+\dfrac{3x-5}{x^2-6x+5}$. 19. $\dfrac{1-2x+x^2}{1-x^6}$ by $\dfrac{1-3x+3x^2-x^3}{1+x^2+x^4}$.

20. $y^2+1+\dfrac{1}{y^2}$ by $\dfrac{1}{y^2}-\dfrac{1}{y}+1$. 21. $z^2+1+\dfrac{1}{z^2}$ by $z+1+\dfrac{1}{z}$.

22. $\dfrac{3x^2}{8b^2}+\dfrac{a}{2b}-\dfrac{4a^2}{x^2}$ by $\dfrac{2a}{x}+\dfrac{x}{2b}$. 23. $x^2+\dfrac{1}{x^2}-\left(x+\dfrac{1}{x}\right)$ by $\left(x-\dfrac{1}{x}\right)^2$.

24. $x^4-\dfrac{1}{x^4}-2\left(x^2-\dfrac{1}{x^2}\right)-\left(x+\dfrac{1}{x}\right)$ by $x^2-\dfrac{1}{x^2}-3\left(x-\dfrac{1}{x}\right)-1$

25. $\dfrac{a^2x^3}{bd}+\dfrac{abx^3}{c^2d}-\dfrac{acx^2}{d^2}-\dfrac{b^2x}{cd^2}+\dfrac{a^2x}{bc}-\dfrac{a}{d}$ by $\dfrac{ax}{c}-\dfrac{b}{d}$

179. Complex Fractions A fraction whose numerator or denominator, or both, are fractions, is called a *Complex Fraction.*

Thus $\dfrac{\frac{a}{b}}{c}$, $\dfrac{a}{\frac{b}{c}}$, $\dfrac{\frac{a}{x}}{\frac{y}{a}}$, are Complex Fractions.

Hence in reducing complex fractions the Rules of the foregoing articles will apply.

Examples

Ex. 1. $\dfrac{\frac{a}{x}}{\frac{y}{a}}=\dfrac{a}{x}\div\dfrac{y}{a}=\dfrac{a}{x}\times\dfrac{a}{y}=\dfrac{a^2}{xy}$.

Ex. 2 $\dfrac{1-x^2}{\frac{1}{x}-1}=(1-x^2)\div\left(\dfrac{1}{x}-1\right)=(1-x^2)\div\dfrac{1-x}{x}=(1-x^2)\times\dfrac{x}{1-x}$

$$=\frac{x(1-x^2)}{1-x}=x(1+x).$$

Ex 3. $\dfrac{a+\frac{ab}{a-b}}{\frac{a}{a-b}-\frac{b}{a}}=\left(a+\dfrac{ab}{a-b}\right)\div\left(\dfrac{a}{a-b}-\dfrac{b}{a}\right)$

$$=\frac{a(a-b)+ab}{a-b}\div\frac{a^2-b(a-b)}{a(a-b)}=\frac{a^2}{a-b}\div\frac{a^2-ab+b^2}{a(a-b)}$$

$$=\frac{a^2}{a-b}\times\frac{a(a-b)}{a^2-ab+b^2}=\frac{a^3(a-b)}{(a-b)(a^2-ab+b^2)}=\frac{a^3}{a^2-ab+b^2}.$$

Ex. 4 Reduce $\dfrac{\dfrac{1}{1+x}+\dfrac{x}{1-x}}{\dfrac{1}{1-x}-\dfrac{x}{1+x}}+\dfrac{1+\dfrac{1}{x}}{1+x}$.

(1) First fraction $=\left\{\dfrac{1}{1+x}+\dfrac{x}{1-x}\right\}\div\left\{\dfrac{1}{1-x}-\dfrac{x}{1+x}\right\}$

$$=\frac{1-x+x+x^2}{1-x^2}\div\frac{1+x-x+x^2}{1-x^2}=\frac{1+x^2}{1-x^2}\div\frac{1+x^2}{1-x^2}$$

$$=\frac{1+x^2}{1-x^2}\times\frac{1-x^2}{1+x^2}=1.$$

(2) Second fraction $=\left(1+\dfrac{1}{x}\right)\div(1+x)=\dfrac{x+1}{x}\div(1+x)$

$$=\frac{x+1}{x}\times\frac{1}{1+x}=\frac{x+1}{x(x+1)}=\frac{1}{x}$$

∴ Required fraction $=1+\dfrac{1}{x}=\dfrac{1+x}{x}$.

Ex. 5 $\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c}}}=\dfrac{1}{a+\dfrac{1}{\dfrac{bc+1}{c}}}=\dfrac{1}{a+\dfrac{c}{bc+1}}=\dfrac{1}{\dfrac{abc+a+c}{bc+1}}=\dfrac{bc+1}{abc+a+c}$

Simplify

6 $\dfrac{x-1\frac{1}{3}}{2x}$ 7 $\dfrac{\frac{3}{2}a}{3-\frac{3}{4}a}$ 8 $\dfrac{4m-\dfrac{1}{m}}{2+\dfrac{1}{m}}$ 9 $\dfrac{25x-\dfrac{9}{x}}{5+\dfrac{3}{x}}$ 10 $\dfrac{x-\dfrac{2}{3}}{\dfrac{3x}{2}-\dfrac{2}{3x}}$

11 $\dfrac{\dfrac{a}{x}+\dfrac{b}{y}}{\dfrac{a}{x}\times\dfrac{b}{y}}$ 12 $\dfrac{\dfrac{m}{a}-\dfrac{n}{a}}{\dfrac{x}{a}+\dfrac{y}{a}}$ 13 $\dfrac{x-\dfrac{y}{z}}{z-\dfrac{y}{x}}$ 14 $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{a}{x}+\dfrac{b}{x}}$ 15 $\dfrac{\dfrac{2ax}{a^2-x^2}}{1-\dfrac{a}{a+x}}$

16 $\dfrac{\dfrac{xyz}{a}-mb}{ma-\dfrac{xyz}{b}}$ 17 $\dfrac{1-\dfrac{x-y}{x+y}}{2+\dfrac{2y}{x-y}}$ 18 $\dfrac{x^3-\dfrac{1}{x}}{x+\dfrac{1}{x}+1}$. 19 $\dfrac{x+y+\dfrac{y^2}{x}}{y-\dfrac{x^3}{y^2}}$.

20. $\dfrac{1+\dfrac{2mn}{m^2+n^2}}{\dfrac{m^3+n^3}{m+n}+3mn}$ 21. $\dfrac{x+\dfrac{y-x}{1+xy}}{1-x\dfrac{y-x}{1+xy}}$ 22. $\dfrac{\dfrac{(a+b)^2}{4ab}-1}{\dfrac{(a-b)^2}{4ab}+1}\times\dfrac{a+b}{a-b}$.

Simplify

23. $1-\dfrac{x-\frac{1+x^2}{1-2x}}{x+2\frac{1+x^2}{1-2x}}$ 24. $\dfrac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}$. 25 $\dfrac{\frac{x^2+y^2}{y}-x}{\frac{1}{y}-\frac{1}{x}}\div\dfrac{x^3+y^3}{x^2-y^2}$.

26 $\dfrac{\frac{1}{x}-\frac{1}{y+z}}{\frac{1}{y+z}+\frac{1}{x}}$. 27 $\dfrac{x-\frac{1}{y}}{1-\frac{(1-x)(1+y)}{y-x}}$ 28 $\dfrac{(1-x^2)(1-x^3)}{x(1+x)(1-x)^2}-\dfrac{x^3+\frac{1}{x^3}}{x^2+\frac{1}{x^2}-1}$

29. $\dfrac{3}{x+1}-\dfrac{2x-1}{x^2+\frac{x}{2}-\frac{1}{2}}$ 30. $\dfrac{a-\frac{ab}{a+b}}{a^2+\frac{a^2b^2}{a^2-b^2}}\times\dfrac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a}-\frac{1}{b}}$

31 $\dfrac{\frac{1}{a}+\frac{1}{b+c}}{\frac{1}{a}-\frac{1}{b+c}}-\left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$. 32. $2-\dfrac{1}{1+\dfrac{1}{1+\frac{1}{x}}}$

33 $\dfrac{1}{a-b+\dfrac{1}{a-\frac{1}{b}}}$ 34 $\dfrac{1}{x-1+\dfrac{1}{1+\frac{x}{4-x}}}$. 35 $2+\dfrac{3}{2-\dfrac{3}{2+\dfrac{3}{2-\frac{3+4x}{2x-1}}}}$.

180 Fractions with certain denominators. If $a-b$, $b-c$, $c-a$, where a, b, c, follow one another in *cyclic order* [§ 133], enter in pairs the denominators of fractions, it is often seen that the numerators are so arranged that the algebraic sum of the new numerator assumes either entirely, or in part, one of these forms.

$$\left.\begin{array}{l} a^2(b-c)+b^2(c-a)+c^2(a-b), \\ bc(b-c)+ca(c-a)+ab(a-b), \\ a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2) \end{array}\right\}$$

the first two of which are each equal to $-(b-c)(c-a)(a-b)$, and the third to $(b-c)(c-a)(a-b)$ [see § 117] Hence it is generally advantageous, in simplifying these fractions, to transform the denominators so as to give $(b-c)(c-a)(a-b)$, which we shall denote in this article by D*, for their L. C. M

* We shall also represent the lowest common denominator by D, when *any other letters* stand for a, b and c, for example

$$D=(y-z)(z-x)(x-y), \text{ or } (\beta-\gamma)(\gamma-a)(a-\beta), \text{ \&c.}$$

Examples

Ex. 1 Simplify $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$

Change the signs of the fractors $a-c$, $b-a$, $c-b$, thus the given fraction

$$=-\frac{a^2}{(a-b)(c-a)}-\frac{b^2}{(b-c)(a-b)}-\frac{c^2}{(c-a)(b-c)}$$
$$=-\left\{\frac{a^2}{(a-b)(c-a)}+\frac{b^2}{(a-b)(b-c)}+\frac{c^2}{(b-c)(c-a)}\right\}$$

Thus the L. C. M. of the denominators now is $(b-c)(c-a)(a-b)=D$, and therefore the numerator of the reduced fraction within the bracket

$$=a^2(b-c)+b^2(c-a)+c^2(a-b)$$
$$=-(b-c)(c-a)(a-b)\ [\S\ 117]=-D\ ,$$
$$\therefore\quad \text{value required}=-\frac{-D}{D}=1.$$

Ex 2 Simplify $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}$.

The given fraction

$$=-\left\{\frac{bc}{(a-b)(c-a)}+\frac{ca}{(b-c)(a-b)}+\frac{ab}{(c-a)(b-c)}\right\}\ \text{[see Ex 1]}$$

Numerator of the reduced fraction within the bracket

$$=bc(b-c)+ca(c-a)+ab(a-[illegible])=-D$$

$$\text{val[illegible] required}=-[illegible]=[illegible]$$

Ex 3. Simpl[illegible] $\dfrac{y+z}{[illegible](x-z)}+\dfrac{z+[illegible]}{zx(y-[illegible])([illegible]-x)}+\dfrac{x+y}{xy(z-[illegible])([illegible])}$.

The given fraction

$$=-\left\{\frac{y+z}{yz(x-y)(z-[illegible])}+[illegible]+[illegible](y-z)}\right\}.$$

The L. C. D. here is xy[illegible] numerator within the bracket

$$=x(y+z)(y-z)+y(z+x)(z-[illegible])+[illegible]+y)(x-y)$$
$$=x(y^2-z^2)+y(z^2-x^2)+z(x^2-[illegible])[illegible]\ D\ [\S\ 117]\ ,$$
$$\text{value required}=-\frac{D}{xyzD}=[illegible]\frac{1}{[illegible]yz}$$

Ex. 4. Simplify $\dfrac{a^2+bc+1}{(a-b)(a-c)}+\dfrac{b^2+ca+1}{(b-c)(b-a)}+\dfrac{c^2+ab+1}{(c-a)(c-b)}$.

The given fraction

$$=-\left\{\frac{a^2+bc+1}{(a-b)(c-a)}+\frac{b^2+ca+1}{(b-c)(a-b)}+\frac{c^2+ab+1}{(c-a)(b-c)}\right\},$$

$\therefore$ reduced numerator of the expression within the bracket

$$=(a^2+bc+1)(b-c)+(b^2+ca+1)(c-a)+(c^2+ab+1)(a-b)$$

$$=\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}+\{bc(b-c)+ca(c-a)+ab(a-b)\}$$
$$+\{(b-c)+(c-a)+(a-b)\}$$

$$=-D-D+0=-2D,$$

$$\therefore \quad \text{value required}=-\frac{-2D}{D}=2$$

REMARK. This fraction is in fact equivalent to three fractions, viz. Ex. 1 and 2 of this article, and Ex 8 of § 174 For

$$\frac{b^2+bc+1}{(a-b)(a-c)}=\frac{a^2}{(a-b)(a-c)}+\frac{bc}{(a-b)(a-c)}+\frac{1}{(a-b)(a-c)},$$

$$\frac{b^2+ca+1}{(b-c)(b-a)}=\frac{b^2}{(b-c)(b-a)}+\frac{ca}{(b-c)(b-a)}+\frac{1}{(b-c)(b-a)},$$

$$\frac{c^2+ab+1}{(c-a)(c-b)}=\frac{c^2}{(c-a)(c-b)}+\frac{ab}{(c-a)(c-b)}+\frac{1}{(c-a)(c-b)};$$

thus the given fraction $=\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$

$$+\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)}$$

$$+\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}.$$

Simplify

5. $\dfrac{a(b+c)}{(c-a)(a-b)}+\dfrac{b(c+a)}{(a-b)(b-c)}+\dfrac{c(a+b)}{(b-c)(c-a)}$

6. $\dfrac{a+b-c}{(b-c)(c-a)}+\dfrac{b+c-a}{(c-a)(a-b)}+\dfrac{c+a-b}{(a-b)(b-c)}$.

7. $\dfrac{a^2-bc}{(c-a)(a-b)}+\dfrac{b^2-ca}{(a-b)(b-c)}+\dfrac{c^2-ab}{(b-c)(c-a)}$.

8. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-x)(y-z)}+\dfrac{1}{z(z-x)(z-y)}$

9. $\dfrac{x^2+k^2}{(x-y)(x-z)}+\dfrac{y^2+k^2}{(y-z)(y-x)}+\dfrac{z^2+k^2}{(z-x)(z-y)}$

Simplify

10 $\dfrac{a^2+a+1}{(a-b)(a-c)}+\dfrac{b^2+b+1}{(b-c)(b-a)}+\dfrac{c^2+c+1}{(c-a)(c-b)}$

11 $\dfrac{bc(x-a)^2}{(a-b)(a-c)}+\dfrac{ca(x-b)^2}{(b-c)(b-a)}+\dfrac{ab(x-c)^2}{(c-a)(c-b)}$

12. $\dfrac{x+k}{(x-y)(x-z)yz}+\dfrac{y+k}{(y-z)(y-x)zx}+\dfrac{z+k}{(z-x)(z-y)xy}$

13 $\dfrac{x^2+x+1}{(x-y)(x-z)}yz+\dfrac{y^2+y+1}{(y-z)(y-x)}zx+\dfrac{z^2+z+1}{(z-x)(z-y)}xy$

14 $\dfrac{(x-a)(x-b)}{(x-y)(x-z)}+\dfrac{(y-a)(y-b)}{(y-z)(y-x)}+\dfrac{(z-a)(z-b)}{(z-x)(z-y)}$

15. $\dfrac{(1+ab)(1+bc)}{(a-b)(b-c)}+\dfrac{(1+bc)(1+ca)}{(b-c)(c-a)}+\dfrac{(1+ca)(1+ab)}{(c-a)(a-b)}.$

16 $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}.$

17 $\dfrac{a^2+ah+h^2}{(a-b)(a-c)}+\dfrac{b^2+bh+h^2}{(b-a)(b-c)}+\dfrac{c^2+ch+h^2}{(c-a)(c-b)}$

18 $\dfrac{x^2+x+1}{(x-y)(x-z)}(y+z)+\dfrac{y^2+y+1}{(y-z)(y-x)}(z+x)+\dfrac{z^2+z+1}{(z-x)(z-y)}(x+y)$

19 $\dfrac{(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{(b+c)(b+a)}{(b-c)(b-a)}+\dfrac{(c+a)(c+b)}{(c-a)(c-b)}$

20 $\dfrac{1}{(a-b)(a-c)(x+a)}+\dfrac{1}{(b-c)(b-a)(x+b)}+\dfrac{1}{(c-a)(c-b)(x+c)}.$

21 $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-c)(b-a)(x-b)}+\dfrac{c}{(c-a)(c-b)(x-c)}.$

22 $\dfrac{a^2}{(a-b)(a-c)(x+a)}+\dfrac{b^2}{(b-c)(b-a)(x+b)}+\dfrac{c^2}{(c-a)(c-b)(x+c)}.$

23 Prove that $\dfrac{a^2+a+1}{(a-b)(a-c)(x-a)}+\dfrac{b^2+b+1}{(b-c)(b-a)(x-b)}$

$$+\frac{c^2+c+1}{(c-a)(c-b)(x-c)}=\frac{x^2+x+1}{(x-a)(x-b)(x-c)}.$$

181 Fractions with Symmetrical Denominators The student has perhaps noticed that in working out the examples of the last article, much unnecessary trouble has been saved by the method of that article We may now remark that this method will be advantageous generally, when the denominators has a *certain symmetry in their formation*

Examples

Simplify

1. $\dfrac{b+c}{(a-b)(a-c)}+\dfrac{c+a}{(b-a)(b-c)}+\dfrac{a+b}{(c-a)(c-b)}$.

2. $\dfrac{x}{(x-y)(x-z)}+\dfrac{y}{(y-z)(y-x)}+\dfrac{z}{(z-x)(z-y)}$.

3. $\dfrac{x+a}{(a-b)(a-c)}+\dfrac{x+b}{(b-c)(b-a)}+\dfrac{x+c}{(c-a)(c-b)}$.

4. $\dfrac{a-b}{(b+c)(c+a)}+\dfrac{b-c}{(c+a)(a+b)}+\dfrac{c-a}{(a+b)(b+c)}$.

5. $\dfrac{x+a}{(x-b)(x-c)}+\dfrac{x+b}{(x-c)(x-a)}+\dfrac{x+c}{(x-a)(x-b)}$.

6. $\dfrac{x^2(y-z)}{(x+y)(x+z)}+\dfrac{y^2(z-x)}{(y+z)(y+x)}+\dfrac{z^2(x-y)}{(z+x)(z+y)}$.

7. $\dfrac{x^2-yz}{(x+y)(x+z)}+\dfrac{y^2-zx}{(y+z)(y+x)}+\dfrac{z^2-xy}{(z+x)(z+y)}$.

8. $\dfrac{x^2-yz}{(x-y)(x-z)}+\dfrac{y^2+zx}{(y+z)(y-x)}+\dfrac{z^2+xy}{(z-x)(z+y)}$.

9. $\dfrac{x+y}{(x^2-yz)(y^2-zx)}+\dfrac{y+z}{(y^2-zx)(z^2-xy)}+\dfrac{z+x}{(z^2-xy)(x^2-yz)}$.

10. $\dfrac{1}{(a+b)(b-c)}+\dfrac{1}{(b+c)(c-a)}+\dfrac{1}{(c+a)(a-b)}$.

11. $\dfrac{abx}{(ax-by)(ax-cz)}+\dfrac{acy}{(by-ax)(by-cz)}+\dfrac{bcz}{(cz-ax)(cz-by)}$.

12. Prove that $\dfrac{x^2}{(x+y)(x+z)}+\dfrac{y^2}{(y+z)(y+x)}+\dfrac{z^2}{(z+x)(z+y)}$

$$=\frac{yz}{(x+y)(x+z)}+\frac{zx}{(y+z)(y+x)}+\frac{xy}{(z+x)(z+y)}$$

$$=1-\frac{2xyz}{(x+y)(y+z)(z+x)}.$$

182 Examination upon Chapter XV.

1 Define a *fraction*, and from the definition, prove that $\dfrac{a}{b}\times b=a$.

2 What are the *terms* of a fraction? Define the terms *numerator* and *denominator* of a fraction.

3. What are Mixed Quantities? How can a Mixed Quantity be reduced to an Improper Fraction, and *vice versa*?

4. Shew that $\frac{a}{b}=\frac{am}{bm}=\frac{a\div m}{b\div m}$, whatever m may be

5. When is a fraction said to be in its *lowest terms* ?

6. What is the use of reducing fractions to others having a *common denominator* ? Shew that the common denominator will be the *lowest* when it is the L. C. M. of the denominators

7. Prove that $\frac{x}{a}+\frac{y}{a}=\frac{x+y}{a}$ and $\frac{x}{a}-\frac{y}{a}=\frac{x-y}{a}$

8. Prove that

$$\frac{a}{b}\times m=\frac{am}{b}=\frac{a}{b\div m}\text{, and that }\frac{a}{b}\div m=\frac{a}{bm}=\frac{a\div m}{b}.$$

9. Shew that $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$, and that $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$

***183. Some examples worked out.** We shall conclude this chapter by working a few examples

Ex. 1. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$

$$\begin{aligned}\text{Given expression }&=\frac{(x+2a)(x-2b)+(x-2a)(x+2b)}{(x-2a)(x-2b)}\\&=\frac{x^2+2(a-b)x-4ab+x^2-2(a-b)x-4ab}{x^2-2(a+b)x+4ab}\\&=\frac{2(x^2-4ab)}{x^2-2(a+b)x+4ab}\\&=\frac{2\{x^2-(a+b)x\}}{x^2-2(a+b)x+(a+b)x},\quad\because (a+b)x=4ab,\\&=\frac{2\{x^2-(a+b)x\}}{x^2-(a+b)x}=2\end{aligned}$$

Ex. 2. If $y=x+\frac{1}{x}$, shew that $x^3+\frac{1}{x^3}=y^3-3y$.

We have $x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3\,x\,\frac{1}{x}\left(x+\frac{1}{x}\right)$ [§ 103, *Cor* 2] $=y^3-3y$.

Ex. 3. Simplify $\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(c-a)^2}{(a+b)^2-c^2}+\frac{c^2-(a-b)^2}{(b+c)^2-a^2}$

Given expression

$$=\frac{(a+b-c)(a-b+c)}{(a+b+c)(a-b+c)}+\frac{(b+c-a)(b-c+a)}{(a+b+c)(a+b-c)}+\frac{(c+a-b)(c-a+b)}{(a+b+c)(b+c-a)}$$

$$=\frac{a+b-c}{a+b+c}+\frac{b+c-a}{a+b+c}+\frac{c+a-b}{a+b+c}=\frac{a+b+c}{a+b+c}=1$$

Ex. 4 Shew that $1-\left\{\dfrac{a^2+b^2-c^2}{2ab}\right\}^2$

$$=\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4a^2b^2}$$

$$=\frac{4s(s-a)(s-b)(s-c)}{a^2b^2},\text{ if } 2s=a+b+c.$$

Given expression $=\left\{1+\dfrac{a^2+b^2-c^2}{2ab}\right\}\left\{1-\dfrac{a^2+b^2-c^2}{2ab}\right\}$

$$=\frac{2ab+a^2+b^2-c^2}{2ab}\times\frac{2ab-a^2-b^2+c^2}{2ab}$$

$$=\frac{(a+b)^2-c^2}{2ab}\times\frac{c^2-(a-b)^2}{2ab}=\frac{(a+b+c)(a+b-c)}{2ab}\times\frac{(c+a-b)(c-a+b)}{2ab}$$

$$=\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4a^2b^2}.$$

Again $2s=a+b+c$, $2s-2a=a+b+c-2a$, or $2(s-a)=b+c-a$; similarly $2(s-b)=c+a-b$ and $2(s-c)=a+b-c$,

$$\therefore \text{ given expn.}=\frac{2s\,2(s-a)\,2(s-b)\,2(s-c)}{4a^2b^2}=\frac{4s(s-a)(s-b)(s-c)}{a^2b^2}.$$

Ex 5. Find the value of

$$\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{c+a}{ca}(c^2+a^2-b^2)+\frac{a+b}{ab}(a^2+b^2-c^2)$$

Given expression

$$=\left(\frac{1}{b}+\frac{1}{c}\right)(b^2+c^2-a^2)+\left(\frac{1}{c}+\frac{1}{a}\right)(c^2+a^2-b^2)+\left(\frac{1}{a}+\frac{1}{b}\right)(a^2+b^2-c^2)$$

$$=\frac{1}{b}(b^2+c^2-a^2+a^2+b^2-c^2)+\frac{1}{c}(b^2+c^2-a^2+c^2+a^2-b^2)$$

$+\dfrac{1}{a}(c^2+a^2-b^2+a^2+b^2-c^2)$, bracketing the coefficients of $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$,

$$=\frac{1}{b}\times 2b^2+\frac{1}{c}\times 2c^2+\frac{1}{a}\times 2a^2=2b+2c+2a=2(a+b+c).$$

Ex. 6 Shew that $\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}=\left(\dfrac{1}{a-b}+\dfrac{1}{b-c}+\dfrac{1}{c-a}\right)^2$

Right side $=\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}$

$$+2\left\{\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}+\frac{1}{(c-a)(a-b)}\right\}$$

$=\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}$; $\because$ the expression within the braces $=0$ [see § 174, Ex 8]

Ex 7 Reduce to its simplest form, the expression

$$\frac{(a-b)^2}{(b-c)(c-a)}+\frac{(b-c)^2}{(c-a)(a-b)}+\frac{(c-a)^2}{(a-b)(b-c)}.$$

The L. C. M of denominators $=(b-c)(c-a)(a-b)$, thus reduced numerator $=(a-b)^3+(b-c)^3+(c-a)^3=3(b-c)(c-a)(a-b)$ [§ 130 Ex. 5],

$$\therefore \text{ value required } =3$$

Ex. 8 Reduce to its simplest form

$$\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a-b)(b-c)(c-a)}.$$

The L C D $=2(a-b)(b-c)(c-a)$, thus reduced numerator $=2(b-c)(c-a)+2(c-a)(a-b)+2(a-b)(b-c)+(a-b)^2+(b-c)^2+(c-a)^2=\{(a-b)+(b-c)+(c-a)\}^2$ [§ 116] $=0$, &c

Ex. 9. Prove that

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$\text{Left side}=\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{c^2}{a^2}+2+\frac{a^2}{c^2}+\frac{a^2}{b^2}+2+\frac{b^2}{a^2}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{1}{a^2}(b^2+c^2)+a^2\left(\frac{1}{c^2}+\frac{1}{b^2}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{bc}{a^2}\left(\frac{b^2+c^2}{bc}\right)+\frac{a^2}{bc}\left(\frac{bc}{c^2}+\frac{bc}{b^2}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{bc}{a^2}\left(\frac{b}{c}+\frac{c}{b}\right)+\frac{a^2}{bc}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\frac{b}{c}+\frac{c}{b}+\frac{bc}{a^2}+\frac{a^2}{bc}\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\left(\frac{ac}{ab}+\frac{a^2}{bc}\right)+\left(\frac{bc}{a^2}+\frac{ab}{ac}\right)\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\frac{a}{b}\left(\frac{c}{a}+\frac{a}{c}\right)+\frac{b}{a}\left(\frac{c}{a}+\frac{a}{c}\right)\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$$

Ex 10 If $s=a+b+c+\ldots$ to n terms, shew that

$$\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}+\ldots \text{ to } n \text{ terms}=n-1.$$

$$\text{Left side}=\left(1-\frac{a}{s}\right)+\left(1-\frac{b}{s}\right)+\left(1-\frac{c}{s}\right)+\ldots\ldots\text{to } n \text{ terms}$$

$$=(1+1+1+\ldots\ldots\text{to } n \text{ terms})-\left(\frac{a}{s}+\frac{b}{s}+\frac{c}{s}+\ldots\ldots\text{to } n \text{ terms}\right)$$

$$=n-\frac{1}{s}(a+b+c+\ldots\ldots\text{ to } n \text{ terms})=n-\frac{1}{s}\times s=n-1.$$

Ex. 11 If $x=\dfrac{ab-cd}{(a-b)-(c-d)}$, prove that $(x+a)(x-b)=(x+c)(x-d)$; and for this value of x, shew that $\dfrac{x+a}{x-b}=\dfrac{(a-c)(a+d)}{(b+c)(b-d)}$.

We have $\{(a-b)-(c-d)\}x=ab-cd$,

whence $(a-b)x-ab=(c-d)x-cd$, by transp. [§ 143],

and x^2, thus $x^2+(a-b)x-ab=x^2+(c-d)x-cd$,

$\therefore\quad (x+a)(x-b)=(x+c)(x-d)$ [§ 108]

$$\text{Again } \frac{x+a}{x-b}=\frac{\dfrac{ab-cd}{(a-b)-(c-d)}+a}{\dfrac{ab-cd}{(a-b)-(c-d)}-b}=\frac{a^2-ac+ad-cd}{b^2+bc-bd-cd}=\frac{(a-c)(a+d)}{(b+c)(b-d)}.$$

Ex. 12 Prove that

$$\frac{a^2}{(a-b)(a-c)(b+c)}+\frac{b^2}{(b-c)(b-a)(c+a)}+\frac{c^2}{(c-a)(c-b)(a+b)}$$
$$=\frac{(a+b+c)^2}{(b+c)(c+a)(a+b)}.$$

Change the sign of each term; thus

L. C. M. $=-(b^2-c^2)(c^2-a^2)(a^2-b^2)$, and therefore reduced numerator

$$=a^2(b-c)(c+a)(a+b)+b^2(c-a)(a+b)(b+c)+c^2(a-b)(b+c)(c+a)$$
$$=a^2(b-c)\{a^2+a(b+c)+bc\}+b^2(c-a)\{b^2+b(c+a)+ca\}$$
$$+c^2(a-b)\{c^2+c(a+b)+ab\}$$
$$=a^2(b-c)\{a^2+a(b+c)\}+a^2bc(b-c)+b^2(c-a)\{b^2+b(c+a)\}$$
$$+ab^2c(c-a)+c^2(a-b)\{c^2+c(a+b)\}+abc^2(a-b)$$
$$=a^3(b-c)(a+b+c)+b^3(c-a)(a+b+c)+c^3(a-b)(a+b+c)$$
$$+abc\{a(b-c)+b(c-a)+c(a-b)\}$$
$$=(a+b+c)\{a^3(b-c)+b^3(c-a)+c^3(a-b)\}+abc\times 0 \text{ [§ 116]}$$
$$=(a+b+c)\times-(a+b+c)(b-c)(c-a)(a-b) \text{ [§ 132, Ex 2]}$$
$$=-(a+b+c)^2(b-c)(c-a)(a-b),$$

$$\therefore\quad \text{left side}=\frac{-(a+b+c)^2(b-c)(c-a)(a-b)}{-(b^2-c^2)(c^2-a^2)(a^2-b^2)}=\frac{(a+b+c)^2}{(b+c)(c+a)(a+b)}.$$

Ex 13 Prove that

$$\frac{x}{x^2-1}+\frac{x^2}{x^4-1}+\frac{x^4}{x^8-1}+\frac{x^8}{x^{16}-1}=\frac{1}{2}\left(\frac{x+1}{x-1}-\frac{x^{16}+1}{x^{16}-1}\right).$$

We have $\dfrac{x}{x^2-1}=\dfrac{1}{2}\times\dfrac{2x}{x^2-1}=\dfrac{1}{2}\times\dfrac{(x+1)^2-(x^2+1)}{x^2-1}$

$$=\frac{1}{2}\left\{\frac{(x+1)^2}{x^2-1}-\frac{x^2+1}{x^2-1}\right\}$$

$$=\frac{1}{2}\left(\frac{x+1}{x-1}-\frac{x^2+1}{x^2-1}\right),$$

similarly

$$\frac{x^2}{x^4-1}=\frac{1}{2}\left(\frac{x^2+1}{x^2-1}-\frac{x^4+1}{x^4-1}\right),$$

$$\frac{x^4}{x^8-1}=\frac{1}{2}\left(\frac{x^4+1}{x^4-1}-\frac{x^8+1}{x^8-1}\right),$$

$$\frac{x^8}{x^{16}-1}=\frac{1}{2}\left(\frac{x^8+1}{x^8-1}-\frac{x^{16}+1}{x^{16}-1}\right),$$

therefore by addition, the left side of the proposed expression

$$=\frac{1}{2}\left(\frac{x+1}{x-1}-\frac{x^{16}+1}{x^{16}-1}\right)$$

Miscellaneous Examples VI

1 Find the value of $\dfrac{3\frac{1}{4}-\frac{1}{3}(x-2)}{1\frac{1}{12}+(x-\frac{3}{2})}$ when $x=3\frac{1}{4}$

2 Multiply $a+b+\dfrac{b^2}{a}+\dfrac{a^2}{b}$ by $a-b+\dfrac{b^2}{a}-\dfrac{a^2}{b}$

3 Reduce $\dfrac{x^2+\left(\frac{a}{b}+\frac{b}{a}\right)xy+y^2}{x^2+\left(\frac{a}{b}-\frac{b}{a}\right)xy-y^2}$ to its lowest terms.

4 If $x=b+c$, $y=c+a$, $z=a+b$, shew that

$$\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}=2$$

5 Prove that $\dfrac{a+b}{ab}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)+\dfrac{b+c}{bc}\left(\dfrac{1}{b}-\dfrac{1}{c}\right)=\dfrac{a+c}{ac}\left(\dfrac{1}{a}-\dfrac{1}{c}\right)$.

6 Find the value of $\left(1-\dfrac{x}{y}\right)y+\left(1-\dfrac{y}{z}\right)z+\left(1-\dfrac{z}{x}\right)x$

7 Multiply $\dfrac{a}{a+x}+\dfrac{3a}{a-x}+\dfrac{2ax}{x^2-a^2}$ by $\dfrac{a+x}{2(a-x)}$

8. Resolve into factors $\frac{a^2+b^2-c^2}{2ab}-1$

9 Having given $\frac{yz}{x^2}=y+z,\ \frac{zx}{y^2}=z+x,\ \frac{xy}{z^2}=x+y,$
shew that $yz+zx+xy+2xyz=1$

10. If $a+b+c=0$, prove that

$$3+\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=0$$

11. Find the value of $\frac{x-a}{b}+\frac{x-b}{a}$ when $x=\frac{a^2}{a+b}$.

12. Divide $\frac{1}{a+b}+\frac{a-b}{a^2-ab+b^2}+\frac{ab-a^2}{a^3+b^3}$ by $\frac{a}{a^2-ab+b^2}$.

13 Reduce $\frac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$ to its lowest terms

14 Shew that

$$\left\{\frac{2bc}{b+c}-b\right\}\div\left\{\frac{1}{c}+\frac{1}{b-2c}\right\}+\left\{\frac{2bc}{b+c}-c\right\}\div\left\{\frac{1}{b}+\frac{1}{c-2b}\right\}=bc.$$

15 If $\frac{x^2}{y^2}=\frac{2a^2}{b^2}+1$, prove that $\frac{2y^2}{x^2+y^2}+\frac{a^2}{a^2+b^2}=1.$

16 Prove that $(a+b)\left(\frac{1}{a}-\frac{1}{b}\right)+(b+c)\left(\frac{1}{b}-\frac{1}{c}\right)+(c+a)\left(\frac{1}{c}-\frac{1}{a}\right)$

$$=\frac{a(b-c)}{bc}+\frac{b(c-a)}{ca}+\frac{c(a-b)}{ab}$$

17. Find the value of $\frac{a^2+ax+x^2}{a^3-x^3}-\frac{a^2-ax+x^2}{a^3+x^3}$, when $x=\frac{2a^2b}{a^2+b^2}$

18 Reduce $\dfrac{\left(\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2-b^2}\right)(a^2+b^2)^2}{\frac{a}{a+b}+\frac{b}{a-b}}$ to its simplest form

19. Simplify $\frac{1}{\left(1-\frac{c}{a}\right)\left(1-\frac{b}{a}\right)}+\frac{1}{\left(1-\frac{a}{b}\right)\left(1-\frac{c}{b}\right)}+\frac{1}{\left(1-\frac{b}{c}\right)\left(1-\frac{a}{c}\right)}$.

20. Find the simplest form of $1-\frac{a^2-(b+c)^2}{(a+b+c)^2}$.

21. Prove that $\frac{a^2+b^2+c^2+d^2}{(ab-cd)^2+(ad+bc)^2}=\frac{1}{a^2+c^2}+\frac{1}{b^2+d^2}$.

22 If $x+\dfrac{1}{y}=1$, $z+\dfrac{1}{x}=1$, then $y+\dfrac{1}{z}=1$

23 If $x+y+z=0$, prove that

$$\frac{1}{y^2+z^2-x^2}+\frac{1}{z^2+x^2-y^2}+\frac{1}{x^2+y^2-z^2}=0$$

24 Simplify $\dfrac{(a+b)(1-ab)}{(1-ab)^2-(a+b)^2}-\dfrac{a(1-b^2)+b(1-a^2)}{(1-a^2)(1-b^2)-4ab}$

25 Reduce $\dfrac{x+y+\dfrac{y^2}{x}}{x-y+\dfrac{y^2}{x}}\times\dfrac{x+\dfrac{y^3}{x^2}}{x-\dfrac{y^3}{x^2}}-\left(\dfrac{x+y}{x-y}\right)^2$

26 Divide $a^2-b^2-c^2-2bc$ by $\dfrac{a-b-c}{a+b-c}$.

27 Find the value of $\dfrac{(x+y)^2(x^4-y^4)\{(x^2+y^2)^2-x^2y^2\}}{(x^6-y^6)\{(x^2+y^2)^2+2xy(x^2+y^2)\}}$.

28 Prove that $\left(\dfrac{x}{a}-\dfrac{y}{b}\right)\dfrac{z}{c}+\left(\dfrac{x}{a}-\dfrac{z}{c}\right)\dfrac{y}{b}+\left(\dfrac{y}{b}-\dfrac{z}{c}\right)\dfrac{x}{a}=\dfrac{2y}{b}\left(\dfrac{x}{a}-\dfrac{z}{c}\right)$.

29 If $x=a+b+\dfrac{(a-b)^2}{4(a+b)}$, $y=\dfrac{a+b}{4}+\dfrac{ab}{a+b}$, shew that

$$(x-a)^2-(y-b)^2=b^2$$

30 Reduce $\dfrac{1-a^2}{(1+ax)^2-(a+x)^2}$, and find its value when $x=\dfrac{m-n}{m+n}$

31 Multiply $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2}$ by $\dfrac{x^2+xy-2y^2}{x^2-5xy+y^2}$

32 If $a^2\dfrac{a'-b}{a-b'}=a'^2\dfrac{a-b}{a'-b'}$, then $\dfrac{1}{a}+\dfrac{1}{a'}=\dfrac{1}{b}+\dfrac{1}{b'}$

33 Resolve into factors $1-\left\{\dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right\}^2$

34 If $x+y+z=0$, shew that $\dfrac{x(y^3-z^3)}{y-z}+\dfrac{y(z^3-x^3)}{z-x}+\dfrac{z(x^3-y^3)}{x-y}=0$

35 Divide $\dfrac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}$ by $\dfrac{x-c}{x-b}$.

36. Find the value of $\dfrac{x^2-y^2+x}{y^2-x^2+y}$, when $x=\dfrac{a-b}{a+b}$ and $y=\dfrac{a+b}{a-b}$

37 Simplify $\dfrac{\left(\dfrac{x}{a}+1\right)^2}{\dfrac{x}{a}-\dfrac{a}{x}} \times \dfrac{\dfrac{x}{a}+\dfrac{a}{x}-1}{\dfrac{x^3}{a^3}+1} \div \dfrac{\dfrac{x^2}{a^2}+\dfrac{x}{a}+1}{\dfrac{x^3}{a^3}-1}$.

38. If $x=\dfrac{b-c}{a}$, $y=\dfrac{c-a}{b}$, $z=\dfrac{a-b}{c}$, shew that

$$xyz+x+y+z=0$$

39. If $x=\dfrac{a+b-c}{a+b+c}$, then $\dfrac{a+bx^2}{b+ax^2}=\dfrac{(a-b+c)^2+4ab}{(b+c-a)^2+4ab}$

40. Simplify $\dfrac{1}{x}\left(x+\dfrac{1}{1-\dfrac{1}{x}}\right)-\dfrac{1}{x+1}-\dfrac{x}{x^2-1}$.

41 Reduce $\dfrac{6x^3-19x^2y+18xy^2-5y^3}{2x^2-3xy+y^2}$

42 Shew that $\dfrac{a}{b-c}+\dfrac{b}{c-a}+\dfrac{c}{a-b}+\dfrac{abc}{(b-c)(c-a)(a-b)}$

$$=\left\{1+\frac{a}{b-c}\right\}\left\{1+\frac{b}{c-a}\right\}\left\{1+\frac{c}{a-b}\right\}$$

43. If $y=\dfrac{nx}{1+(1-n)x^2}$, prove that $\dfrac{x-y}{1+xy}=(1-n)x$

44. Reduce to lowest terms $\dfrac{a^4+a^3b+ab^3+b^4}{a^4-3a^3b+4a^2b^2-3ab^3+b^4}$

45 Divide $a^6+\dfrac{1}{a^6}+a^4+\dfrac{1}{a^4}+a^2+\dfrac{1}{a^2}+2$ by $a^3+\dfrac{1}{a^3}+a+\dfrac{1}{a}$.

46. Find the value of $\dfrac{b-c}{b+c-2a}+\dfrac{c-a}{c+a-2b}+\dfrac{a-b}{a+b-2c}$

47 If $(x+a)(x+b)=(a+b)^2$, shew that $\dfrac{ax-b^2}{x-a}=\dfrac{bx-a^2}{x-b}$.

48. Simplify $\dfrac{ab^2-(ab-b^2)x+(a-b)x^2+x^3}{ab^2+(ab+b^2)x+(a+b)x^2+x^3}$.

49 Prove that

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$$

50. If $x+y+z=0$, and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$, shew that

$$bx^2+(a+b-c)xy+ay^2=0.$$

51. Reduce $\dfrac{\dfrac{x-y}{1+xy}+\dfrac{y-z}{1+yz}}{1-\dfrac{(x-y)(y-z)}{(1+xy)(1+yz)}}$ to a simple form.

52. Reduce $\dfrac{\{(ax+by)^2+(ay-bx)^2\}\{(ax+by)^2-(ay+bx)^2\}}{x^4-y^4}$.

53 Simplify $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

54 Reduce $\dfrac{3x^2-(4a+2b)x+2ab+a^2}{x^3-(2a+b)x^2+(2ab+a^2)x-a^2b}$

55. Shew that

$$\frac{2ay+bx}{2bx}-\frac{1}{2}\left(\frac{a}{x}+\frac{x}{a}\right)\left(\frac{b}{y}+\frac{y}{b}\right)+\frac{1}{2}\left(\frac{a}{x}-\frac{x}{a}\right)\left(\frac{b}{y}-\frac{y}{b}\right)+\frac{2bx-ay}{2ay}=0$$

56. Prove that $\dfrac{(a+b)^3-c^3}{a+b-c}+\dfrac{(b+c)^3-a^3}{b+c-a}+\dfrac{(c+a)^3-b^3}{c+a-b}$

$$=2(a+b+c)^2+a^2+b^2+c^2$$

57. Simplify $\dfrac{x^2+1}{x^2-1}-1-\left(\dfrac{2x}{2x-1}-\dfrac{2}{2-x}\right)$

58 Find the value of $\dfrac{1}{2a-3b+\dfrac{1}{2a-3b-\dfrac{1}{2a-3b}}}$.

59 Simplify $\dfrac{(1-ab)^2+(a+b-2)(a+b-2ab)}{(1+ab)^2-(a+b)^2}$.

60 Add together $\dfrac{x^2+y^2+x+y-xy+1}{x-y-1}$ and $\dfrac{x^2+y^2+x-y+xy+1}{x+y-1}$.

61 Prove that

$$\frac{1}{x(x-a)(x-b)}=\frac{1}{abx}+\frac{1}{a(a-b)(x-a)}+\frac{1}{b(b-a)(x-b)}$$

62 If $\dfrac{(1+a^2)(1+b^2)}{(1+ab)^2}=\dfrac{(1+c^2)(1+d^2)}{(1+cd)^2}$, shew that

(i) $\dfrac{a+d}{1-ad}=\dfrac{b+c}{1-bc}$, (ii) $\dfrac{(a-b)^2}{(1+a^2)(1+b^2)}=\dfrac{(c-d)^2}{(1+c^2)(1+d^2)}$

63. Simplify $\dfrac{x^4-2mx^3+4m^2x^2-16m^3x+16m^4}{x^4-6mx^3+4m^2x^2+16m^3x-16m^4}$

64. Find the value of the expression

$$\frac{1-x}{1+x}+\frac{(1-x)(1-x^2)}{(1+x)(1+x^2)}+\frac{(1-x)(1-x^2)(1-x^3)}{(1+x)(1+x^2)(1+x^3)}.$$

65. Reduce $\dfrac{e^{2x}x^3+e^{2x}-x^3-1}{e^{2x}x^3+2e^xx^2-e^{2x}-2e^x+x^2-1}$.

66. Shew that $\dfrac{(2x-y-z)^2+(2y-z-x)^2+(2z-x-y)^2}{(y-z)^2+(z-x)^2+(x-y)^2}=3.$

67. Simplify $\dfrac{\dfrac{a+bx}{a-bx}+\dfrac{b+ax}{b-ax}}{\dfrac{a+bx}{a-bx}-\dfrac{b+ax}{b-ax}}-1\div\dfrac{2\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}{\dfrac{x}{a-b}}$.

68. If $x=\dfrac{2ab+b^2}{a^2+ab+b^2}$, $y=\dfrac{a^2-b^2}{a^2+ab+b^2}$, then $x^3+y=y^3+x$.

69. Shew that

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-1=\left(\frac{x}{y}+\frac{x}{z}\right)\left(\frac{y}{z}+\frac{y}{x}\right)\left(\frac{z}{x}+\frac{z}{y}\right).$$

70. Prove that

$$\frac{(x^2+y^2)(1+x^2y^2)-2(1-x^2)(1-y^2)xy-4x^2y^2}{(x^2+y^2)(1+x^2y^2)-2(1+x^2)(1+y^2)xy+4x^2y^2}=\left(\frac{1+xy}{1-xy}\right)^2.$$

71. Shew that $(1+x)(1+x^2)(1+x^4)(1+x^8)=\dfrac{1-x^{16}}{1-x}$. [See *App.*]

72. Prove that

$$a\left(\frac{b^3-c^3}{b-c}\right)+b\left(\frac{c^3-a^3}{c-a}\right)+c\left(\frac{a^3-b^3}{a-b}\right)=(a+b+c)(bc+ca+ab).$$

73. Simplify $\dfrac{3xyz}{yz+zx+xy}-\dfrac{\dfrac{x-1}{x}+\dfrac{y-1}{y}+\dfrac{z-1}{z}}{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}$.

74. If $x^2\dfrac{(s-b)(s-c)}{bc}=y^2\dfrac{(s-c)(s-a)}{ca}=z^2\dfrac{(s-a)(s-b)}{ab}$,

then $$x^2\left(\frac{1}{b}-\frac{1}{c}\right)+y^2\left(\frac{1}{c}-\frac{1}{a}\right)+z^2\left(\frac{1}{a}-\frac{1}{b}\right)=0.$$

75 Reduce $\dfrac{x}{x-y}-\dfrac{x}{x+y}-\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}$

76 Shew that $\dfrac{1}{x+\dfrac{1}{y+\dfrac{1}{z}}}\div\dfrac{1}{x+\dfrac{1}{y}}-\dfrac{1}{y(xyz+x+z)}=1$

77. Find the value of

$$\frac{x^2+xy+y^2}{x^2y^2}\left(\frac{1}{x}+\frac{1}{y}\right)+\frac{y^2+yz+z^2}{y^2z^2}\left(\frac{1}{y}+\frac{1}{z}\right)-\frac{x^2+xz+z^2}{x^2z^2}\left(\frac{1}{x}+\frac{1}{z}\right)$$

78 If $2s=a+b+c$, shew that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

79. If $x^2=\dfrac{ab(c^2+d^2)-cd(a^2+b^2)}{ab-cd}$, decompose

$1-\left\{\dfrac{a^2+b^2-x^2}{2ab}\right\}^2$ into simple factors

80. Prove that

$$\frac{la^2+ma+n}{(a-b)(a-c)(x+a)}+\frac{lb^2+mb+n}{(b-c)(b-a)(x+b)}+\frac{lc^2+mc+n}{(c-a)(c-b)(x+c)}$$
$$=\frac{lx^2-mx+n}{(x+a)(x+b)(x+c)}.$$

81. Shew that

$$\frac{pa^2+qa+r}{(a-b)(a-c)(x-a)}+\frac{pb^2+qb+r}{(b-c)(b-a)(x-b)}+\frac{pc^2+qc+r}{(c-a)(c-b)(x-c)}$$
$$=\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$$

82 Prove that

$$\frac{y^2+z^2-x^2}{yz}+\frac{z^2+x^2-y^2}{zx}+\frac{x^2+y^2-z^2}{xy}=2+\frac{(y+z-x)(z+x-y)(x+y-z)}{xyz}$$

83 If $ax=y+z+u$, $by=z+u+x$, $cz=u+x+y$, $du=x+y+z$,

shew that $$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=1.$$

CHAPTER XV.

INVOLUTION AND EVOLUTION

184. Involution. It is the method of finding any proposed power of a given quantity. Hence this operation is nothing but Multiplication where the *multiplicand* and the *multiplier* are the *same*.

185. Powers of Quantities. Any *power of a power* of a given quantity is obtained by *multiplying* together the indices of the two powers. For

$$(x^2)^3 = x^2 \times x^2 \times x^2 = x^{2+2+2} \text{ [§ 68]} = x^6;$$
$$(x^3)^2 = x^3 \times x^3 = x^{3+3} \text{ [§ 68]} = x^6,$$
$$(a^m)^n = a^{m+m+\ldots\ \text{to } n \text{ terms}} = a^{mn};$$
$$\begin{aligned}(ab)^n &= ab \times ab \times ab \times \ldots\ldots \text{ to } n \text{ factors}\\ &= (a \times a \times a \times \ldots\ldots \text{ to } n \text{ factors})\\ &\quad \times (b \times b \times b \times \ldots\ldots \text{ to } n \text{ factors})\\ &= a^n \times b^n \text{ [§ 20]},\end{aligned}$$

and generally $(a^m b^n c^p \ldots\ldots)^r = a^{mr} b^{nr} c^{pr} \ldots$

Corollary. From the first two examples, it is seen that $(x^2)^3 = (x^3)^2$. In a similar way, it can be generally proved that $(x^m)^n = (x^n)^m$, that is, *the* n^{th} *power of the* m^{th} *power of a quantity is equal to the* m^{th} *power of its* n^{th} *power*.

186. Sign of powers. RULE.—If the quantity to be *involved* have a *positive* sign, the sign of *any power* will be obviously *positive*; but if it have a *negative* sign, the sign of the *even* powers will be *positive* and that of the *odd* powers will be *negative*.

For $(-a)(-a) = a^2$, [§ 64, *Cor.*], $(-a)(-a)(-a) = -a^3$; $(-a)(-a)(-a)(-a) = a^4$; $(-a)(-a)(-a)\ldots\ldots$ to m factors $= (-a)^m = \pm a^m$, according as m is *even* or *odd*. Again since $-a = -1 \times a$, $(-a)^m = (-1 \times a)^m = (-1)^m a^m$ [§ 185].

187. Involution of Monomials. The examples here given depend on the two preceding articles.

Examples

Ex. 1. $(-a^2)^3 = -a^6$, the sign is $-$, for the index is *odd*.

Ex. 2. $(-a^2x)^4 = a^8x^4$, the sign is $+$, the index being *even*.

Ex. 3. $\left(\dfrac{a}{b}\right)^2 = \dfrac{a}{b} \times \dfrac{a}{b} = \dfrac{aa}{bb} = \dfrac{a^2}{b^2}$. Ex. 4. $\left(-\dfrac{2a^3b^2}{x^5yz^4}\right)^9 = -\dfrac{512a^{27}b^{18}}{x^{45}y^9z^{36}}$.

REMARK. The last two examples shew that when the quantity involved is a *fraction*, both of its terms are raised to the proposed power.

Ex 5 $(-1)^{2m}=1$, for $2m$ is *always even* whatever m may be, since $2m$ is always divisible by 2 [see foot-note p 45]

Ex 6 $(-1)^{2m+1}=-1$, for $2m+1$ is *always odd* whatever m may be.

Find the value of

7 $(-a)^6$; $(a^3)^7$; $(x^2y^3)^3$, $(-3a^3x^2)^5$, $(-a^2x^4y^5)^9$

8. $\left(\frac{x}{y^2}\right)^4$, $\left(-\frac{3a}{x}\right)^3$, $\left(-\frac{a^2b^3c}{2}\right)^5$, $\left(-\frac{x^3y^4}{a^2cz^5}\right)^8$

9. $(a^2)^5(-b)^2(-3ac)^4$; $(x^{2m}y^{3n}z^p)^p$, $(-ax^my^p)^n$

10 $(-a)^{2m}$, $(-a)^{m+1}$, $\left(-\frac{a}{b}\right)^{2m-1}$, $\left(-\frac{a}{b}\right)^{2m+2}$

188 Square of Polynomials We know [§ 99] that

$$(a+b)^2=a^2+2ab+b^2$$

$$(a+b+c)^2=a^2+2a(b+c)+(b+c)^2$$

$$=a^2+2a(b+c)+b^2+2bc+c^2 \qquad (a)$$

$$\text{or}=a^2+b^2+c^2+2ab+2ac+2bc \qquad (\beta)$$

$$(a+b+c+d)^2=a^2+2a(b+c+d)+b^2+2b(c+d)+c^2+2cd+d^2 \qquad (a)$$

$$\text{or}=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd \; (\beta).$$

.. ..

We have thus *two forms* (a) and (β), of the results, which enable us to state the RULE in two ways —

(i) *The square of any polynomial is equal to the sum of the squares of all the terms together with twice the product of each term into the sum of all the others which follow it* That is

$$(a+b+c+d+\dots+l)^2=a^2+2a(b+c+d+\dots+l)$$
$$+b^2+2b(c+d+\dots+l)$$
$$+c^2+2c(d+\dots+l)+\dots$$

(ii) *The square of any polynomial is equal to the sum of the squares of all the terms together with twice the product of every two of them* [See § 99, Ex 5]

REMARK Any one or more of the quantities a, b, c, .may be *negative*, and then the sign of each term of the result will be determined by the Law of Signs [§ 64]

Examples.

Find the value of

1. $\left(\frac{x}{2}+\frac{y}{3}\right)^2$. 2. $\left(\frac{a}{b}-\frac{5b}{a}\right)^2$. 3. $\left(\frac{x}{2}-\frac{2}{3x^2}\right)^2$.

Find the value of

4 $\left(1+\frac{x}{2}-\frac{x^2}{3}\right)^2$. 5 $\left(\frac{x}{a}+\frac{y}{b}-\frac{z}{c}\right)^2$. 6 $\left(\frac{a^2}{x}-\frac{2b^2}{y}+\frac{3c^2}{z}\right)^2$.

7. $(ax-2by+3cz)^2+(ax+2by-3cz)^2$.

8 $(a^m+b^n)^2$. 9. $(a^m-2b^n+c)^2$. 10 $(e^{2x}+e^x+1)^2$.

11. $\{(a+b)x-(a-b)y\}^2$. 12 $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2$. 13 $\left(\frac{ax}{m^2}+\frac{by}{n^2}-1\right)^2$.

14. $(x^3-2x^2+x-2)^2$ 15 $(a+bx+cx^2+dx^3+\ldots.)^2$

189 Cube of Polynomials The method of finding the cube of a polynomial has already been explained [see §§ 103 and 104] We shall add only a few more examples

Examples.

Find the value of

1. $\left(\frac{2a}{b}-1\right)^3$. 2 $\left(\frac{4}{x}-\frac{3}{y}\right)^3$ 3. $(a^2-2b^2+3c^2)^3$.

4 $(1-2x+3x^2)^3$. 5 $(a+bx+cx^2)^3$. 6 $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^3$.

7 $\left(\frac{3x^4}{2y^4}-\frac{2y^3}{3x^3}\right)^3$ 8. $\left(\frac{x^m}{a^n}+\frac{y^n}{b^m}\right)^3$ 9 $\left(x^4-\frac{2ax^2}{c}+\frac{a^2}{c^2}\right)^3$.

190 Other Powers of Polynomials. The other powers of polynomials may be found either by direct multiplication, or by the method explained in the COROLLARY to § 185. For example, $a^4=(a^2)^2$, $a^6=(a^2)^3=(a^3)^2$, $a^9=(a^3)^3$, $a^8=\{(a^2)^2\}^2$, &c

Examples

Ex. 1. $(3x+1)^4=\{(3x+1)^2\}^2=(9x^2+6x+1)^2$
$=81x^4+108x^3+54x^2+12x+1$.

Ex. 2. $\left(\frac{a}{x}-\frac{x}{a}\right)^6=\left\{\left(\frac{a}{x}-\frac{x}{a}\right)^3\right\}^2=\left(\frac{a^3}{x^3}-\frac{3a}{x}+\frac{3x}{a}-\frac{x^3}{a^3}\right)^2$.

$$=\frac{a^6}{x^6}+\frac{9a^2}{x^2}+\frac{9x^2}{a^2}+\frac{x^6}{a^6}-\frac{6a^4}{x^4}+\frac{6a^2}{x^2}-20+\frac{6x^2}{a^2}-\frac{6x^4}{a^4}$$

$$=\frac{a^6}{x^6}-\frac{6a^4}{x^4}+\frac{15a^2}{x^2}-20+\frac{15x^2}{a^2}-\frac{6x^4}{a^4}+\frac{x^6}{a^6}.$$

Ex. 3 $(a+b-c)^4=\{(a+b-c)^2\}^2$
$=(a^2+b^2+c^2+2ab-2ac-2bc)^2$
$=a^4+b^4+c^4+4a^2b^2+4a^2c^2+$&c

Find the value of

4. $\left(1+\frac{3x}{2}\right)^4$. 5. $\left(\frac{2a}{b}-1\right)^6$ 6. $\left(\frac{x}{a}+\frac{a}{x}\right)^5$.

7. $\left(\frac{x}{y}-\frac{y}{x}\right)^4$ 8. $(a-x)^6$

191 Rule for Expanding a Power of a Binomial The method of the last article is tedious We shall therefore give a rule for expanding a Binomial raised to a proposed power, which is practically very useful This rule is derived from the BINOMIAL THEOREM and consequently no proof can at present be given

The value of an expression raised to a certain power is called its *development* or *expansion* Thus the development or expansion of $(a+b)^2$ is $a^2+2ab+b^2$ The general **Rule** by which such expansions are obtained without multiplication is as follows —

In the expansion of $A+B$ raised to any power—

The first and last terms are A and B each raised to that power and in each successive term the index of A is less by 1 and that of B greater by 1, than those of A and B in the next preceding term

The coefficient of the first term is unity and that of each successive term is obtained by multiplying the coefficient of the next preceding term by the index of A in that term and then dividing the product by the number of terms preceding the term whose coefficient is sought

The same theorem also furnishes us with the following tests to examine the accuracy of our work —

(1) *The number of terms in a binomial expansion is one greater than the index of its power*

(2) *The coefficients of terms equidistant from the beginning and the end are equal*

Ex. 1 Expand $(a+b)^4$

The expansion will contain 4+1 or 5 terms, of which the first and last are a^4 and b^4 respectively

in the *second* term, the index of a is $4-1$ or 3, and that of b is 1,

... ... *third* .... $3-1$ or 2, ... $1+1$ or 2,

... *fourth* ... . $2-1$ or 1, ... $2+1$ or 3,

The coefficient of *second* term $=\frac{1\times 4}{1}=4$,

... ... *third* ... ... $=\frac{4\times 3}{2}=6$,

....*fourth* .. .. $=\frac{6\times 2}{3}=4$.

The *fifth* or last term has already been found Thus

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

The above work may be more concisely shewn, thus :—

$$(a+b)^4=a^4+\frac{1\times 4}{1}a^{4-1}b+\frac{4\times 3}{2}a^{3-1}b^2+\frac{6\times 2}{3}a^{2-1}b^3+b^4$$
$$=a^4+4a^3b \qquad +6a^2b^2 \qquad +4ab^3 \qquad +b^4,$$

each term of the *second* line being put down in a *simplified form* to facilitate calculation as soon as the *corresponding* term has been put down in the *first* line according to the Rule

Note *Every term* of the expansion of $(a+b)^4$ must be *homogeneous* and of the *fourth degree* [§ 90] , hence we have another test for the accuracy of a binomial expansion See Exs 2, 3, 4 and 5 below

Ex 2. Expand $(a+b)^5$

$$(a+b)^5=a^5+\frac{1\times 5}{1}a^{5-1}b+\frac{5\times 4}{2}a^{4-1}b^2+\frac{10\times 3}{3}a^{3-1}b^3+\frac{10\times 2}{4}a^{2-1}b^4+b^5$$
$$=a^5+5a^4b \qquad +10a^3b^2 \qquad +10a^2b^3 \qquad +5ab^4 \qquad +b^5.$$

Ex 3 Develop $(ax+by)^7$

$$(ax+by)^7=(ax)^7+\frac{1\times 7}{1}(ax)^{7-1}(by)+\frac{7\times 6}{2}(ax)^{6-1}(by)^2$$
$$+\frac{21\times 5}{3}(ax)^{5-1}(by)^3+\frac{35\times 4}{4}(ax)^{4-1}(by)^4+\frac{35\times 3}{5}(ax)^{3-1}(by)^5$$
$$+\frac{21\times 2}{6}(ax)^{2-1}(by)^6+(by)^7$$
$$=(ax)^7+7(ax)^6(by) \qquad +21(ax)^5(by)^2 \qquad +35(ax)^4(by)^3$$
$$+35(ax)^3(by)^4 \qquad +21(ax)^2(by)^5 \qquad +7(ax)(by)^6 \qquad +(by)^7$$
$$=a^7x^7+7a^6x^6by+21a^5x^5b^2y^2+35a^4x^4b^3y^3+35a^3x^3b^4y^4$$
$$+21a^2x^2b^5y^5+7axb^6y^6+b^7y^7.$$

Ex 4. Develop $(a-b)^4$

Change the sign of b in Ex 1 ; thus

$$(a-b)^4=a^4+4a^3(-b)+6a^2(-b)^2+4a(-b)^3+(-b)^4$$
$$=a^4-4a^3b+6a^2b^2-4ab^3+b^4.$$

Corollary. From this example, it is easily seen that the terms involving the *odd powers* of b have the $-$ sign ; or in other words—*the terms of the expansion of any power of $A-B$ beginning from the first, are alternately $+$ and $-$*

Ex 5 Expand $(2x-3y)^6$.

$$(2x-3y)^6=(2x)^6-6(2x)^{6-1}(3y)+\frac{6\times 5}{2}(2x)^{5-1}(3y)^2$$
$$-\frac{15\times 4}{3}(2x)^{4-1}(3y)^3+\frac{20\times 3}{4}(2x)^{3-1}(3y)^4-\frac{15\times 2}{5}(2x)^{2-1}(3y)^5+(3y)^6$$
$$=64x^6-576x^5y+2160x^4y^2-4320x^3y^3+4860x^2y^4-2916xy^5+729y^6$$

Ex 6 Expand $(3x-1)^7$

$$(3x-1)^7=(3x)^7-7(3x)^{7-1}+\frac{7\times 6}{2}(3x)^{6-1}-\frac{21\times 5}{3}(3x)^{5-1}$$
$$+\frac{35\times 4}{4}(3x)^{4-1}-\frac{35\times 3}{5}(3x)^{3-1}+\frac{21\times 2}{6}(3x)^{2-1}-1$$
$$=2187x^7-5103x^6+5103x^5-2835x^4+945x^3-189x^2+21x-1.$$

[*Expand examples* 1–8 *of* § 190 *by this Rule*]

192 Root The Root* of a quantity is that quantity which being raised to a certain power gives the proposed quantity It is expressed by means of the *radical sign* $\sqrt{\ }$ placed before the quantity [§ 22]

In § 22, we have defined the *square root* and the *cubic root* of a quantity Similarly other roots may be defined Thus generally, the m^{th} *Root* of a quantity is that quantity whose m^{th} power gives the proposed quantity Thus a is an m^{th} root of a^m.

To express a given root of any quantity, we place before the quantity the radical sign *with the number that denotes the root* Thus $\sqrt[4]{a}$ signifies a *fourth* root of a, $\sqrt[5]{a}$ signifies a *fifth* root of a, $\sqrt[m]{a}$, an m^{th} root of a, &c

Another mode of expressing the root of a quantity is by means of *the fractional indices which are the reciprocals of the numbers denoting the root* Thus the above roots are expressed respectively by $a^{\frac{1}{4}}$, $a^{\frac{1}{5}}$, $a^{\frac{1}{m}}$, &c [See § 207 *post*]

193 Evolution (It is the method of finding any proposed root of a given quantity) *Involution* and *Evolution* are thus two inverse processes

194 Extraction of Roots *Any root* of a given quantity is obtained by *dividing* the index of the power of that quantity by the number expressing the required root For

$(a^m)^n=a^{mn}$ [§ 185], an n^{th} root of a^{mn} must be a^m

* The student should notice the different senses in which the word '*Root*' has been used [see § 140]

195. Sign of Roots From § 186, it is seen that

$a^3=aaa$ and $-a^3=(-a)(-a)(-a)$;

$a^5=aaaaa$ and $-a^5=(-a)(-a)(-a)(-a)(-a)$; &c.

Hence if an *odd root** of a quantity be extracted *the sign of the root will be the same as the sign of the quantity itself.*

Again $a^2=aa$, or $=(-a)(-a)$,

$a^4=aaaa$, or $=(-a)(-a)(-a)(-a)$; &c

That is, the *square* root of a^2 is either $+a$ or $-a$, the *fourth* root of a^4 is $+a$ or $-a$; and so on Hence if an *even root** of *a positive* quantity be extracted, *the root may be either positive or negative*

When the proposed quantity is *negative*, no *even root* can be extracted, for *no quantity raised to an even power can give a negative quantity* In such cases the roots are said to be *Imaginary* or *Impossible* Hence $\sqrt{-a}$ is an Imaginary Quantity.

196. Evolution of Monomials Any root of a monomial can be extracted by the principles of § 194

Examples

Ex 1 Find the square root of a^6b^4.

$$\sqrt{a^6b^4}=a^3b^2, \text{ for } 6\div 2=3 \text{ and } 4\div 2=2.$$

Ex. 2 Find the cube root of $-27a^3b^3$

$\sqrt[3]{-27a^3b^3}=-3ab$, for $3\div 3=1$ and $-27a^3b^3$ being negative the root is also negative.

Ex 3 $\sqrt[m]{a^{2m}b^mc^{3m+1}}=a^2bc^{3+\frac{1}{m}}$; for $2m\div m=2$, and $(3m+1)\div m$ $=3+\dfrac{1}{m}$.

Find the square root of

4 a^4b^2 5. $25x^2y^4z^6$ 6. $49m^8n^4z^2$. 7. $144x^{12}y^{16}z^4$.

8. $64x^{2m}y^{4n}$. 9. $81a^{2x+2}b^{3x+2}$ 10 $169x^{4m}y^{8n}z^{m}$. 11. a^2b^2.

12 $\dfrac{15a^2x^4}{9y^3z^6}$ 13 $\dfrac{36x^{2m}y^2}{49y^{2n}z^4}$ 14. $\dfrac{75a^4x^3}{48y^2z^6}$ 15 $\dfrac{63a^3}{175b^4}$.

Find the value of

16 $\sqrt[3]{a^3x^3y^6}$. 17 $\sqrt[3]{-8x^3y^{12}}$ 18 $\sqrt[4]{x^{4m}y^{2n}}$.

* A root expressed by an *odd* number, may be called an *odd root*; and that expressed by an *even* number, may be called an *even root*

Find the value of

19 $\sqrt[5]{-a^5x^{10}y^{15}}$ 20 $\sqrt[3]{\left(\frac{125x^6y^3}{8a^3b^3}\right)}$. 21 $\sqrt[4]{\frac{16m^4n^8}{81p^{12}}}$.

22 $\sqrt[m]{5x^{am}y^{3m}}$ 23. $\sqrt[n]{a^{mn}x^{2n+an}}$

197 Square root found by inspection We know that $a^2+2ab+b^2=(a+b)^2$, therefore the square root of $a^2+2ab+b^2$ is $\pm(a+b)$ Similarly the square root of $a^2-2ab+b^2$ is $\pm(a-b)$ Thus when two of the terms of a *trinomial* are the squares of any two quantities, and the third term is twice their product with the sign + or −, the trinomial is the square of the sum or difference of those two quantities Hence when a trinomial of this form is arranged according to the ascending or descending powers of a particular symbol, its square root is found by taking the square root of the extreme terms and connecting them by + or −, according as twice their product has the sign + or −

Thus to find the square root of $x^2+10xy+25y^2$ Here the expression is arranged according to the descending powers of x, and the extreme terms are x^2 and $25y^2$ whose square roots are $\pm x$ and $\pm 5y$, also the term $10xy$ is equal to twice the product of x and $5y$ and has the + sign, therefore the square root is $\pm(x+5y)$

REMARK. In the examples that follow we shall give only the square root with the + sign, the square root with the − sign, will be obtained by changing the sign of every term of this root Thus the above root will be written $x+5y$, the other root will evidently be $-x-5y$

Examples (i)

Ex 1 Find the square root of $x^2-4xy+4y^2$

Now, given expression $=x^2-2x(2y)+(2y)^2=(x-2y)^2$,

square root required $=x-2y$

Ex 2 Find the square root of $9a^2+12ax+4x^2$

Given expression $=(3a)^2+2(3a)(2x)+(2x)^2=(3a+2x)^2$,

$\therefore$ square root required $=3a+2x$

Ex 3 Find the square root of $a^2-ax+\frac{x^2}{4}$

Given expression $=a^2-2a\frac{x}{2}+\left(\frac{x}{2}\right)^2=\left(a-\frac{x}{2}\right)^2$;

square root required $=a-\frac{x}{2}$

Extract the square root of

4. $9a^2+6ab+b^2$. 5 $16x^4-24x^2+9$ 6 $x^4-8ax^3+16a^2x^2$.

7. $64a^2x^2+48ab^2x+9b^4$. 8 $36a^4x^4-24a^3bx^3+4a^2b^2x^2$

9. $25x^4+30x^3y+9x^2y^2$. 10 $4a^2b^2-12abc+9c^2$

11. $9x^4y^2+66x^2yz+121z^2$. 12 $9a^2-102abc+289b^2c^2$

13 $x^2+xy+\frac{y^2}{4}$ 14. $x^2-x+\frac{1}{4}$. 15 $9x^2-2xy+\frac{y^2}{9}$.

16. $4a^2-ax+\frac{x^2}{16}$ 17. $x^2y^4+xy^2z^2+\frac{z^4}{4}$.

18. $\frac{a^2}{c^2}+\frac{2ax}{c}+x^2$ 19 $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}$. 20 $\frac{x^2}{a^2}-\frac{x}{y}+\frac{a^2}{4y^2}$

21. $a^6+2+\frac{1}{a^6}$ 22 $\frac{25}{4x^2}+\frac{x^2}{25}-1$ 23 $9a^8+\frac{a^4b^4}{36}-a^6b^2$.

Note. When an expression, which is a *complete square*, consists of *several* terms, its square root may be found by inspection, *if only two different powers of some one letter occur in it*; for when arranged according to the powers of that letter, the expression will be easily seen to be a perfect square. The following examples will illustrate the truth of this remark.

Examples (11)

Ex 1. Find the square root of $x^2+y^2+z^2+2yz+2zx+2xy$

Arrange the expression according to the descending powers of some *one* letter, say x; thus the given expression

$$=x^2+2x(y+z)+(y^2+2yz+z^2)$$
$$=x^2+2x(y+z)+(y+z)^2=\{x+(y+z)\}^2;$$
$$\therefore \text{ square root required}=x+y+z$$

Ex. 2. Find the square root of

$a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd$ [*Cal*, 1868]

Arrange according to powers of a; thus

given expression $=a^2-2a(b-c+d)+(b^2+c^2+d^2-2bc+2bd-2cd)$

$$=a^2-2a(b-c+d)+(b-c+d)^2=\{a-(b-c+d)\}^2;$$
$$\therefore \text{ required square root}=a-b+c-d$$

Ex 3 Extract the square root of $(bc+ca+ab)^2-4abc(a+c)$.
[*Cal*, 1888]

Given expression
$$=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c)-4abc(a+c) \quad [\S\,115, \textit{Cor } 2]$$
$$=b^2c^2+c^2a^2+a^2b^2-2a^2bc+2ab^2c-2abc^2$$
$$=a^2(b^2-2bc+c^2)+2a(b^2c-bc^2)+b^2c^2$$
[when arranged according to powers of a]
$$=a^2(b-c)^2+2a(b-c)bc+(bc)^2$$
$$=\{a(b-c)+bc\}^2,$$
∴ square root required $=ab-ac+bc$

Ex. 4 Find the square root of $4-4c+2b+c^2-bc+\dfrac{b^2}{4}$ [*Cal*, 1876]

Arrange according to powers of c, thus

given expression $=c^2-(4+b)c+\left(4+2b+\dfrac{b^2}{4}\right)$
$$=c^2-2\left(2+\frac{b}{2}\right)c+\left(2+\frac{b}{2}\right)^2$$
$$=\left\{c-\left(2+\frac{b}{2}\right)\right\}^2;$$
∴ square root required $=c-2-\dfrac{b}{2}$.

Ex 5 Find the square root of $x^4-2bx^3-(2a^2-b^2)x^2+2a^2bx+a^4$.
Arrange according to powers of a, thus

given expression $=a^4+2a^2(bx-x^2)+b^2x^2-2bx^3+x^4$
$$=a^4+2a^2(bx-x^2)+(bx-x^2)^2$$
$$=\{a^2+(bx-x^2)\}^2,$$
∴ square root required $=a^2+bx-x^2$

[Try this example by arranging according to powers of b]

Extract the square root of

6. $a^2+b^2+c^2-2ab+2ac-2bc$ 7. $9x^2-12xy+4y^2+6xz-4yz+z^2$.

8. $x^2-2ax+a^2+2xy-2ay+y^2$ 9. $x^2+\dfrac{2ax}{3}-bx+\dfrac{a^2}{9}-\dfrac{ab}{3}+\dfrac{b^2}{4}$.

10. $4x(1+x)-2a(1+2x)+a^2+1$.

11. $b^2x^4-2abx^3+(a^2+2b)x^2-2ax+1$.

12. $x^3(x-2a)+a^2b(b-2x)+(a^2+2ab)x^2$

Extract the square root of

13. $\frac{a^2}{x^2}+\frac{ab}{xy}+\frac{b^2}{4y^2}-\frac{6ac}{xz}-\frac{3bc}{yz}+\frac{9c^2}{z^2}$.

14. $x^4+(x^2-a)x-2(a-\frac{1}{8})x^2+a^2$

15. $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4$ [*Cal.*, 1870]

198. General Rule for finding the square root The expansion of $(a+b+c+\ldots\ldots+l)^2$ given in § 188, may be put in the following form

$$a^2+(2a+b)b+\{2(a+b)+c\}c+\{2(a+b+c)+d\}d+\&c$$

This form of the expansion furnishes us with the following general rule for the extraction of the square root—

RULE —Arrange the proposed expression according to the ascending or descending powers of a symbol of reference

Take the square root of the first term and set it down as the *first term* of the root .

Subtract the square of the first term from the proposed expression, and bring down the next two terms take *twice* the first term of the root and put this product as the first term of the divisor ; divide the first term of the quantity, brought down, by the first term of the divisor and put the quotient as the *second term* of the root and also as the last term of the divisor, and multiply the divisor by the second term of the root, and subtract the result

Bring down this remainder and the next two terms , and continue the process to find the *third, fourth,* &c , *terms,* till the required root is finally obtained

Examples

Ex 1. Find the square root of $x^4-2x^3+3x^2-2x+1$

$$\begin{array}{r|l}
 & x^4-2x^3+3x^2-2x+1\ (\ x^2-x+1 \\
 & x^4 \\
\hline
2x^2-x & -2x^3+3x^2 \\
 & -2x^3+\ \ x^2 \\
\hline
2x^2-2x+1 & 2x^2-2x+1 \\
 & 2x^2-2x+1.
\end{array}$$

The same result will of course follow if the given expression be arranged according to the *ascending* powers of x.

Otherwise —The given expression

$$=x^4-2x^3+2x^2+x^2-2x+1$$
$$=(x^2)^2-2x^2(x-1)+(x-1)^2=(x^2-\overline{x-1})^2\ ;$$

square root required $=x^2-x+1$

Ex. 2 Extract the square root of

$$4x^6-4x^5+13x^4-22x^3+17x^2-24x+16.$$

$$\begin{array}{r|l}
 & 4x^6-4x^5+13x^4-22x^3+17x^2-24x+16\ (\,2x^3-x^2+3x-4 \\
 & 4x^6 \\ \hline
4x^3-x^2 & -4x^5+13x^4 \\
 & -4x^5+x^4 \\ \hline
4x^3-2x^2+3x & 12x^4-22x^3+17x^2 \\
 & 12x^4-6x^3+9x^2 \\ \hline
4x^3-2x^2+6x-4 & -16x^3+8x^2-24x+16 \\
 & -16x^3+8x^2-24x+16. \\ \hline
\end{array}$$

Ex 3. Find the square root of $16(a^4+1)-24a(a^2+1)+41a^2$. Remove the brackets and arrange according to powers of a.

$$\begin{array}{r|l}
 & 16a^4-24a^3+41a^2-24a+16\ (\,4a^2-3a+4 \\
 & 16a^4 \\ \hline
8a^2-3a & -24a^3+41a^2 \\
 & -24a^3+9a^2 \\ \hline
8a^2-6a+4 & 32a^2-24a+16 \\
 & 32a^2-24a+16 \\ \hline
\end{array}$$

Ex. 4 Extract the square root of $x^4-x^3+\dfrac{x^2}{4}+4x-2+\dfrac{4}{x^2}$

Here the expression is already arranged according to the descending powers of x

$$\begin{array}{r|l}
 & x^4-x^3+\dfrac{x^2}{4}+4x-2+\dfrac{4}{x^2}\left(x^2-\dfrac{x}{2}+\dfrac{2}{x}\right. \\
 & x^4 \\ \hline
2x^2-\dfrac{x}{2} & -x^3+\dfrac{x^2}{4} \\
 & -x^3+\dfrac{x^2}{4} \\ \hline
2x^2-x+\dfrac{2}{x} & 4x-2+\dfrac{4}{x^2} \\
 & 4x-2+\dfrac{4}{x^2} \\ \hline
\end{array}$$

Otherwise —The given expression

$$=x^4-x^3+\frac{x^2}{4}+4(x-\tfrac{1}{2})+\frac{4}{x^2}$$

$$=\left(x^2-\frac{x}{2}\right)^2+2\frac{2}{x}\left(x^2-\frac{x}{2}\right)+\left(\frac{2}{x}\right)^2$$

$$=\left\{\left(x^2-\frac{x}{2}\right)+\frac{2}{x}\right\}^2,$$

square root required $=x^2-\dfrac{x}{2}+\dfrac{2}{x}$

Ex 5 Find the square root of

$$x^{-\frac{4}{3}}+x^{-\frac{2}{3}}y^{\frac{8}{3}}-2x^{-1}y^{\frac{4}{3}}+y^{\frac{4}{3}}-2x^{-\frac{2}{3}}y^{\frac{2}{3}}+2x^{-\frac{1}{3}}y^2.$$

Arrange according to the ascending powers of x

$$\begin{array}{r|l}
 & x^{-\frac{4}{3}}-2x^{-1}y^{\frac{4}{3}}+x^{-\frac{2}{3}}y^{\frac{8}{3}}-2x^{-\frac{2}{3}}y^{\frac{2}{3}}+2x^{-\frac{1}{3}}y^2+y^{\frac{4}{3}}\left(x^{-\frac{2}{3}}-x^{-\frac{1}{3}}y^{\frac{4}{3}}-y^{\frac{2}{3}}\right. \\
 & x^{-\frac{4}{3}} \\ \hline
2x^{-\frac{2}{3}}-x^{-\frac{1}{3}}y^{\frac{4}{3}} & -2x^{-1}y^{\frac{4}{3}}+x^{-\frac{2}{3}}y^{\frac{8}{3}} \\
 & -2x^{-1}y^{\frac{4}{3}}+x^{-\frac{2}{3}}y^{\frac{8}{3}} \\ \hline
2x^{-\frac{2}{3}}-2x^{-\frac{1}{3}}y^{\frac{4}{3}}-y^{\frac{2}{3}} & -2x^{-\frac{2}{3}}y^{\frac{2}{3}}+2x^{-\frac{1}{3}}y^2+y^{\frac{4}{3}} \\
 & -2x^{-\frac{2}{3}}y^{\frac{2}{3}}+2x^{-\frac{1}{3}}y^2+y^{\frac{4}{3}}. \\ \hline
\end{array}$$

Ex 6 Extract the square root of $\left(x^2+\dfrac{1}{x^2}\right)^2-4\left(x+\dfrac{1}{x}\right)^2+12$.

[*Cal*, 1866]

Simplify the expression and proceed as in the above examples.

Otherwise :—The given expression

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)-8+12$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)+4=\left\{\left(x^2+\frac{1}{x^2}\right)-2\right\}^2;$$

$\therefore$ square root required $=x^2+\dfrac{1}{x^2}-2.$

Ex. 7. Extract the square root of $\dfrac{(a^2+b^2)^2}{a^4+b^4-2a^2b^2}+4\times\dfrac{a}{a+b}\times\dfrac{b}{a-b}$.

[*Cal*, 1886]

Proposed expn $=\dfrac{(a^2-b^2)^2+4a^2b^2}{(a^2-b^2)^2}+\dfrac{4ab}{a^2-b^2}$ [§ 101]

$$=1+\frac{4a^2b^2}{(a^2-b^2)^2}+\frac{4ab}{a^2-b^2}$$

$$=1+2\times\frac{2ab}{a^2-b^2}+\left(\frac{2ab}{a^2-b^2}\right)^2=\left(1+\frac{2ab}{a^2-b^2}\right)^2,$$

∴ square root required $=1+\dfrac{2ab}{a^2-b^2}$

Extract the square root of

8 $9x^4+6x^3+7x^2+2x+1$

9 $x^4+4x^3-2x^2-12x+9$

10 $4x^4-12x^3+25x^2-24x+16$

11 $1-2x+5x^2-4x^3+4x^4$

12 $16x^4-24x^3-31x^2+30x+25$

13 $25x^4-50x^3+85x^2-60x+36$

14 $81x^4+108x^3-24x+4$

15 $9-24a-68a^2+112a^3+196a^4$

16 $a^{12}-8a^9+18a^6-8a^3+1$

17 $x^4+4x^3y+12x^2y^2+16xy^3+16y^4$

18 $x^6+8x^4-2x^3+16x^2-8x+1$

19. $4+12ax-6ax^3+9a^2x^2+4x^2+x^4$.

20. $64p^4-48p^3q+41p^2q^2-12pq^3+4q^4$

21. x^4-4x^3+8x+4

22 $4x^4-16x^3-8x^2+48x+36$

23 $4a^2x^4-12a^3x^3+13a^4x^2-6a^5x+a^6$

24 $x^2+3x-2x^{\frac{1}{2}}-2x^{\frac{3}{2}}+1$

25 $x^2+6x-12x^{\frac{1}{3}}+4x^{\frac{2}{3}}-4x^{\frac{4}{3}}+9$

26 $4x^{-2}+9x-24x^{\frac{1}{2}}+12x^{-\frac{1}{2}}-16x^{-1}+16$

27 $x^{-\frac{2}{3}}+x^{\frac{2}{3}}y^{-1}+2y^{-\frac{1}{2}}(1-x^{\frac{1}{3}})-2x^{-\frac{1}{3}}+1$

28 $4xy^{-\frac{1}{2}}-11x^{\frac{2}{3}}y^{-\frac{2}{3}}+4x^{\frac{4}{3}}-6x^{\frac{1}{3}}y^{-1}+9y^{-\frac{4}{3}}$

29 $x^{2m}-2x^{m+n}+x^{2n}$

30 $4x^{2m+2}-12x+9x^{-2m}$

31. $11-6(x^n+x^{-n})+x^{2n}+x^{-2n}$

32 $x^4+2x^3-x+\frac{1}{4}$

33 $x^2+\dfrac{1}{x^2}+2\left(x-\dfrac{1}{x}\right)-1$

34 $\left(x+\dfrac{1}{x}\right)^2-4\left(x-\dfrac{1}{x}\right)$

35. $x^2+8x-\dfrac{64}{x}+\dfrac{64}{x^2}$

36 $x^2-2+\dfrac{2}{x}+\dfrac{1}{x^2}-\dfrac{2}{x^3}+\dfrac{1}{x^4}$.

37 $x^4-3x^3+\dfrac{9x^2}{4}+\dfrac{x}{2}-\dfrac{3}{4}+\dfrac{1}{16x^2}$

38 $\dfrac{x^2}{y^2}\left(\dfrac{x^2}{4y^2}+1\right)+\dfrac{4y^2}{x^2}\left(\dfrac{y^2}{x^2}+1\right)+3$

39 $\dfrac{x^2}{y^2}+\dfrac{y^2}{4x^2}-\dfrac{x}{y}+\dfrac{y}{2x}-\dfrac{3}{4}$

40 $16a^2x^2-16a^2bx+4a^2b^2-24abxy+12ab^2y+9b^2y^2$

Extract the square root of

41. $x^6+4x^5+10x^4+10x^3+5x^2-6x+1$

42. $16a^6-24a^5+25a^4-20a^3+10a^2-4a+1$.

43. $a^4-a^2b+\dfrac{2a^2c^2}{3}-\dfrac{a^2d^3}{2}+\dfrac{b^2}{4}-\dfrac{bc^2}{3}+\dfrac{bd^3}{4}+\dfrac{c^4}{9}-\dfrac{c^2d^3}{6}+\dfrac{d^6}{16}$.

44. $x(x+1)(x+2)(x+3)+1$ 45. $\left(\dfrac{x}{x-1}\right)^2-\dfrac{2x^2}{x^2-1}+\left(\dfrac{x}{x+1}\right)^2$.

46. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$.

47. $(a^2+b^2)(x^2+y^2)-2(a^2-b^2)xy+2ab(x^2-y^2)$.

199. Fourth, Eighth, &c, Roots By the preceding article, we can find the *fourth*, *eighth*, *sixteenth* &c, roots of polynomials; for the fourth root is the square root of the square root, the eighth root is the square root of the fourth root, and so on

Examples

Ex. 1 Find the fourth root of $1-12x+54x^2-108x^3+81x^4$

$$
\begin{array}{r|l}
 & 1-12x+54x^2-108x^3+81x^4\ (\,1-6x+9x^2 \\
 & 1 \\
2-6x & -12x+54x^2 \\
 & -12x+36x^2 \\
2-12x+9x^2 & 18x^2-108x^3+81x^4 \\
 & 18x^2-108x^3+81x^4
\end{array}
$$

Thus the square root of the proposed expression $=1-6x+9x^2$, and the required fourth root is the square root of this root

$$
\begin{array}{r|l}
 & 1-6x+9x^2\ (\,1-3x \\
 & 1 \\
2-3x & -6x+9x^2 \\
 & -6x+9x^2;
\end{array}
$$

$\therefore$ root required $=1-3x$

2. Find the fourth root of $x^4+4x^3+6x^2+4x+1$.

3. Find the fourth root of $a^4-8a^3+24a^2-32a+16$

4. Find the fourth root of $4a^2b^2+(a^2+b^2)^2-4ab(a^2+b^2)$.

5. Find the fourth root of $x^4-4x^2+6-\dfrac{4}{x^2}+\dfrac{1}{x^4}$.

6. Find the eighth root of $256a^8-1024a^7x^2+1792a^6x^4-1792a^5x^6$
$+1120a^4x^8-448a^3x^{10}+112a^2x^{12}-16ax^{14}+x^{16}$.

CHAPTER XVI

Theory of Indices

200 The Index Law In § 68, we have proved that

$$\text{I} \quad a^m \times a^n = a^{m+n}$$

where m *and* n *are positive integers* We have called this result the *Index Law,* for this is the *fundamental law* from which all other laws regulating Indices are derived This we now proceed to shew

II. To prove that $a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\ldots}$

Left side $=(a^m \times a^n) \times a^p \times \ldots = a^{m+n} \times a^p \times \ldots$

$$= a^{m+n+p} \times \ldots = a^{m+n+p+\ldots}$$

III. To prove that $a^m \div a^n = a^{m-n}$, where $m > n$ [§ 75]

We have $\qquad a^{m-n} \times a^n = a^{m-n+n} = a^m$ [II],

$$a^{m-n} = a^m \div a^n \; [\S\ 69] = \frac{a^m}{a^n} \; [\S\ 167]$$

IV. To prove that $(a^m)^n = a^{mn}$, where m may have *any value*

Now $\qquad (a^m)^n = a^m \times a^m \times \ldots$ to n factors [§ 20]

$$= a^{m+m+\ldots \text{ to } n \text{ terms}} \; [\text{II}] = a^{mn}$$

V To prove that $(a^n)^m = a^{mn} = (a^m)^n$, where m and n are *both positive integers*

As in the last case, we can shew that $(a^n)^m = a^{mn}$

$$(a^n)^m = (a^m)^n = a^{mn}$$

VI To prove that $(ab)^n = a^n b^n$ [For proof, see § 185]

Just in the same way, it may be proved that

$(abc)^n = a^n b^n c^n$, and *generally* that $(abc\ldots)^n = a^n b^n c^n \ldots\ldots$

VII To prove that $(a^m b^n)^r = a^{mr} b^{nr}$

Now $\qquad (a^m b^n)^r = (a^m)^r \times (b^n)^r$ [VI] $= a^{mr} b^{nr}$ [IV]

And *generally* $\qquad (a^m b^n c^p \ldots)^r = a^{mr} b^{nr} c^{pr} \ldots$

VIII To prove that $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Now $\qquad \left(\dfrac{a}{b}\right)^n = \dfrac{a}{b} \times \dfrac{a}{b} \times \dfrac{a}{b} \times \ldots$ to n factors

$$= \frac{a \times a \times a \times \ldots \text{ to } n \text{ factors}}{b \times b \times b \times \ldots \text{ to } n \text{ factors}} = \frac{a^n}{b^n}$$

Thus all the above Laws are proved for *positive integers.* We shall see [§ 208] that *fractional and negative indices* also obey the same Laws Thus they hold *universally.*

201. Examples [*Application of Laws I and II.*]

Ex. 1 $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$, for here $m = \frac{p}{q}$, $n = \frac{r}{s}$.

Ex. 2 $x^{-p} \times x^{-q} = x^{-p-q}$, for here $m = -p$, $n = -q$

Ex. 3 $m^{x-1} \times m^{\frac{x}{3}+\frac{1}{2}} = m^{x-1+\frac{x}{3}+\frac{1}{2}} = m^{\frac{4x}{3}-\frac{1}{2}}$

Ex 4 $(a+b)^{\frac{m}{2}-n}(a+b) = (a+b)^{\frac{m}{2}-n+1}$.

Find the value of

5 $2^{\frac{1}{2}} 2^{\frac{1}{4}}$ 6. $3.3^{\frac{4}{5}}$. 7 $4^{\frac{1}{3}} 4^{-\frac{1}{5}}$. 8 $7.8^2 - 8.7^2$.

9 $2^3.2^{-4}$ 10. $4^{-3} 4^{-\frac{1}{5}}$ 11 $a a^{m-1}$ 12. $m^{2a-b} m^{3b-a}$.

13. $x^{\frac{c}{2}-1} x^{\frac{c}{3}-\frac{1}{2}}$. 14 $y^{-\frac{1}{5}} y^{3-\frac{1}{2}}$. 15 $(a+b)^{-2}(a+b)^3$

16. $(x-y)(x-y)^{-\frac{1}{4}}$ 17. $(m+n)^{\frac{m}{2}-1}(m+n)^{-\frac{1}{2}}$.

18 $a^{m-n+1} a^{m+n-1}$. 19. $x^{a+b-2c} x^{a-b+c}$.

20. $b^{3+2m-p} b^{m+n-3}$ 21 $y^{\frac{a}{3}-\frac{c}{2}} y^{a+\frac{c}{4}}$ 22 $z^{\frac{m}{a}+\frac{n}{b}} z^{\frac{n}{b}-\frac{1}{3}}$.

23 $4b^{-m-n} 5b^{-m-p}$. 24. $(a^3 - 3a^2b + 3ab^2 - b^3)(a-b)^{-4}$.

25. $(a^{2m} + 2a^{m+n} + a^{2n})(a^m + a^n)$

Ex. 26. $a^2 a^{-3} a^4 = a^{2-3+4} = a^3$

Ex 27 $z^{\frac{m}{2}-\frac{1}{2}} z^{\frac{n}{3}+1} z^{\frac{p}{4}-\frac{3}{2}} = z^{\frac{m}{2}+\frac{n}{3}+\frac{p}{4}-\frac{1}{2}+1-\frac{3}{2}} = z^{\frac{m}{2}+\frac{n}{3}+\frac{p}{4}-1}$.

Find the value of $2^2, 2^3, 2^{-4} = 2$

28 $2^3 2^2.2$ 29 $4.8.2^{-4}$ 30. $81.27.3^{-7}$

31 $z z^{m-3} z^3$ 32 $2a^{-m} 5a^{-2} 3a^{2m}$ 33 $a^{-\frac{1}{2}} a^{-\frac{1}{3}} a^3 3a^{\frac{1}{3}}$.

34. $5x^{-\frac{1}{m}} 8x^{-\frac{2}{m}} 7x^{-1}$ 35 $4p^{x-a} 3p^{\frac{1}{2}} p^{-\frac{3}{2}} 10p^{a-x}$

202 Examples [*Application of Law III.*]

Ex 1. $a^m \div a = \frac{a^m}{a} = a^{m-1}$

Ex 2. $m^{\frac{1}{2}} \div m^{\frac{1}{4}} = m^{\frac{1}{2}-\frac{1}{4}} = m^{\frac{1}{4}}$

Ex 3 $\frac{y^{-3}}{y^{-4}} = y^{-3-(-4)} = y^{-3+4} = y$, for here $m = -3$, $n = -4$

Ex 4 $c^{\frac{x}{2}-1} \div c^{-\frac{x}{5}} = c^{\frac{x}{2}-1+\frac{x}{5}} = c^{\frac{7x}{10}-1}$

Ex 5 $z^{-\frac{m}{a}} \div z^{\frac{m}{a}-\frac{n}{b}} = z^{-\frac{m}{a}-\left(\frac{m}{a}-\frac{n}{b}\right)} = z^{-\frac{2m}{a}+\frac{n}{b}}$.

Find the value of

6. $a^{x-y} \div a^{x+y}$ 7 $3x^{a-c} \div x^{2a+4c}$ 8 $15a^{m-2} \div 3a^{-3}$.

9 $\frac{9x^{-3}}{y^{-m}} \div \frac{3x^{-4}}{y^{z}}$. 10. $m \div m^{x}$ 11 $x^{-2a} \div x^{-3a-1}$

12 $3z^{c-a} \div z^{a-c}$ 13 $(a+b)^{m-\frac{1}{2}} \div (a+b)^{m+\frac{1}{2}}$

14. $\frac{x^{6m-2n}}{y^{4m-7n}} \times \frac{y^{5n-8m}}{x^{3m+2n}}$ 15 $\frac{a^{p+q}b^{x+y}}{a^{-3}b^{-p-3q}} \div \frac{a^{8-x}}{a^{-5p-1}b^{2(x-1)}}$.

16 $\frac{(x+y)^{a-m}}{(x-y)^{-1}} \div \frac{(x+y)^{-m}}{(x-y)^{3p-1}}$

203. Examples [*Application of Laws IV and V*]

Ex. 1 $(a^4)^3 = a^{4\times 3} = a^{12}$ Ex 2 $\{(x^2)^3\}^{\frac{1}{4}} = (x^2)^{\frac{3}{4}} = x^{2\times\frac{3}{4}} = x^{\frac{3}{2}}$

Ex 3 $(-a^2)^4 = (-1)^4(a^2)^4$ [§ 186] $= a^{2\times 4} = a^8$

Ex 4. $(a^4x^2y^{\frac{1}{2}})^6 = a^{4\cdot 6}x^{2\cdot 6}y^{\frac{1}{2}\times 6} = a^{24}x^{12}y^3$

Find the value of

5 $(3^3)^3$ 6. $(5^3)^2-(124)^2$ 7 $\{(2^2)^3\}^2$. 8 $(x^3)^{\frac{1}{2}}$

9 $\{(256a^4)^{\frac{1}{3}}\}^{\frac{3}{8}}$ 10 $(-x^2)^2$ 11 $(-x^2)^3$ 12 $(-x^3)^{2m}$

13 $(-x^3)^{p+1}$ 14 $(-x^4)^{\frac{1}{4}}$ 15 $(-x^5)^4$. 16 $(a^3b^3)^{2m}$

17 $(4a^{-\frac{1}{3}}b^{\frac{1}{2}})^6$ 18 $(5a^{2m-4}x^{2n}y^{-4})^{-\frac{1}{2}}$ 19 $\{(2a^{-m}b^{n})^{-2}\}^{-m}$.

20. $(a^3+3b^3)^2$. 21 $(a^{n}-b^{\frac{n}{3}})^3$ 22 $(a^2+b^2)^2(a^2-b^2)^2$

204 Examples. [*Application of Laws VI and VII*]

Ex. 1 $2^3\times 3^3 = (2\times 3)^3 = 6^3 = 216$

Ex. 2. $4^5\times 10^5\times 15^5 = (4\times 10\times 15)^5 = (600)^5 =$ &c

Ex 3 $(3x)^8\times(10y)^8 = (3x\times 10y)^8 = (30xy)^8 =$ &c

Ex 4. $a^m(a+b)^m b^m = \{a(a+b)b\}^m = (a^2b+ab^2)^m$.

Ex 5 $(a+1)^p(a^2+1)^p(a-1)^p = \{(a+1)(a^2+1)(a-1)\}^p = (a^4-1)^p$.

Find the value of

6. $3^4 \times 4^4$. 7. $9^{\frac{1}{3}} \times 24^{\frac{1}{3}}$ 8. $7^{-5} \times 10^{-5}$. 9. $12^{\frac{1}{2}} \times 48^{\frac{1}{2}}$.

10 $13^3 \times 10^3 \times 15^3$. 11 $(2a)^4(3b)^4$ 12 $(5a^{-1}b^{-1})^{m+1}(3a^2b^2)^{m+1}$.

13 $(9x^2y^{-m})^a(10x^{-2}y)^a$. 14 $(a+b)^2(a-b)^2$

15. $(x^2-ax+a^2)^3(x+a)^3$. 16 $(x-a)^4(x-b)^4$

17. $(2x+3y)^{-4}(2x-3y)^{-4}$ 18 $(a+1)^{-2}(b+1)^{-2}$.

19. $(x+y)^{-2m}(x-1)^{-2m}$. 20 $\left(\frac{3x}{4a}\right)^{-m}\left(\frac{2y}{5a^2}\right)^{-m}\left(\frac{10a}{3xy}\right)^{-m}$.

21 $\left(\frac{a+b}{a-b}\right)^x\left(\frac{a-b}{a^2+b^2}\right)^x\left(\frac{a^2+b^2}{a+b}\right)^x$

205 Examples [*Application of Law VIII*]

Ex. 1 $\frac{8^5}{10^5} = \left(\frac{8}{10}\right)^5 = \left(\frac{4}{5}\right)^5 = \frac{1024}{3125}$.

Ex 2 $\left(\frac{2}{3}\right)^m\left(\frac{9}{10}\right)^m = \left(\frac{2}{3}\times\frac{9}{10}\right)^m = \left(\frac{3}{5}\right)^m = \frac{3^m}{5^m}$.

Ex 3 $\left(\frac{5ab}{3c^2}\right)^3 \times \left(\frac{cx}{a^2y}\right)^3 \div \left(\frac{10a^2b}{3cxy}\right)^3 = \left(\frac{5ab}{3c^2}\times\frac{cx}{a^2y}\times\frac{3cxy}{10a^2b}\right)^3 = \frac{x^6}{8a^9}$.

Find the value of

4 $\frac{(25)^3}{(30)^3}$ 5 $\frac{(10a^3x^3y^2)^m}{(16abx^3y)^m}$ 6. $\left(\frac{a}{b^4}\right)^4\left(\frac{2}{a}\right)^4$.

7. $\left(\frac{4a^3x^{m+2}}{by^2}\right)^{\frac{3}{m}} \div \left(\frac{4a^2y}{bc}\right)^{\frac{3}{m}} \times \left(\frac{y^3}{acx^2}\right)^{\frac{3}{m}}$. 8. $\left(\frac{a+b}{a-b}\right)^{2m}\left(\frac{b-a}{b+a}\right)^{2m}$.

9 $\left(\frac{ax+ay}{m^2-n^2}\right)^{2n-1}\left(\frac{m+n}{2x+2y}\right)^{2n-1}(n-m)^{2n-1}$.

10. $\left(\frac{3xy}{5ab}\right)^4\left(\frac{5a}{6x}\right)^2\left(\frac{4y}{3b}\right)^2$. 11 $\left(\frac{a^2-b^2}{ax+ay}\right)^{3m}\left(\frac{bx+by}{2a-2b}\right)^{2n-1}\left(\frac{2x+2y}{a+b}\right)$.

206 Fractional and Negative Indices The definition of § 20, is quite sufficient so long as the Index is a *positive integer*, but becomes meaningless when such quantities as $a^{\frac{1}{2}}$, a^{-1}, &c, are considered. It is necessary therefore to find *new meanings* of fractional and negative indices.

Now to preserve generality of reasoning in Algebra, it is necessary that algebraical symbols, whatever their values may be, should always be subject to the same laws Hence to find meanings of fractional and negative indices, our *meaning should in every case be such that the Index Law*

$$a^m \times a^n = a^{m+n}$$

shall always be obeyed We must therefore limit our choice to such meanings only as are consistent with the above law

To give a meaning to $a^{\frac{1}{2}}$ The meaning, to be consistent with the Index Law, should be such that

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} \text{ must be} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$

Hence $a^{\frac{1}{2}}$ is a quantity whose square is a [§ 22], i e, $a^{\frac{1}{2}} = \sqrt{a}$

Similarly it may be shewn that $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$, &c

We shall now investigate the meanings of $a^{\frac{1}{n}}$, $a^{\frac{m}{n}}$, a^0 and a^{-p}.

207 I. Meaning of $a^{\frac{1}{n}}$, where n is a positive integer

By the Index Law,

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots \text{to } n \text{ factors}$$
$$= a^{\frac{1}{n}+\frac{1}{n}+\ldots \text{ to } n \text{ terms}} = a^{\frac{1}{n}\times n} = a^1 = a,$$

therefore $a^{\frac{1}{n}}$ must be such a quantity that its n^{th} power is a.

Hence $a^{\frac{1}{n}} = \sqrt[n]{a}$

II Meaning of $a^{\frac{m}{n}}$, where m and n are both positive integers

By the Index Law, we have

$$x^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots\ldots \ldots \text{to } n \text{ factors}$$
$$= a^{\frac{m}{n}+\frac{m}{n}+\ldots\ldots \text{to } n \text{ terms}} = a^{\frac{m}{n}\times n} = a^m$$

Therefore $a^{\frac{m}{n}}$ must be an n^{th} root of a^m, that is, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Again

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots \text{to } m \text{ factors}$$
$$= a^{\frac{1}{n}+\frac{1}{n}+\ldots\ldots \text{to } m \text{ terms}} = \left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}} \text{ [§ 200, IV]}$$

Thus $a^{\frac{m}{n}}$ must be the m^{th} power of an n^{th} root of a, i e, $a^{\frac{m}{n}}=(\sqrt[n]{a})^m$

Therefore $$a^{\frac{m}{n}}=\sqrt[n]{a^m}=(\sqrt[n]{a})^m$$

Hence $a^{\frac{m}{n}}$ may be considered as an n^{th} root of the m^{th} power of a, or as the m^{th} power of an n^{th} root of a Thus *when the index is a fraction, the numerator denotes a power and the denominator a root*

Corollary. Hence $a^{\frac{m}{n}}=a^{\frac{mp}{np}}$ For if $x=a^{\frac{m}{n}}$, we have $x^n=a^m$, or $(x^n)^p=(a^m)^p$, that is, $x^{np}=a^{mp}$; thus $x=a^{\frac{mp}{np}}$, or $a^{\frac{m}{n}}=a^{\frac{mp}{np}}$.

III. Meaning of a^0, where a may have any value [§ 75, *Cor*]

We have by the Index Law

$$a^0\times a^p=a^{0+p}=a^p\text{; therefore } a^0=a^p\div a^p\ [\S\ 69]=1.$$

IV Meaning of a^{-p}, where p is any positive integer

By the Index Law

$$a^{-p}\times a^p=a^{-p+p}=a^0=1\ [\text{by III}],$$

$$\therefore\quad a^{-p}=1\div a^p=\frac{1}{a^p}\text{; also } a^p=1\div a^{-p}=\frac{1}{a^{-p}}\ [\S\ 69].$$

Thus *the negative index indicates the reciprocal of the power denoted by the positive index* [§ 72 *Def*]

Hence $$a^{-1}=\frac{1}{a},\ a^{-2}=\frac{1}{a^2},\ a^{-3}=\frac{1}{a^3},\ \ldots\ldots$$

also $$a=\frac{1}{a^{-1}},\ a^2=\frac{1}{a^{-2}},\ a^3=\frac{1}{a^{-3}},\ \ldots\ldots$$

Corollary We have seen [§ 87] how an expression may be arranged according to the powers of a letter having *positive integral* indices We may now arrange an expression in which the *symbol of reference* has *fractional* and *negative* indices Thus

$$x^2+x+x^{\frac{1}{2}}+x^{\frac{1}{3}}+x^{\frac{1}{4}}+1$$

is arranged in *descending* powers of x, and the same in a *reversed* order will be arranged in *ascending* powers of x

Again the expression

$$x^3+2x^2+5x+3+4x^{-1}+6x^{-2}+x^{-5}$$
$$=x^3+2x^2+5x+3x^0+4x^{-1}+6x^{-2}+x^{-5}$$

is arranged in *descending* powers of x, as 3, 2, 1, 0, -1, -2, -5 are

in descending order of magnitude [§ 45] And since $x^{-1}=\frac{1}{x}$, $x^{-2}=\frac{1}{x^2}$ and $x^{-5}=\frac{1}{x^5}$, the same expression is equivalent to

$$x^3+2x^2+5x+3+\frac{4}{x}+\frac{6}{x^2}+\frac{1}{x^5},$$

this therefore is also arranged in descending powers of x Of course each of these expressions when written in a *reversed* order will be arranged in *ascending* powers of x

Examples

Ex 1. $\frac{m^3}{m^{-2}}=m^3\,m^2=m^5$ or $=m^3\,\frac{1}{m^{-2}}=\frac{1}{m^{-3}}\,\frac{1}{m^{-2}}=\frac{1}{m^{-5}}$

Ex 2 $\frac{a^{-(x+1)}}{a^{-3x}}=a^{-(x+1)}\times a^{3x}=a^{2x-1}$ or $=\frac{1}{a^{1-2x}}$

Ex 3. $\frac{y^{-2a+1}}{y^{-(2a-1)}}=y^{-3a+1}y^{2a-1}=y^{-a}$ or $=\frac{1}{y^a}$

Ex 4. $(1-x^2)(1+x)^{-1}=(1-x^2)\times\frac{1}{1+x}=1-x$ or $=\frac{1}{(1-x)^{-1}}$

Ex. 5 $\frac{a^{-3}b^{-2}}{c^{-4}d^{-1}}\times\frac{c^3d^{-3}}{a^{-2}b^3}\times\frac{c^{-3}}{a^{-1}b^{-3}}=\frac{c^4d}{a^3b^2}\times\frac{c^3a^2}{b^3d^3}\times\frac{ab^3}{c^3}=\frac{c^3}{b^2d^2}$ or $=\frac{b^{-2}d^{-2}}{c^{-3}}$.

Ex 6 $\left(\frac{a+b}{a-b}\right)^{-m}=\frac{(a+b)^{-m}}{(a-b)^{-m}}=\frac{(a-b)^m}{(a+b)^m}=\left(\frac{a-b}{a+b}\right)^m$

Simplify

7. $\frac{8}{3^{-2}}$ 8 $\frac{20}{(-5)^2}$ 9 $\frac{18}{(-4)^{-2}}$ 10 $\left(\frac{1}{8}\right)^{-2}-\left(\frac{1}{5}\right)^{-3}$.

11. $x^2\left(\frac{x}{y}\right)^{-5}$ 12 $c^{-\frac{3}{2}}\left\{\frac{x^{-3}}{c^2y^{-4}}\right\}^{-\frac{1}{2}}$ 13 $\left(\frac{1}{x}\right)^{-1}\left(\frac{y}{x}\right)^{-m}$

14. $\{(2a)^{-1}\}^{-4}$ 15 $\{(-3x^2)^{-1}\}^3$ 16 $(2a^2b^{-1})^{-2}$

17. $\left\{\frac{a^mb^{-n}}{cd^{-2}}\right\}^{-x}$ 18 $\left\{-\frac{a^mb^{-x}}{b^{-y}c^x}\right\}^{-3}$

19 $\left(\frac{1}{a^{-m}}\right)^2-\left(\frac{1}{a}\right)^{2m}$ 20 $\left\{\frac{1}{x^{-2}}\right\}^2\left\{\frac{m^{-1}}{x^{-2}}\right\}^{-1}$

208. Theorem *To prove that $a^m \times a^n = a^{m+n}$ for all values of* m *and* n

The Theorem has already been proved for positive integers [§ 20]

Now let $m = \frac{p}{q}$, $n = \frac{r}{s}$, where p, q, r and s are all positive integers

$$\text{Thus } a^m \times a^n = a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}} \text{ [§ 207, II, Cor.]}$$

$$= \sqrt[qs]{(a^{ps})} \times \sqrt[qs]{(a^{qr})} = \sqrt[qs]{(a^{ps} \times a^{qr})}$$

$$= \sqrt[qs]{a^{ps+qr}} = a^{\frac{ps+qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}} = a^{m+n}.$$

Next let *both* m and n be negative, and let $m = -p$, $n = -q$; then

$$a^m \times a^n = a^{-p} \times a^{-q} = \frac{1}{a^p} \times \frac{1}{a^q} = \frac{1}{a^{p+q}} = a^{-p-q} = a^{m+n}$$

Secondly, let *one* of m and n be negative, and let $m = p$ and $n = -q$. Here we have to consider two cases according as $p >$ or $< q$.

(i) Let $p > q$, i e, let $p - q$ be positive, then

$$a^{p-q} \times a^q = a^{p-q+q} = a^p,$$

$$\text{or } a^{p-q} = a^p \div a^q \text{ [§ 69]} = a^p \times \frac{1}{a^q} \text{ [§ 72]} = a^p \times a^{-q};$$

$$\therefore a^m \times a^n = a^p \times a^{-q} = a^{p-q} = a^{m+n}$$

(ii) Let $p < q$, i e, let $p - q$ be negative, then

$$a^m \times a^n = a^p \times a^{-q} = \frac{1}{a^{-p}} \times \frac{1}{a^q} = \frac{1}{a^{q-p}} = a^{p-q} = a^{m+n}.$$

Thus the Theorem is proved for all values of m and n

We have remarked in § 209, that all the Index Laws are *corollaries to the fundamental Index Law*, which is now proved to be true for all values of m and n Hence the other Index Laws hold for all values of m and n

Corollary. Since $a^{m-n} \times a^n = a^m$ for all values of m and n, we have $a^{m-n} = a^m \div a^n$ for *all values* of m and n [§ 69]

Miscellaneous Examples. VII.

Add together

1 $4a^{\frac{2}{5}} - 5a^{\frac{3}{5}} + 6a^{\frac{1}{5}}$, $3a^{\frac{3}{5}} - 8a^{\frac{2}{5}} - 4a^{\frac{1}{5}}$, $a^{\frac{1}{5}} - 6a^{\frac{3}{5}} + 7a^{\frac{2}{5}}$, and $4a^{\frac{3}{5}} - 3a^{\frac{1}{5}}$.

2 $2x^{\frac{4}{3}} - 3x^{\frac{2}{3}}y^{\frac{2}{3}} - 4y^{\frac{4}{3}}$, $3x^{\frac{2}{3}}z^{\frac{2}{3}} - 2y^{\frac{4}{3}} - z^{\frac{4}{3}}$, $x^{\frac{4}{3}} - 2y^{\frac{2}{3}}z^{\frac{2}{3}} + 5z^{\frac{4}{3}}$, $3x^{\frac{2}{3}}y^{\frac{2}{3}} - 6x^{\frac{2}{3}}z^{\frac{2}{3}} - 3x^{\frac{4}{3}}$, and $3x^{\frac{2}{3}}z^{\frac{2}{3}} - 2z^{\frac{4}{3}} + 5y^{\frac{2}{3}}z^{\frac{2}{3}}$

Add together

3 $x^{-2} - x^{-3}y^{\frac{1}{2}} - 3x^{-4}y^{\frac{1}{3}}$, $2x^{-3}y^{\frac{1}{2}} - 2x^{-2} + 5x^{-4}y^{\frac{1}{3}}$, and $-x^{-4}y^{\frac{1}{3}} + x^{-2} - 3x^{-3}y^{\frac{1}{2}}$

4 $\frac{3}{4}a^{-\frac{5}{2}} - \frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}} + \frac{2}{3}y^{\frac{1}{4}}$, $\frac{3}{8}a^{\frac{3}{2}}x^{-\frac{1}{2}} + 2a^{-\frac{5}{2}} - \frac{4}{3}y^{\frac{1}{4}}$, $\frac{5}{6}y^{\frac{1}{4}} - \frac{4}{5}a^{\frac{4}{3}}x^{-\frac{1}{2}} - \frac{5}{2}a^{-\frac{5}{2}}$ and $\frac{2}{5}a^{\frac{4}{3}}x^{-\frac{1}{2}} - \frac{1}{8}y^{\frac{3}{4}}$

5 $x^{-\frac{4}{5}} - 2ax^{-\frac{3}{5}} + 2bx^{-\frac{2}{5}}$, $3ax^{-\frac{3}{5}} - mx^{-\frac{4}{5}} - cx^{-\frac{2}{5}}$, $bx^{-\frac{2}{5}} + dx^{-\frac{3}{5}} - 2x^{-\frac{4}{5}}$ and $-nx^{-\frac{3}{5}} + 5x^{-\frac{4}{5}} - 4cx^{-\frac{2}{5}}$

6 $-5ax^{\frac{m}{3}} - 4by^{-\frac{m}{2}} + cz^{-\frac{m}{4}}$, $2by^{-\frac{m}{2}} + cz^{-\frac{3m}{4}} + 8x^{\frac{m}{3}}$, $13ax^{\frac{m}{3}} - 5z^{-\frac{m}{4}}$ and $7cz^{-\frac{m}{4}} - 6y^{-\frac{m}{2}}$.

7 From $5x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}}a^{-\frac{1}{2}} + 4$ take $2a^{-\frac{1}{2}}x^{-\frac{1}{3}} - x^{-\frac{2}{3}} - 2\frac{1}{2}$

8 Take $a^{-\frac{m}{n}} + \frac{3}{8}a^{\frac{m}{n}}b^{-3} - \frac{2}{5}a^{-2m}$ from $\frac{3}{4}a^{-\frac{m}{n}} - \frac{5}{16}a^{\frac{m}{n}}b^{-3} + \frac{1}{10}a^{-m}$

Multiply

9 $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$

10. $x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2$ by $x - x^{\frac{1}{2}}$.

11 $2a^{-4}b^{-1} - 3a^{-5}$ by $3a^{-4}b^{2} - 2a^{-3}$

12 $\frac{3}{4}a^{-1}x^{-3} - \frac{2}{5}b^{-2}y$ by $\frac{2}{3}a^{-1}x^{-3} + \frac{1}{3}b^{-2}y$

13. $a^{\frac{m}{2}} + 2a^{\frac{m}{4}}b^{-1} + b^{-2}$ by $a^{\frac{m}{2}} - 2a^{\frac{m}{4}}b^{-1} + b^{-2}$

14 $ax^{\frac{3}{4}} - by^{-\frac{1}{2}}$ by $ax^{\frac{3}{4}} + by^{-\frac{1}{2}}$

15 $a^{\frac{m}{2}} - a^{\frac{m}{4}}b^{\frac{n}{4}} + b^{\frac{n}{2}}$ by $a^{\frac{m}{2}} + a^{\frac{m}{4}}b^{\frac{n}{4}} + b^{\frac{n}{2}}$

16 $\frac{3}{2}\left(x^{\frac{3}{2}} + x^{-\frac{3}{2}}\right) + \frac{1}{2}\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)$ by $\frac{2}{3}(x + x^{-1}) + 1$

17 $x^{-\frac{3}{p}} - 2x^{-\frac{2}{p}} + 3x^{\frac{1}{p}}$ by $2x^{\frac{2}{p}} - x^{\frac{1}{p}} + 2$

Divide

18. $a^{\frac{3}{2}} + b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

19 $27 - a^{-\frac{3}{5}}$ by $3 - a^{-\frac{1}{5}}$

20 $x - 2x^{\frac{1}{2}} + 1$ by $x^{\frac{1}{3}} - 2x^{\frac{1}{6}} + 1$

21 $x^{4} + x^{-4} + 1$ by $x^{2} + x^{-2} + 1$

22 $a^{\frac{4}{3}} - 2 + a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}} - a^{-\frac{2}{3}}$.

23 $x^{\frac{5}{2}} - x^{2} - 4x^{\frac{3}{2}} + 6x - 2x^{\frac{1}{2}}$ by $x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2$

24 $x + y - z + 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}} - z^{\frac{1}{3}}$.

Divide

25. $1-2abxy^{-1}+3acx^2y^{-2}-a^{-1}x^my+2bx^{m+1}-3cx^{m+2}y^{-1}$ by $ax^{-m}-y$.

26. $(1+3b)a^{-5}(3-2a^{-2}b^4)+5a^{-4}b^{-5}(2b^4-3a^2)$ by $a^{-3}b^2(2a^{-2}-3b^{-4})$.

Simplify

27. $\left\{\dfrac{ab^{\frac{1}{2}}}{a^{\frac{1}{3}}b^{\frac{1}{3}}}\right\}^{\frac{3}{4}}$. 28. $\left(\dfrac{x}{y}\right)^{2m}\left(-\dfrac{y}{x}\right)^{2m-1}$. 29. $\left(\dfrac{ay}{x}\right)^{\frac{1}{2}}\left(\dfrac{bx}{y^2}\right)^{\frac{1}{3}}\left(\dfrac{y^2}{a^3b^2}\right)^{\frac{1}{6}}$.

30. $\left\{\left(\dfrac{a^{-2m}}{b^{-2n}}\right)^{\frac{p}{m}}\right\}^{\frac{q}{2n}}$. 31. $\dfrac{\{(a^m)^{\frac{1}{r}}(a^q)^{\frac{1}{n}}\}^{nr}}{\left(b^{\frac{n}{q}}b^{\frac{r}{m}}\right)^{mq}}$.

32. $\left\{(x^a)^{1-\frac{1}{a}}\right\}^{\frac{1}{a-1}}$. 33. $\left\{(ax^x)^{a+\frac{1}{x}}\right\}^{1-\frac{ax}{1+ax}}$.

34. $\left\{aa^{-\frac{m+n}{n}}\right\}^{\frac{n}{m+n}}\left\{a^{m+n}a^{-n}\right\}^{\frac{1}{m-n}}$.

*35. $\left(-\dfrac{x}{a}\right)^{2m-1}\left(-\dfrac{a}{y}\right)^{2m+1}$. 36. $\dfrac{x}{y^m}-\dfrac{x}{y^{m-1}}$.

37. $\left(\dfrac{x}{y}\right)^m+\left(\dfrac{x}{y}\right)^{m-1}$. 38. $\dfrac{a^n}{(a-b)^n}-\dfrac{a^{n-1}}{(a-b)^{n-1}}$.

*39. $\left(\dfrac{3x}{a}+\dfrac{2y^{-1}}{b^{-2}}\right)^{-2}\left(\dfrac{2y^{-1}}{b^{-2}}-\dfrac{3x}{a}\right)^{-2}$. *40. $\left(\dfrac{x-y}{a-b}\right)^{2m}\left(\dfrac{b-a}{x-y}\right)^{2m+1}$.

*41. $\left(\dfrac{z-y}{x-a}\right)^{2m+1}\left(\dfrac{3y}{y-z}\right)^{2m}\left(\dfrac{a-x}{z-y}\right)^2$. 42. $\dfrac{x^{a+b}}{x^{2c}}\times\dfrac{x^{a+c}}{x^{2b}}\times\dfrac{x^{b+c}}{x^{2a}}$.

Find the value of

43. $\left(\frac{1}{2}x^{-\frac{1}{2}}y^{-1}-\frac{3}{4}x^{-1}y^{-\frac{1}{2}}\right)^2$. 44. $\left(a^{\frac{1}{3}}x^{-1}-a^{-\frac{1}{3}}x\right)^3$.

Resolve into factors

*45. $x^{4m}-y^{2n}$. 46. $\left(\dfrac{x}{a}\right)^{\frac{3}{m}}-\left(\dfrac{y}{b}\right)^{\frac{3}{m}}$. 47. $x^{-6}-\left(\dfrac{1}{y^{-1}}\right)^{-2}$.

Extract the square root of

48. $a^{-2}x^{-2}-2a^{-1}x^{-1}+3-2ax+a^2x^2$.

49. $x^{m}+\frac{1}{4}(b^px^{2n})^{\frac{2}{np}}-b^{\frac{1}{n}}x^{\frac{mp+4}{2p}}$.

Reduce to a simple form

50. $\dfrac{\dfrac{1}{4^{-3}} - \dfrac{2}{10^{-2}}}{\dfrac{1}{2^{-2}} + \dfrac{1}{4^{-1}}}$. 51. $\left\{\dfrac{a^{-1}+b^{-1}}{a^{-2}-b^{-2}}\right\}^{-2} - \left\{\dfrac{a^{-2}+b^{-2}}{a^{-4}-b^{-4}}\right\}^{-3}$

52. $\dfrac{x^{\frac{3}{2}}+y^{\frac{3}{2}}}{x-y}$. 53. $\dfrac{x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}}{x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}}$ 54. $\dfrac{x^{\frac{2m}{3}}-x^{\frac{m}{3}}y^{\frac{m}{3}}-2y^{\frac{2m}{3}}}{x^{\frac{m}{3}}-y^{\frac{m}{3}}}$

55. $\dfrac{2x^{\frac{3}{m}}+3x^{\frac{2}{m}}+4x^{\frac{1}{m}}-3}{6x^{\frac{3}{m}}+x^{\frac{2}{m}}-1}$ 56. $\dfrac{e^{2x}+e^{-x}-e^{x}-1}{e^{2x}-e^{-x}+e^{x}-1}$

57. $\dfrac{x^{\frac{2}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{3}}-y^{\frac{2}{3}}}{x^{\frac{4}{3}}-8x^{\frac{2}{3}}y^{\frac{2}{3}}-9x^{\frac{1}{3}}y-2y^{\frac{4}{3}}}$. 58. $\dfrac{a^2b^{-2}+(xy^{-1}+yx^{-1})ab^{-1}+1}{a^2b^{-2}+(xy^{-1}-yx^{-1})ab^{-1}-1}$.

59. $\dfrac{a^2+2ax+3x^2+2a^{-1}x^3}{a^2+x^2-2a^{-1}x^3}$. 60. $\dfrac{4a^3b^{-3}+4a^2b^{-2}-7ab^{-1}+2}{4a^3b^{-3}+5a^2b^{-2}-7ab^{-1}-2}$.

Divide

61. $x^{2^n}-1$ by $x^{2^{n-1}}+1$. 62. $\left(\frac{4}{3}a^{\frac{3}{2}}\right)^{\frac{1}{2}} - \left(\frac{3}{4}a^{\frac{3}{2}}\right)^{\frac{1}{2}}$ by $3^{\frac{3}{2}}a^{-\frac{1}{2}}$.

63. If $x^2=y^3$, shew that $\left(\dfrac{x}{y}\right)^{\frac{3}{2}} + \left(\dfrac{y}{x}\right)^{\frac{2}{3}} = x^{\frac{1}{2}}+y^{-\frac{1}{3}}$

64. Evaluate $\dfrac{(a^{n+x}-a^n)\times(a^n-a^{n-x})}{(a^{n+x}-a^n)-(a^n-a^{n-x})}$.

CHAPTER XVII.

SURDS

209. Surds. When any proposed root of a quantity cannot be exactly (*i e*, in any finite number of terms) obtained, it is called a SURD or IRRATIONAL QUANTITY, or simply a SURD Thus $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt{a}$, $\sqrt[3]{a^3+b^3}$, &c, are surds

A surd is expressed either by means of the radical sign as above or by means of its equivalent the fractional index [§ 207] Thus $\sqrt{a^2+b^2}=(a^2+b^2)^{\frac{1}{2}}$, $\sqrt{(a-b)^3}=(a-b)^{\frac{3}{2}}$ [§ 207], &c.

Hence *surds are subject to all the rules of operation explained in the Chapter on Indices* [Chapter XVI]

210. We have

(i). $x=x^{\frac{2}{2}}=\sqrt{x^2}=x^{\frac{m}{m}}=\sqrt[m]{x^m}=$&c. [207]

(ii). $a-x=(a-x)^{\frac{3}{3}}=\sqrt[3]{(a-x)^3}=\sqrt[n]{(a-x)^n}=$&c.

Thus *a rational quantity may be reduced to the form of a given surd, by raising it to the power whose root the surd expresses*

211 We have

(i) $3\sqrt{3}=\sqrt{3^2}\sqrt{3}=\sqrt{3^2\times 3}=\sqrt{27}$ or $=3^{\frac{2}{2}}\times 3^{\frac{1}{2}}=3^{\frac{3}{2}}=\sqrt{3^3}$.

(ii) $a\sqrt{x}=a^{\frac{2}{2}}\times x^{\frac{1}{2}}=(a^2)^{\frac{1}{2}}x^{\frac{1}{2}}=(a^2x)^{\frac{1}{2}}$ [§ 200] $=\sqrt{a^2x}$.

(iii) $x\sqrt[m]{(a-x)^n}=x^{\frac{m}{m}}(a-x)^{\frac{n}{m}}=\{x^m(a-x)^n\}^{\frac{1}{m}}$.

Thus *to introduce a coefficient of a surd under the radical, reduce the coefficient to the form of the surd and then multiply the quantity so reduced by that under the radical*

Examples

Introduce under the radical, the co efficient of by art. of § 211.

1 $5\sqrt{2}$ 2 $3\sqrt[3]{4}$ 3 $10\sqrt{ax^3}$ 4 $a^3x\sqrt[3]{ax^4}$ 5. $q^2r^3\sqrt[m]{q^n r^t}$

212 It is easy to see that

(i) $\sqrt{18}=\sqrt{9\times 2}=\sqrt{3^2\times 2}=3\sqrt{2}$.

(ii) $\sqrt{a^3x^4}=\sqrt{a^2ax^2x^2}=ax^2\sqrt{a}$.

(iii) $(a^2-x^2)^{\frac{1}{2}}=\left\{a^2\left(1-\frac{x^2}{a^2}\right)\right\}^{\frac{1}{2}}=(a^2)^{\frac{1}{2}}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}=a\sqrt{1-\frac{x^2}{a^2}}$

(iv). $(a+x)^{\frac{1}{n}}=\left\{a\left(1+\frac{x}{a}\right)\right\}^{\frac{1}{n}}=a^{\frac{1}{n}}\left(1+\frac{x}{a}\right)^{\frac{1}{n}}=a^{\frac{1}{n}}\sqrt[n]{1+\frac{x}{a}}$;

$$\text{or}=\left\{x\left(\frac{a}{x}+1\right)\right\}^{\frac{1}{n}}=x^{\frac{1}{n}}\left(\frac{a}{x}+1\right)^{\frac{1}{n}}=x^{\frac{1}{n}}\sqrt[n]{\frac{a}{x}+1}.$$

Thus *any quantity may be made the co-efficient of a surd, if every part under the radical be divided by this quantity raised to the power whose root the radical expresses.*

Corollary. Hence *a given surd may be reduced to its simplest form by first resolving into factors the quantity under the radical and then removing from under the sign those whose indices are equal to, or are multiples of, the denominator of the surd-index*

Thus $\sqrt{252}=\sqrt{9\times 4\times 7}=\sqrt{3^2\times 2^2\times 7}=3\times 2\sqrt{7}=6\sqrt{7}$.

Examples

Simplify

1 $\sqrt{72}$ 2 $\sqrt[3]{1080}$ 3 $\sqrt{10368}$ 4 $\sqrt[5]{3a^{10}b^8c^4}$.

5 $\sqrt{2a^4x+4a^3x^2+2a^2x^3}$

213 Addition and Subtraction of Surds It is easily seen that

(i) $3\sqrt{2}+5\sqrt{2}=(3+5)\sqrt{2}=8\sqrt{2}$

(ii) $8\sqrt{3}-3\sqrt{3}=(8-3)\sqrt{3}=5\sqrt{3}$

(iii) $a\sqrt{x}\pm\sqrt{x}=(a\pm 1)\sqrt{x}=(a\pm 1)\sqrt{x}$

(iv) $5a\sqrt{x}-3a\sqrt{x}+a\sqrt{x}=(5a-3a+a)\sqrt{x}=3a\sqrt{x}$

Thus *the sum of similar surds is found by prefixing the algebraic sum of the coefficients to the irrational part*

Definition SIMILAR or LIKE surds are those which have the *same irrational part* Thus $\sqrt{x}$ and $3\sqrt{x}$ are similar surds, so are $a\sqrt[3]{x}$ and $2\sqrt[3]{m^3x}$, for $2\sqrt[3]{m^3x}=2m\sqrt[3]{x}$, &c

214 Multiplication and Division of Surds It is evident that

(i) $\sqrt{5}\times\sqrt{3}=5^{\frac{1}{2}}\times 3^{\frac{1}{2}}=(5\times 3)^{\frac{1}{2}}$ [§ 200] $=(15)^{\frac{1}{2}}=\sqrt{15}$

(ii) $\sqrt[m]{a}\times\sqrt[m]{c}=(a)^{\frac{1}{m}}(c)^{\frac{1}{m}}=(ac)^{\frac{1}{m}}=\sqrt[m]{ac}$

Thus *if the surds have the same index, their product is found by taking the product of the quantities under the radical and retaining the common index.*

(iii) $x^{\frac{3}{m}}\times y^{\frac{2}{m}}=(x^3)^{\frac{1}{m}}(y^2)^{\frac{1}{m}}=(x^3y^2)^{\frac{1}{m}}=\sqrt[m]{x^3y^2}$.

Thus *if the indices of the surds have the* SAME *denominator, their product is found by taking the product of the powers expressed by the numerators and affixing the radical expressed by the denominator*

(iv) $\sqrt[m]{x}\sqrt[n]{y}=x^{\frac{1}{m}}y^{\frac{1}{n}}=x^{\frac{n}{mn}}y^{\frac{m}{nm}}=(x^n)^{\frac{1}{mn}}(y^m)^{\frac{1}{nm}}=(x^ny^m)^{\frac{1}{mn}}$

(v) $(a+x)^{\frac{1}{2}}(a-x)^{\frac{1}{3}}=(a+x)^{\frac{3}{6}}(a-x)^{\frac{2}{6}}=\{(a+x)^3\}^{\frac{1}{6}}\{(a-x)^2\}^{\frac{1}{6}}$

$=\{(a+x)^3(a-x)^2\}^{\frac{1}{6}}=\sqrt[6]{(a+x)^3(a-x)^2}$

Thus *if the indices have* DIFFERENT *denominators, reduce them to a common denominator and proceed as in Ex* (iii)

REMARK. Since to divide a by b is the same as to multiply a by $\frac{1}{b}$ [§ 72], i e, by b^{-1} [§ 207, iv], *the rules of this Article will likewise apply to examples of division of one surd by another.*

Examples.

Simplify

1. $\sqrt{15}\times\sqrt{3}$ 2. $(3+2\sqrt{5})(2-\sqrt{5})$. 3. $\sqrt{ab^2}\times\sqrt[4]{a^2b^3}$.

4. $\sqrt[4]{\frac{ab^3}{x^3}}\times\sqrt[6]{\frac{cx^5}{a^3b}}$. 5. $\sqrt{x\sqrt[3]{x^2}}\times\sqrt{\sqrt[3]{x}}$.

6. $\sqrt{ab+bc}\times\sqrt{a^2+ac}$. 7. $\sqrt{ax+x^2}\times\sqrt{ab-bx}$.

8. $\sqrt[3]{54a^7}\div\sqrt[3]{2a}$. 9. $\sqrt{a^2x}\div\sqrt{ax^5}$.

10. $\left(\frac{x}{a^2}-y\right)\div\left(\frac{1}{a}\sqrt{x}+\sqrt{y}\right)$. 11. $\sqrt{\frac{a^3}{b^5}}\div\sqrt{\frac{a}{b^3}}$.

12. $\sqrt{\frac{54a^3}{x}}\div\sqrt[4]{\frac{8x^5}{9a}}$. 13. $(x+y)\div\frac{1}{5}\sqrt{x^2-y^2}$.

215. Definitions A QUADRATIC SURD is that of which the index is $\frac{1}{2}$; as $\sqrt{a}$, $\sqrt{(a+x)}$, $\sqrt{a^3+b^3+c^3}$, &c

A SIMPLE QUADRATIC SURD is that which consists of a *single* term, as $\sqrt{x}$, $10\sqrt{y}$, $5a\sqrt{x^3}$, &c.

A BINOMIAL QUADRATIC SURD is that which consists of *two* terms, one or both of which are simple quadratic surds; as $a+\sqrt{b}$, $\sqrt{x}-3\sqrt{y}$, &c

Two binomial quadratic surds are said to be CONJUGATE or COMPLEMENTARY when they have the same terms connected respectively by the sign + and −, as $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$.

216. Rationalisation of Surds To rationalise a surd is to find a quantity by which the surd must be multiplied to give a *rational product.*

Thus $\sqrt{a}$ is rationalised when it is multiplied by $\sqrt{a}$, for $\sqrt{a}\sqrt{a}=a$, $\sqrt[3]{a}$ is rationalised by multiplying it by $\sqrt[3]{a^2}$, for $\sqrt[3]{a}\sqrt[3]{a^2}=\sqrt[3]{a^3}=a$; and so on.

Again $\sqrt{x}+\sqrt{y}$ is rationalised when it is multiplied by $\sqrt{x}-\sqrt{y}$, for $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})=x-y$ Hence *a binomial quadratic surd is rationalised by multiplying by its conjugate.*

To rationalise $a^{\frac{1}{3}}+b^{\frac{1}{3}}$, the multiplier is evidently $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$, for $(a^{\frac{1}{3}}+b^{\frac{1}{3}})(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})=a+b$.

To rationalise $\sqrt{a}+\sqrt{b}+\sqrt{c}$

Multiply first by $\sqrt{a}+\sqrt{b}-\sqrt{c}$; the product is $(\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2$ $=a+b-c+2\sqrt{(ab)}$, next multiply by $a+b-c-2\sqrt{(ab)}$; the product is $(a+b-c)^2-4(\sqrt{ab})^2=(a+b-c)^2-4ab$; thus the required multiplier is $(\sqrt{a}+\sqrt{b}-\sqrt{c})(a+b-c-2\sqrt{ab})$.

We proceed in this way in the case of any other surd.

217 Fractions with surd denominators The principal use of the last article is to find without much labour the values of fractions having *irrational denominators.* This will be seen from the following examples

Examples.

Ex. 1. Find the value of $\frac{2}{\sqrt{3}}$ to 5 decimal places

$$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}}\ [\S\ 168]=\frac{2\sqrt{3}}{3}=1.15470$$

REMARK If, instead of rationalising the denominator, we had found the square root of 3 first, and then divided 2 by it, the operation would have been long and tedious

Ex. 2. Find the value of $\frac{\sqrt{3}}{\sqrt{2}+1}$ to 5 places of decimals

Multiply the numerator and denominator by the conjugate surd $\sqrt{2}-1$; thus

$$\frac{\sqrt{3}}{\sqrt{2}+1}=\frac{\sqrt{3}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}=\frac{\sqrt{6}-\sqrt{3}}{2-1}=\sqrt{6}-\sqrt{3}=.71743.$$

Ex 3 Simplify $\frac{2+\sqrt{3}}{2-\sqrt{3}}+\frac{2-\sqrt{3}}{2+\sqrt{3}}$.

$$\text{Given surd}=\frac{(2+\sqrt{3})^2}{(2-\sqrt{3})(2+\sqrt{3})}+\frac{(2-\sqrt{3})^2}{(2+\sqrt{3})(2-\sqrt{3})}$$

$$=\frac{(2+\sqrt{3})^2}{4-3}+\frac{(2-\sqrt{3})^2}{4-3}$$

$$=4+4\sqrt{3}+3+4-4\sqrt{3}+3=14$$

Ex. 4 Simplify $\frac{2}{\sqrt{(xy)}-y}-\frac{1}{x+\sqrt{(xy)}}-\frac{\sqrt{x}+\sqrt{y}}{x\sqrt{y}-y\sqrt{x}}$.

$$\text{Given expn}=\frac{2}{\sqrt{y}(\sqrt{x}-\sqrt{y})}-\frac{1}{\sqrt{x}(\sqrt{x}+\sqrt{y})}-\frac{\sqrt{x}+\sqrt{y}}{\sqrt{(xy)}(\sqrt{x}-\sqrt{y})}$$

$$=\frac{2\sqrt{x}(\sqrt{x}+\sqrt{y})-\sqrt{y}(\sqrt{x}-\sqrt{y})-(\sqrt{x}+\sqrt{y})^2}{\sqrt{(xy)}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}$$

$$=\frac{2x+2\sqrt{(xy)}-\sqrt{(xy)}+y-x-y-2\sqrt{(xy)}}{\sqrt{(xy)}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}$$

$$=\frac{x-\sqrt{(xy)}}{\text{Dnr}}=\frac{\sqrt{x}(\sqrt{x}-\sqrt{y})}{\sqrt{x}\sqrt{y}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}=\frac{1}{\sqrt{(xy)}+y}.$$

Ex 5 Prove that $\sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}}=\sqrt{2a+2\sqrt{a^2-b}}$.

Let $x=\sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}}$;

$\therefore \quad x^2=a+\sqrt{b}+a-\sqrt{b}+2\sqrt{a^2-b}=2a+2\sqrt{a^2-b}$,

whence $x=\sqrt{2a+2\sqrt{a^2-b}}$; &c

Rationalise the denominator of

6. $\dfrac{2\sqrt{3}}{\sqrt{3}+1}$ 7 $\dfrac{\sqrt{7}+\sqrt{3}}{5-\sqrt{21}}$. 8. $\dfrac{1+2\sqrt{3}}{3-\sqrt{2}}$. 9 $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$

10 Rationalise the numerator of $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

Simplify

11 $\dfrac{1}{2-\sqrt{3}}+\dfrac{1}{3-2\sqrt{2}}$.

12 $\dfrac{1}{a-\sqrt{b}}-\dfrac{1}{a+\sqrt{b}}$

13 $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}+\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

14 $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}\times(8+2\sqrt{15})$.

15 $\dfrac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(1+\sqrt{3})}-\dfrac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$.

16 $\dfrac{\sqrt{18}}{\sqrt{3}+\sqrt{2}}+\dfrac{\sqrt{12}}{\sqrt{3}-\sqrt{2}}$.

17. $\sqrt{\dfrac{a+x}{a-x}}+\sqrt{\dfrac{a-x}{a+x}}$

18. $\dfrac{1}{1-\sqrt{1-x^2}}\pm\dfrac{1}{1+\sqrt{1-x^2}}$.

19. $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}-\dfrac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}$.

20 $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

21 $\sqrt{\left\{\dfrac{1-x-\sqrt{2x+x^2}}{1-x+\sqrt{2x+x^2}}\right\}}+\sqrt{\left\{\dfrac{1-x+\sqrt{2x+x^2}}{1-x-\sqrt{2x+x^2}}\right\}}$.

22 Simplify $\dfrac{a}{a-\sqrt{a^2-x^2}}-\dfrac{a}{a+\sqrt{a^2-x^2}}$; and find its value when $x=\dfrac{\sqrt{3}}{2}a$.

Find the value of

23 $\frac{1+\sqrt{3}}{1-\sqrt{3}}+\frac{2+\sqrt{3}}{2-\sqrt{3}}-\frac{2\sqrt{3}+1}{2+\sqrt{3}}$ 24 $\frac{1}{2(a+\sqrt{x})}+\frac{1}{2(a-\sqrt{x})}+\frac{a}{a^2+x}$

25. x^2+ax+b, when $x=\sqrt{\left(\frac{a^2}{4}-b\right)}-\frac{a}{2}$

26 $\frac{x}{y}-\sqrt{\frac{1+x}{1-y}}$, when $x=\frac{1}{2}$, $y=\frac{1}{3}$

27 $\sqrt{\frac{3}{4}-x}+\sqrt{2x}-\frac{3}{2}\sqrt{1-4x}$, when $x=\frac{1}{12}$

28 If $y=\frac{1}{x+\sqrt{x^2-1}}$, shew that $2x=y+y^{-1}$

29 Find the value of $x^3+y^3+z^3-xyz$, when $x=\sqrt{q}-\sqrt{r}$ $y=\sqrt{r}-\sqrt{p}$, $z=\sqrt{p}-\sqrt{q}$

218 Extraction of the Square Root In the present Article we shall shew how to find the square root of binomial quadratic surds by *inspection*.

Examples

Ex. 1. Extract the square root of $5+2\sqrt{6}$

Here $2\sqrt{6}=2\sqrt{3\times 2}=2\sqrt{3}\sqrt{2}$ (1),

and $5=3+2=(\sqrt{3})^2+(\sqrt{2})^2$ (2)

Now (1) is *twice the product* of two numbers, the *sum of whose squares* is (2), therefore by § 99, we have

$$5+2\sqrt{6}=(\sqrt{3})^2+(\sqrt{2})^2+2\sqrt{3}\sqrt{2}=(\sqrt{3}+\sqrt{2})^2,$$

$\therefore$ square root required $=\sqrt{3}+\sqrt{2}$

Ex. 2 Extract the square root of $4-2\sqrt{3}$

Here $2\sqrt{3}=2\times\sqrt{3}\times 1$, and $4=3+1$,

$\therefore$ $4-2\sqrt{3}=(\sqrt{3})^2+1^2-2\sqrt{3}\times 1=(\sqrt{3}-1)^2$,

whence required square root $=\sqrt{3}-1$

Ex. 3 Extract the square root of $2a\pm 2\sqrt{a^2-x^2}$

Given expression $=(a+x)+(a-x)\pm 2\sqrt{(a+x)(a-x)}$

$=\{\sqrt{(a+x)}\pm\sqrt{(a-x)}\}^2$; &c

Remark Hence it appears that the sign before the radical determines whether the required root will be the *sum* or *difference* of the two quantities obtained by factorising the quantity under the radical

Find the square root of

4. $11+6\sqrt{2}$. 5. $14+6\sqrt{5}$. 6. $8+2\sqrt{15}$.

7. $19-8\sqrt{3}$ 8. $9-4\sqrt{5}$ 9. $30+12\sqrt{6}$

10. $9-4\sqrt{2}$. 11. $4\frac{1}{3}-\frac{4}{3}\sqrt{3}$. 12. $2+\sqrt{3}$

13. $2(a+b+\sqrt{a^2+2ab})$ 14. $x+y-2\sqrt{(x+1)(y-1)}$.

15. $1+\sqrt{1-x^2}$. 16. $x+y+z-2\sqrt{xy+xz}$.

17. $ax-2a\sqrt{ax-a^2}$. 18. $a^2+2x\sqrt{a^2-x^2}$.

19. Find the value of $\dfrac{\sqrt{1+x}-1}{\sqrt{1-x}+1}+\dfrac{\sqrt{1-x}+1}{\sqrt{1+x}-1}$, when $x=\dfrac{\sqrt{3}}{2}$.

20. Express $\alpha\beta+\sqrt{(\alpha^2-1)(\beta^2-1)}$ in terms of x and y, when $2\alpha=x+x^{-1}$, $2\beta=y+y^{-1}$. [See App.]

219. Examination upon Chapters XV, XVI and XVII

1. What are *Involution* and *Evolution*?

2. State the Rule for finding the *squares* of polynomials, and illustrate it by expanding $(a-b+c-d)^2$.

3. Define a *root* of a quantity. In what two senses is this word used in Algebra?

4. Define the n^{th} root of an expression.

5. Give the reason for using $a^{\frac{1}{m}}$ to denote the m^{th} root of a.

6. Explain the term *Index* when it is (1) a positive integer, (2) a negative integer, (3) a fraction.

7. What is a *rational* and what an *irrational* quantity? Illustrate by examples.

8. What is an Imaginary Quantity? What quantity is $\sqrt{-a^2}$?

9. Why does an even root of a positive quantity admit of a double sign, e.g., $\sqrt{a^2}=\pm a$?

10. Define a *quadratic* surd, *similar* surds and *conjugate* surds.

11. What is to *rationalise* a surd?

12. Prove that $(ab)^n=a^nb^n$; $(a^m)^n=a^{mn}$; $\left(\dfrac{a}{b}\right)^n=\dfrac{a^n}{b^n}$.

13. Shew that $x^0=1$, and find the values of x^{-0}, $\dfrac{1}{x^{-0}}$, $x^{-2}y^0$, $x^{-0}y^{-0}$ and $\dfrac{1-x^0}{1+x^0}$.

14. Express with *fractional* indices

$$\sqrt{x^3}\,,\ \sqrt[3]{x^5}\,,\ (\sqrt[3]{x^2})^4\,,\ \sqrt{(\sqrt[6]{a^3b^2})^3}\,;\ \{\sqrt[4]{(\sqrt[3]{x^4})^3}\}^6.$$

15. Express with *negative* indices

$$\frac{2}{x^3}\,,\ 4a^2x^5\,,\ \frac{3x^3}{yz^4}\,,\ \frac{4z}{x\sqrt{y}}\,,\ \frac{5}{x}\sqrt{\left(\frac{y}{z}\right)^3}.$$

16 Express with *radicals* $5a^{\frac{2}{5}}$, $3x^{\frac{1}{2}}y^{\frac{2}{3}}z^{\frac{4}{3}}$, $\dfrac{a^{\frac{3}{5}}x^{\frac{1}{5}}}{3y^{\frac{4}{5}}}$

17 Express with *positive* indices $3ab^{-1}$; $a^{-1}x^2y^{-3}$, $\dfrac{x^{-\frac{2}{5}}c}{b^{-\frac{1}{3}}c^{-3}}$

CHAPTER XVIII.

Simple Equations in one Variable

Section II

220 Different Forms of Expressions We have defined an *Expression* as a collection of symbols connected by the signs of operation [§ 23]

An Integral Expression is one in which no letter appears in the denominator of any term, or in other words, in which no letter has a negative index. Thus $x^2+\frac{1}{2}ax+\frac{2}{3}b$ is an integral expression, but not so is $x^2+\dfrac{1}{x}+a$ or $x^2+ax^{-1}+c$

An Integral Expression in a particular letter is one in which that letter occurs in a form having only positive integral index, *i e*, when it does not occur in the denominator of any term; thus $ax^2+\dfrac{x}{b}+\dfrac{c}{d}$ is an integral expression in x, but not in b or d

An integral expression is said to be Complete or Natural when it contains the symbol of reference *in all its powers from the highest to* 0 Thus

$$px+q,$$
$$fx^2+gx+h,$$
$$kx^3+lx^2+mx+n,$$
$$ax^4+bx^3+cx^2+dx+e,$$

are complete expressions in x. The last terms, q, h, n and e, *which do not contain the symbol of reference, i e*, in which the power of

the symbol of reference is *zero* are called ABSOLUTE or CONSTANT TERMS, or simply CONSTANTS.

A RATIONAL EXPRESSION IN A PARTICULAR LETTER is one in which that letter occurs in a rational form, *i e*, in a form free from radicals or fractional indices thus ax^3+bx^2+cx+d is a *rational* expression in x, but not so is $ax+b\sqrt{x}+c$, where the second term is *irrational*.

REMARK It is to be noted that these definitions *refer only to the symbol of reference* as x in the above examples, and therefore any one or more of the *coefficients* a, b, c, &c , may be *surd* or *fractional*; thus $px^3+\frac{x^2}{q}+\sqrt{r}x+s^{-1}$ is a rational and integral expression in x, though some of the coefficients are surds and fractions

221. Definitions When two expressions in the same letter are given equal to each other for some particular value or values and no more, of that letter, the equality is termed an EQUATION. [See § 137]

It is clear from the last article that we cannot ascertain the degree of an equation unless it is in a *rational and integral form* For though the equations

$$\frac{1}{x}+c=0,\ 2\sqrt{x}+3=0,\ \frac{1}{x}+x=0,\ x+2\sqrt{x}+1=0,$$

appear to be of the first degree, they are in reality not all of them of that degree Hence to ascertain the degree of an equation, say in x, we must see whether it is in a rational and integral form as far as x is concerned When the proposed equations are not in rational and integral forms as in the above examples, we must reduce them to that form and then ascertain their degrees

To reduce an equation to an integral form, we multiply by the denominator, thus $x+\frac{2}{x}=3$ becomes $x^2+2=3x$, or $x^2-3x+2=0$, when put in an integral form [§ 224, *post*]

To reduce an equation to a rational form, we transpose and raise to the power expressed by the radical thus $ax+b\sqrt{x}+c=0$ becomes $(ax+c)^2=b^2x$, or $a^2x^2+(2ac-b^2)x+c^2=0$, when put in a rational form [§ 225, *post*]

Hence we have the following

Definition If after reducing an equation (if not already so reduced) to a rational and integral form with respect to the variables, we see that the term or terms of the highest dimensions in the variables are of *one, two, three, four* or n dimensions, it is said to be respectively of the *first, second, third, fourth* or n^{th} degree

An equation of the *first* degree is commonly called a SIMPLE or LINEAR EQUATION, and an equation of the *second* degree is called a QUADRATIC EQUATION Thus

$$ax+b=0,\ a\sqrt{x}+b=0,\ \frac{a}{x}+b=0,$$

are *Simple* or *Linear* equations, for when reduced to *rational and integral forms,* the last two become

$$a^2x-b^2=0, \text{ and } bx+a=0,$$

similarly

$$ax^2+bx+c=0,\ ax+\frac{b}{x}+c=0,\ ax+\sqrt{x}+b=0,$$

are *Quadratic* equations, for the last two may be reduced to the forms

$$ax^2+cx+b=0, \text{ and } a^2x^2+(2ab-1)x+b^2=0,$$

and so on [§ 141]

Examples

Determine the degree of the equations

1 $x-\frac{1}{x}=a$ 2 $x^{-1}-x^2=c.$ 3 $3x+\sqrt{x}=a\sqrt{x}$ 4 $\frac{1}{x}-\frac{1}{y}=\frac{1}{a}$

***222 General Form** *A simple equation in one variable can always be reduced to the general form* $ax+b=0$

For when the equation has been reduced to a rational and integral form [§ 220], the terms involving x may be collected together, when we may bracket the coefficients of x, thus finding the a, and the constant terms being bracketed together, we find the b Thus $\frac{a}{x}+b-\frac{c}{x}=d$ become $a+bx-c=dx$ when we multiply by x; then by transposition, it assumes the form $(b-d)x+(a-c)=0$, hence here $b-d$ is the a, and $a-c$ is the b.

Examples

Reduce the following equations to the general form

1. $5x-\frac{3}{2}x+3+\frac{x}{2}=0.$ 2 $ax+b-cx=d$ 3 $\frac{1}{x}+\frac{3}{2}+\frac{8}{x}=1.$

4 $x=\frac{ax-b^2}{c}.$ 5 $\frac{a}{x}+\frac{b}{c}-\frac{d}{e}=0$ 6. $1-\frac{x-2}{5}=\frac{x+2}{4}.$

*223 **Theorem.** *A simple equation has only one root and no more.*

If possible, let the equation $ax+b=0$ have more than one root, viz., α and β. Now since α and β are its roots, they will severally satisfy it [§ 140]. therefore

$$a\alpha+b=0 \qquad (1),$$
$$a\beta+b=0 \qquad (2)$$

Subtract (2) from (1), thus $a(\alpha-\beta)=0$. Now either $a=0$, or $\alpha-\beta=0$. If $a=0$, the given equation reduces to $b=0$, which cannot be satisfied by any *finite value* of x; therefore (since a root is commonly supposed to be a finite value of the variable) a is not $=0$. Therefore $\alpha-\beta=0$ or $\alpha=\beta$, *i. e.*, α and β are not two *different* quantities but *one* and the *same* quantity.

224 **Solution of Simple Equations.** The rule is to *reduce the given equation to the general form, transpose the constant; and divide by the coefficient of* x [§ 146]. It is, however, often tedious to strictly follow the rule, so we proceed as in the Examples below.

Examples

1. **Solve** $5x-\frac{3}{2}x+3+\frac{x}{2}=0$ [Ex. 1, § 222]

Multiply by 2, thus $\quad 10x-3x+6+x=0,$

$$\therefore\ 8x+6=0,$$

transpose, thus $\quad 8x=-6,$

divide by 8, thus $\quad x=-\frac{6}{8}=-\frac{3}{4}.$

2. **Solve** $\dfrac{5x}{2}+\dfrac{2x}{3}-17=\dfrac{3x}{5}+60$

Instead of multiplying separately by 2, 3, 5, multiply by 30, which is the L. C. D.; thus

$$75x+20x-510=18x+1800,$$

transpose, thus $\quad 75x+20x-18x=1800+510,$

$$77x=2310,$$
$$\therefore\ x=\tfrac{2310}{77}=30.$$

Solve the following equations

3. $\dfrac{x}{12}-\dfrac{x}{6}=2-\dfrac{x}{4}.$ 4. $\dfrac{5x}{9}-\dfrac{2x-1}{3}=\dfrac{4}{15}.$

5. $\dfrac{3x}{4}+\dfrac{7x}{15}+\dfrac{11x}{6}=366.$ 6. $\dfrac{x-2}{4}-\dfrac{3-x}{6}-3\frac{1}{6}=0.$

Solve the following equations

7. $\frac{3-2x}{4}=1-\frac{4x-5}{6}$

8. $\frac{x+6}{4}-\frac{16-3x}{12}=\frac{25}{6}$

9. $\frac{12-3x}{4}-1=\frac{3x-11}{3}$.

10. $1-\frac{x-2}{5}=\frac{x+2}{4}$.

11. $7x+15=23-\frac{1-9x}{2}$.

12. $\frac{x}{3}+2x\left(\frac{3}{2}-\frac{1}{6}\right)-1=2\frac{1}{2}$

13. $3-\frac{x-2}{3}=\frac{4x+7}{5}$.

14. $\frac{x}{4}-\frac{x-4}{6}=\frac{4}{3}+\frac{24-x}{12}$.

15. $\frac{x+1}{2}+\frac{x+2}{3}=14+\frac{5-x}{4}$.

16. $x+\frac{11-x}{3}=\frac{19-x}{2}$

17. $\frac{x}{2}-\frac{5x+4}{3}=\frac{4x-9}{3}$

18. $x-\frac{x-7}{3}+\frac{3x-1}{5}=\frac{2x}{7}+9$.

19. $\frac{x+6}{24}+\frac{16-3x}{12}=3x-4\frac{5}{6}$

20. $\frac{3x+5}{8}-\frac{21+x}{3}=39-5x$

21. $\frac{3x+4}{5}-\frac{7x-3}{2}+\frac{16-x}{4}=0$

22. $\frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36$.

23. $\frac{x-7}{11}-\frac{3x-5}{7}+\frac{125}{77}=2x-17$.

24. $x-\frac{x-2}{3}=5\frac{3}{4}-\frac{10+x}{5}+\frac{x}{4}$

25. $\frac{x-2}{5}-\frac{10-x}{3}=\frac{2x-3}{6}-1\frac{13}{30}$.

26. $\frac{x-1}{7}+\frac{23-x}{5}=7-\frac{4+x}{4}$.

27. $\frac{x}{8}-\frac{x-1}{2\frac{1}{2}}=\frac{3x-4}{15}+\frac{x}{12}$

28. $\frac{5x-1}{2}-\frac{7x-2}{10}=6\frac{3}{5}-\frac{x}{2}$

29. $\frac{3x+7}{14}-\frac{2x-7}{21}+2\frac{3}{4}=\frac{x-4}{4}$.

30. $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$.

31. $\frac{1}{2}(x-\frac{1}{3}a)-\frac{1}{3}(x-\frac{1}{4}a)+\frac{1}{4}(x-\frac{1}{5}a)=0$

32. $\frac{4x-34}{17}-\frac{268-5x}{3}=\frac{69-x}{2}$.

33. $\frac{4x-21}{7}+7\frac{5}{6}+\frac{7x-28}{3}=x+3\frac{3}{4}-\frac{9-7x}{8}+\frac{1}{12}$.

Solve the following equations

34. $\frac{11x-3}{25}+\frac{19x+3}{7}-\frac{5x-25\frac{1}{5}}{4}=28\frac{1}{7}-\frac{17x+4}{21}$.

35. $\frac{x-1\frac{25}{28}}{2}-\frac{2-6x}{13}=x-\frac{5x-\frac{10-3x}{4}}{39}$

36. $\frac{x-a}{3}-\frac{2x-3b}{5}-\frac{a-x}{2}=10a+11b$.

The following are examples of literal equations.

37. Solve $\frac{ax}{b}-\frac{cx}{d}=1$.

Here the L. C. D. is bd, multiply therefore by bd; thus

$$adx-bcx=bd,$$

or

$$(ad-bc)x=bd,$$

$$\therefore x=\frac{bd}{ad-bc}$$

38. Solve $\frac{a}{x}-b=\frac{c}{x}+d$

Multiply by x; thus $\qquad a-bx=c+dx,$

or

$$(b+d)x=a-c.$$

$$\therefore x=\frac{a-c}{b+d}.$$

39. Solve $\frac{a-x}{a}+\frac{2a-x}{2a}+\frac{x-3a}{3a}=0$

Multiply by $6a$; thus

$$6(a-x)+3(2a-x)+2(x-3a)=0,$$

$$\therefore 7x=6a, \text{ or } x=\frac{6a}{7}.$$

Solve the following equations

40. $x=\frac{ax-b^2}{c}$.

41. $\frac{a}{x}+\frac{b}{c}-\frac{d}{c}=0$

42. $\frac{a}{bx}+\frac{b}{ax}=a^2+b^2$.

43. $\frac{a(d^2+x^2)}{dx}=ac+\frac{ax}{d}$

44. $\frac{x-a}{b}+\frac{x-b}{a}=0$.

45. $\frac{a+bx}{a+b}=\frac{c-dx}{c-d}$.

46. $\frac{ax}{b}+g=qx+h$

47. $a+\frac{bx}{a}=\frac{ax-b^2}{b}$.

Solve the following equations

48. $\frac{x}{bc}-\frac{3x-1}{ca}=\frac{x+3a}{ab}$

49. $\frac{a}{b}\left(1-\frac{a}{x}\right)+\frac{b}{a}\left(1-\frac{b}{x}\right)=1$

50. $\frac{ax-b}{m}-\frac{bx-c}{n}+\frac{dx}{2m}=\frac{3}{p}$.

51. $\frac{3bx}{2a^2}-\frac{x-b}{a+b}-\frac{bx-a^2}{a^2-b^2}+\frac{x}{4a}=0$

Note 1 In solving equations, much labour is sometimes saved by a suitable transposition of terms, as for instance by transposing all the terms having *monomial* denominators to one side

52. Solve $\frac{4x-5}{3}+\frac{10x-3}{20x+17}=\frac{12x-11}{9}$.

By transp, $$\frac{10x-3}{20x+17}=\frac{12x-11}{9}-\frac{4x-5}{3}$$

$$=\frac{12x-11}{9}-\frac{12x-15}{9}=\frac{4}{9},$$

whence $9(10x-3)=4(20x+17)$,

or $10x=95$, $\therefore x=9\frac{1}{2}$

Solve the following equations

53. $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$

54. $\frac{2x}{5}+\frac{3x+5}{5x-25}=\frac{6x+13}{15}$

55. $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{4x+6}{7}$

56. $\frac{10x+17}{18}-\frac{12x+2}{11x-8}=\frac{5x-4}{9}$.

57. $\frac{1}{x-1}-\frac{2}{x+7}=\frac{1}{7(x-1)}$

58. $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$

59. $\frac{2x+8\frac{1}{2}}{9}+\frac{x}{3}+\frac{x+16}{36}=\frac{7x}{12}-\frac{2-13x}{17x-32}$.

60. $\frac{6x-7\frac{1}{4}}{13-2x}+2x+\frac{1+16x}{24}=4\frac{5}{12}-\frac{12\frac{5}{8}-8x}{3}$

Note 2 In certain cases, solution is very easily effected by the actual division of numerator by denominator

61. Solve $\frac{2x^2+3x+1}{x+2}=\frac{2x^2-7x+5}{x-3}$.

By actual division,

$$2x-1+\frac{3}{x+2}=2x-1+\frac{2}{x-3} \text{ [§ 96]},$$

$$\therefore \frac{3}{x+2}=\frac{2}{x-3}, \text{ whence } x=13$$

62 Solve $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}$

By division, $1+\dfrac{1}{x-2}-1-\dfrac{1}{x-3}=1+\dfrac{1}{x-6}-1-\dfrac{1}{x-7}$ [§ 96],

or
$$\frac{1}{x-2}-\frac{1}{x-3}=\frac{1}{x-6}-\frac{1}{x-7},$$

$$\frac{x-3-x+2}{(x-2)(x-3)}=\frac{x-7-x+6}{(x-6)(x-7)},$$

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)},$$

$$(x-2)(x-3)=(x-6)(x-7),$$

whence
$$x=4\tfrac{1}{2}$$

Solve the following equations

63. $\dfrac{15x+8}{3x-7}=\dfrac{25x-62}{5x-21}$.

64. $\dfrac{mx-a}{nx-b}=\dfrac{mx-c}{nx-d}$.

65. $\dfrac{a^2x^3+b^2}{ax-b}=\dfrac{a^2x^2+2abx-2b^2}{ax+b}$.

66. $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}$

67. $\dfrac{x^3+2x^2+1}{x^2+2x}=x+\dfrac{3}{x+2}$

68. $\dfrac{x-1}{x-3}+\dfrac{x-9}{x-11}=\dfrac{x-3}{x-5}+\dfrac{x-7}{x-9}$.

69. $\dfrac{ax-2}{ax-3}-\dfrac{ax-3}{ax-4}=\dfrac{ax-5}{ax-6}-\dfrac{ax-6}{ax-7}$

70. $\dfrac{7x-26}{x-4}-\dfrac{4x-21}{x-6}=\dfrac{9x-68}{x-8}-\dfrac{6x-55}{x-10}$.

71. $\dfrac{x+2}{x-2}+\dfrac{x+5}{x-5}=\dfrac{x+3}{x-3}+\dfrac{x+4}{x-4}$

72. $\dfrac{4x-17}{x-4}+\dfrac{10x-13}{2x-3}=\dfrac{8x-30}{2x-7}+\dfrac{5x-4}{x-1}$.

73. $\dfrac{x^2+5x+4}{x+3}+\dfrac{x^2+5x-3}{x+4}=\dfrac{2x^2+7x-3}{x+2}$.

74. $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+20}{x+4}=\dfrac{x^2+4x+6}{x+2}+\dfrac{x^2+6x+12}{x+3}$.

Note 3 Fractional equations are sometimes easily solved by splitting a term into partial fractions

75 Solve $\dfrac{a}{x-3a}-\dfrac{b}{x+3b}=\dfrac{2c}{x-3c}$

Since $$\frac{2c}{x-3c}=\frac{c}{x-3c}+\frac{c}{x-3c},$$

we have $$\frac{a}{x-3a}-\frac{c}{x-3c}=\frac{c}{x-3c}+\frac{b}{x+3b},$$

or $$\frac{(a-c)x}{(x-3a)(x-3c)}=\frac{(b+c)x}{(x-3c)(x+3b)},$$

dividing by $\dfrac{x}{x-3c}$, $\quad \dfrac{a-c}{x-3a}=\dfrac{b+c}{x+3b},$

or $$(a-c)x+3b(a-c)=(b+c)x-3a(b+c),$$

i. e., $$x(a-b-2c)=3(bc-ca-2ab),$$

$$x=\frac{3(bc-ca-2ab)}{a-b-2c}.$$

Solve the following equations

76. $\dfrac{5}{x+15}+\dfrac{8}{x-12}=\dfrac{3}{x-9}$ 77 $\dfrac{11}{x-33}-\dfrac{13}{x+39}=\dfrac{20}{x-30}$

78 $\dfrac{m}{mx+a}+\dfrac{n}{nx+a}=\dfrac{2p}{px+a}$ 79 $\dfrac{x-1}{x+1}+\dfrac{x+3}{x-3}=2\dfrac{x+2}{x-2}$

80 $\dfrac{a}{x+a}-\dfrac{b}{x+b}=\dfrac{a-b}{x-c}$ 81 $\dfrac{a}{bx^2+abx}=\dfrac{2a}{1+abx}-\dfrac{1}{a+bx}.$

82 $\dfrac{m+n}{x+m+n}+\dfrac{m-n}{x+m-n}=\dfrac{2m}{x+n}$

The following are additional examples

83 Solve $\dfrac{7x+1}{x-1}=\dfrac{35}{9}\,\dfrac{x+4}{x+2}+3\frac{1}{9}$

We may proceed in the usual way, or perhaps thus —

By division, $$7+\frac{8}{x-1}=\frac{35}{9}\cdot\frac{x+4}{x+2}+3\tfrac{1}{9},$$

or $$\frac{8}{x-1}=\frac{35}{9}\cdot\frac{x+4}{x+2}+3\tfrac{1}{9}-7=\frac{35}{9}\,\frac{x+4}{x+2}-\frac{35}{9}=\frac{35}{9}\cdot\frac{2}{x+2},$$

or $$\frac{4}{x-1}=\frac{35}{9(x+2)},$$

$$36(x+2)=35x-35, \text{ or } x=-107$$

84 Solve $\dfrac{m(x+a)}{x+b}+\dfrac{n(x+b)}{(x+a)}=m+n.$

From the given equations, we have

$$\frac{m(x+a)}{x+b}-m+\frac{n(x+b)}{x+a}-n=0,$$

or

$$\frac{m(a-b)}{x+b}+\frac{n(b-a)}{x+a}=0,$$

$\therefore$ dividing both sides by $a-b$, we have

$$\frac{m}{x+b}-\frac{n}{x+a}=0,$$

or

$$\frac{m}{x+b}=\frac{n}{x+a}, \text{ whence } x=\frac{nb-ma}{m-n}$$

85 Solve $\dfrac{x+3}{x+1}-\dfrac{x+4}{x+2}+\dfrac{x-6}{x-4}=\dfrac{x^2-2x-15}{x^2-9},$

$$\frac{x+3}{x+1}-\frac{x+4}{x+2}+\frac{x-6}{x-4}=\frac{(x+3)(x-5)}{x^2-9}=\frac{x-5}{x-3},$$

$$\frac{x+3}{x+1}-\frac{x+4}{x+2}=\frac{x-5}{x-3}-\frac{x-6}{x-4},$$

by division $$1+\frac{2}{x+1}-1-\frac{2}{x+2}=1-\frac{2}{x-3}-1+\frac{2}{x-4},$$

$$\therefore \quad \frac{1}{x+1}-\frac{1}{x+2}=\frac{1}{x-4}-\frac{1}{x-3},$$

or

$$\frac{1}{(x+1)(x+2)}=\frac{1}{(x-4)(x-3)},$$

whence $$(x+1)(x+2)=(x-4)(x-3),$$

or $$x=1.$$

86 Solve $\dfrac{3x-2}{4}+\dfrac{x}{2}-11\tfrac{5}{6}=\dfrac{x-\dfrac{4x-9}{3}}{6}-5$

Multiply by 12, the L C D ; thus

$$9x-6+6x-142=2x-\frac{8x-18}{3}-60,$$

or

$$13x=88-\frac{8x-18}{3},$$

multiply by 3, thus $$39x=264-8x+18,$$

whence $$x=6.$$

87 Solve $\dfrac{a^2x}{bc}-\dfrac{d^2}{a}+bx=\dfrac{ex}{f}-b+(d+b)x$

Multiply by $abcf$, thus

$$a^3fx-bcd^2f+ab^2cfx=abcex-ab^2cf+(d+b)abcfx,$$

$$(a^3f-abce-abcdf)x=bcd^2f-ab^2cf,$$

$$x=\frac{bcf(d^2-ab)}{a^3f-abce-abcdf}$$

Solve the following equations

88 $x-\dfrac{x}{2}+\dfrac{x}{3}-\dfrac{x}{4}=7$

89 $\dfrac{1}{x}+\dfrac{1}{2x}-\dfrac{1}{3x}=\dfrac{7}{3}$

90 $\dfrac{b}{x}+\dfrac{2}{x}-a=\dfrac{3}{x}+2c$

91 $\dfrac{x}{x-1}-\left(1+\dfrac{2}{x}\right)=0$

92 $a+\dfrac{c-x}{x}=b-1+\dfrac{d}{x}.$

93 $\dfrac{3}{x}-\dfrac{2}{x+1}=\dfrac{5}{4}\cdot\dfrac{1}{x+1}$

94 $\dfrac{30+6x}{x+1}+\dfrac{60+8x}{x+3}=14+\dfrac{48}{x+1}$

95 $5x+\dfrac{7x+9}{4x+3}=9+\dfrac{10x^2-18}{2x+3}$

96 $\dfrac{3x+2}{x-3}-\dfrac{3x-2}{x+3}=\dfrac{4x+36}{x^2-9}$

97 $\dfrac{2x}{3}-\dfrac{1-\dfrac{x}{2}}{4x}=\dfrac{x-1}{2}+\dfrac{x}{6}+\dfrac{7}{12}.$

98 $\dfrac{25-\frac{1}{3}x}{x+1}+\dfrac{16x+4\frac{1}{5}}{3x+2}=5+\dfrac{23}{x+1}.$

99 $\dfrac{6-5x}{15}-\dfrac{7-2x^2}{14(x-1)}=\dfrac{1+3x}{21}-\dfrac{2x-2\frac{1}{3}}{6}+\dfrac{1}{105}$

100 $\dfrac{1}{2}\left(\dfrac{2x}{3}+4\right)-\dfrac{7\frac{1}{2}-x}{3}=\dfrac{x}{2}\left(\dfrac{6}{x}-1\right)$

101 $\dfrac{132x+1}{3x+1}+\dfrac{8x+5}{x-1}=52$

102 $\dfrac{4x-17}{9}-\dfrac{3\frac{2}{3}-22x}{33}=x-\dfrac{6}{x}\left(1-\dfrac{x^2}{54}\right)$

103 $\dfrac{\dfrac{3x}{2}-4}{6}-\dfrac{4x-7}{9}+x=\dfrac{8-\dfrac{x+4}{2}}{3}+2$

104 $\dfrac{3x}{2}-\dfrac{81x^2-9}{(3x-1)(x+3)}=3x-\dfrac{3}{2}\,\dfrac{2x^2-1}{x+3}-\dfrac{57-3x}{2}$

105 $\dfrac{x+a}{a}-\dfrac{2x}{x+a}=3-\dfrac{x^3-x^2a}{a^3-ax^2}$

106 $\dfrac{x+6}{x+2}+\dfrac{x+10}{x+3}=\dfrac{2x^2+10x+49}{x^2+5x+6}.$

107 $\dfrac{3+2x}{1+2x}-\dfrac{5+2x}{7+2x}=1-\dfrac{4x^2-2}{7+16x+4x^2}$

Solve the following equations

108. $\dfrac{1}{x-30}+\dfrac{2}{x+15}+\dfrac{3}{x-10}=\dfrac{6}{x-5}$. 109. $\dfrac{1}{ab-ax}+\dfrac{1}{bc-bx}=\dfrac{1}{ac-ax}$.

110. $\frac{5}{6}ab+\frac{4}{5}ac-\frac{2}{3}cx=\frac{3}{4}ac+2ab-6cx$.

111. $\dfrac{ax}{b}+\dfrac{cx}{f}+g=gx+\dfrac{1}{f}(fh-cx)$.

112. $(a+x)(b+x)-a(b+c)=\dfrac{a^2c}{b}+x^2$.

113. $ax-\dfrac{a^2-3bx}{a}-ab^2=bx+\dfrac{6bx-5a^2}{2a}-\dfrac{bx+4a}{4}$.

114. $\dfrac{20x+36}{25}+\dfrac{5x+20}{9x-16}=\dfrac{4x}{5}+\dfrac{86}{25}$. 115. $\dfrac{7x-29}{5x-12}+\dfrac{4x+3}{9}=\dfrac{8x+19}{18}$.

116. $\dfrac{2x-6}{3x-8}=\dfrac{2x-5}{3x-7}$. 117. $\dfrac{mx-(a+b)}{nx-(c+d)}=\dfrac{mx-(a+c)}{nx-(b+d)}$.

118. $\dfrac{(x+a)(x+b)}{x+a+b}=\dfrac{(x+c)(x+d)}{x+c+d}$. 119. $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}=\dfrac{x-a-b}{x-c-d}$.

120. $\dfrac{1}{1+ax}-\dfrac{1}{1-bx}=\dfrac{a+b}{1+abx}$. 121. $\dfrac{cx^m}{a+bx}=\dfrac{dx^m}{e+fx}$.

122. $\dfrac{x+a}{x+b}=\left(\dfrac{2x+a+c}{2x+b+c}\right)^2$. 123. $\left(\dfrac{x-a}{x-b}\right)^3=\dfrac{x-2a+b}{x-2b+a}$.

124. $\dfrac{x}{2}-\dfrac{\dfrac{2x-3}{3}-\dfrac{3x-1}{4}}{\dfrac{x-1}{2}}=\dfrac{3}{2}\,\dfrac{x^2+2}{3x-2}$.

125. $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}$. 126. $\dfrac{1}{x+6a}+\dfrac{2}{x-3a}+\dfrac{3}{x+2a}=\dfrac{6}{x+a}$.

127. $a-ax\left(1-\dfrac{1}{x}\right)=a(a+x)\left(1+\dfrac{1}{x}\right)+a^2\left(1-\dfrac{1}{x}\right)-a$.

The following are examples of equations involving decimals.

128. Solve $3\cdot5x-4\cdot7=6\cdot5-2\cdot1x$.

Transpose, thus $3\cdot5x+2\cdot1x=4\cdot7+6\cdot5$,

$5\cdot6x=11\cdot2$, $11\cdot2$

129 Solve $1\ 2x-\dfrac{18x-{\cdot}05}{5}=4x+9\ 178$

Multiply by 5, $\quad 6x-18x+\ 05=2x+45\ 89,$

or $\quad 4x-\ 18x=45\ 89-\ 05,$

or $\quad 3\ 82x=45\ 84,$

$$x=12$$

Solve the following equations

130 $\quad 4x-\ 35=\ 34x+\ 01$ $\qquad$ 131 $\quad 1\ 32x+\ 02x=1\ 17-x$

132 $\quad 15x+\ 2-\ 875x=\ 0625x-1\ 375$

133 $\quad \dfrac{5\ 2x}{13}-\dfrac{1-\ 2x}{5}\left(\dfrac{3}{5}-\ 1\right)=\ 1x+\ 028-\dfrac{5x-2}{4}.$

134. $\quad {\cdot}5\left(x-\dfrac{5\ 1}{2\ 6}\right)+\dfrac{02}{13}(3x-1)=x-\dfrac{01}{39}\left(5x-\dfrac{1-\ 3x}{4}\right)$

225 Irrational Equations An IRRATIONAL EQUATION is that in which one or more terms are surds involving the variable.

The principle to be followed in solving such equations, is to *reduce them to a rational form* This is done by leaving *one* of the surd terms on one side of the given equation and transposing *all the rest* to the other side, and then raising both sides to the power whose root the surd expresses, this process being repeated till we finally get rid of all the radicals in the equation

Properly speaking most of these equations are not *simple equations*, for when reduced to a *rational form*, they will often be seen to contain powers of the variable *higher than the first* We have here given a few examples of irrational equations, to shew that some of them can be solved as simple equations

Examples.

1 Solve $\sqrt{x-2}=3$

Here we get rid of the radical by simply squaring,

$$x-2=9, \text{ or } x=11$$

2 Solve $\sqrt[m]{3x+a}=b$

Raise to the m^{th} power; thus

$$3x+a=b^m,$$

$$\therefore\ x=\tfrac{1}{3}(b^m-a)$$

3. **Solve** $\sqrt{a^2-\sqrt{mx+b^4}}=c.$

Square, thus $a^2-\sqrt{mx+b^4}=c^2$

transpose; thus $\sqrt{mx+b^4}=a^2-c^2,$

square again, $mx+b^4=(a^2-c^2)^2,$

$$x=\frac{1}{m}\{(a^2-c^2)^2-b^4\}.$$

4 **Solve** $\sqrt[n]{x+15}=\sqrt[2n]{x^2+75x-135.}$

We have $(x+15)^{\frac{1}{n}}=(x^2+75x-135)^{\frac{1}{2n}};$

raise to $2n^{th}$ power; thus

$$\{(x+15)^{\frac{1}{n}}\}^{2n}=\{(x^2+75x-135)^{\frac{1}{2n}}\}^{2n},$$

$$\therefore (x+15)^2=x^2+75x-135,$$

whence $x=8$

5 **Solve** $\sqrt{7+x}-\sqrt{x}=1$

By transp, $\sqrt{7+x}=1+\sqrt{x}$

squaring, $7+x=1+2\sqrt{x}+x,$

$$\therefore 2\sqrt{x}=6, \text{ or } x=9.$$

6. **Solve** $1+\sqrt{1+x}-\sqrt{1+x+\sqrt{1-x}}=0$

Transpose, thus $\sqrt{1+x+\sqrt{1-x}}=1+\sqrt{1+x},$

square, $1+x+\sqrt{1-x}=1+1+x+2\sqrt{1+x},$

or $\sqrt{1-x}=1+2\sqrt{1+x},$

square again, $1-x=1+4(1+x)+4\sqrt{1+x},$

or $-4\sqrt{1+x}=5x+4,$

$$\therefore 16(1+x)=25x^2+40x+16,$$

$$25x^2=-24x,$$

divide by x, $25x=-24, \text{ or } x=-\frac{24}{25}$

7 Solve $\sqrt{x}+\sqrt{x-a}=\dfrac{a}{\sqrt{x-a}}$.

Multiply by $\sqrt{x-a}$, thus

$$\sqrt{x^2-ax}+x-a=a,$$
$$\sqrt{x^2-ax}=2a-x,$$
$$x^2-ax=4a^2-4ax+x^2,$$
$$3ax=4a^2, \text{ or } x=\frac{4a}{3}$$

Solve the following equations

8 $\sqrt{5x-1}=7$

9. $\sqrt{3x-a}=\sqrt{2x}$.

10 $\sqrt{30+2x}=6-\sqrt{2x}$

11 $\sqrt[3]{3x+7}-1=3$.

12 $\sqrt[m]{3+\sqrt{x}}=2$

13 $\sqrt[m]{8x-1}=\sqrt[m]{3x+a}$

14 $\dfrac{\sqrt[4]{x+12}}{2}+\dfrac{3}{4}=1$

15 $\dfrac{2\sqrt[3]{7x-6}}{3}+\dfrac{3}{4}=\dfrac{25}{12}$.

16 $\sqrt{5}\sqrt{x+2}=\sqrt{5x}+2$

17 $\sqrt{x+9}=1+\sqrt{x}$

18 $\sqrt{x}-\dfrac{\sqrt{x}}{5}=\sqrt{x-9}$

19 $\sqrt{x}-\sqrt{\dfrac{a}{x}}=\sqrt{a+x}$.

20 $a\sqrt[4]{x+m}=\sqrt[3]{x+m}$

21 $\sqrt{x}+\sqrt{3+x}=\dfrac{12}{\sqrt{3+x}}$

22 $\sqrt[m]{a-x}=\sqrt[2m]{3a^2-6ax+x^2}$

23 $\sqrt{x}-\sqrt{x+\sqrt{1-x}}=1$

24 $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{\sqrt{x}+38}{\sqrt{x}+6}$

25 $\sqrt{1+x}+\sqrt{1-x}=4\sqrt[4]{1-x^2}$

26 $\sqrt{4a+x}+\sqrt{x}=2\sqrt{1+x}$

27 $\dfrac{5x-9}{\sqrt{5x}+3}=1+\dfrac{\sqrt{5x}-3}{2}$

28 $\dfrac{4x-9}{2\sqrt{x}-3}=3+\dfrac{2\sqrt{x}+20}{5}$

29 $\dfrac{6x-2}{\sqrt{3x}-1}=4+\dfrac{\sqrt{3x}+1}{2}$

30 $\dfrac{ax-b^2}{\sqrt{ax}+b}=\dfrac{\sqrt{ax}-b}{c}-c$

31 $\sqrt{x}-\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}$.

32 $\sqrt{a+x}+\sqrt{b+x}=2\sqrt{a+b+x}$

33 $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\dfrac{3}{2}\sqrt{\dfrac{x}{x+\sqrt{x}}}$

Solve the following equations

34. $\sqrt{\dfrac{a+x}{x}}+\sqrt{\dfrac{a-x}{x}}=\sqrt{\dfrac{x}{b}}$

35. $\dfrac{1}{a}\sqrt[n]{a+x}+\dfrac{1}{x}\sqrt[n]{a+x}=\dfrac{1}{c}\sqrt[n]{x}$

36. $\dfrac{1+x}{1+\sqrt{1+x}}+\dfrac{1-x}{1+\sqrt{1-x}}=1.$

37 $\dfrac{x-\sqrt{x^2-9}}{x+\sqrt{x^2-9}}=\left(\dfrac{2}{3}x-3\right)^2.$

38 $\dfrac{\sqrt{a}-\sqrt{a-x}}{\sqrt{a}+\sqrt{a-x}}=a$

39 $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{1}{b}.$

40 $\dfrac{\sqrt{a+bx^n}+\sqrt{a-bx^n}}{\sqrt{a+bx^n}-\sqrt{a-bx^n}}=c$

41 $\dfrac{1}{x}+\dfrac{1}{a}=\sqrt{\dfrac{1}{a^2}+\sqrt{\dfrac{4}{a^2x^2}+\dfrac{9}{x^4}}}$

Examples of Irrational Equations in the form of the sum or difference of two cube roots

42 Solve $\sqrt[3]{4+x}+\sqrt[3]{4-x}=2$

Cube both sides, thus

$$4+x+4-x+3\sqrt[3]{16-x^2}(\sqrt[3]{4+x}+\sqrt[3]{4-x})=8,$$

$$8+3\sqrt[3]{16-x^2}\times 2=8, \quad \because \sqrt[3]{4+x}+\sqrt[3]{4-x}=2,$$

$$\therefore \sqrt[3]{16-x^2}=0,$$

or $$16-x^2=0, \text{ whence } x=4$$

Solve the following equations

43. $\sqrt[3]{76+x}+\sqrt[3]{76-x}=8$

44 $\sqrt[3]{a+\sqrt{x}}+\sqrt[3]{a-\sqrt{x}}=\sqrt[3]{b}$

45 $\sqrt[3]{x+a}-\sqrt[3]{x-a}=c$

46 $\sqrt[3]{5x+158}-\sqrt[3]{5x-158}=4$

*226 **Exponential Equations** Let $a^x=c$ be an equation in which the variable x occurs as an *exponent* The student at once sees that these equations are quite different in nature from those which we have treated in the previous articles These are called EXPONENTIAL EQUATIONS and are solved by the aid of Logarithms. They do not properly belong to the present Chapter , but we have given here only a few simple examples to illustrate their nature. Of the constants a and c, the former is called the *base* of x ; and in solving these equations, the principle which we have to follow is to *reduce a given equation to such a form that the two sides may have the same base*

Examples

1 Solve $a^x = \frac{1}{a^{-3}}$ Here $\frac{1}{a^{-3}} = a^3$, $\therefore$ $a^x = a^3$, whence $x = 3$

2 Solve $2 \times 4^x = 8^{x-1}$

Here $2 \times 4^x = 2(2^2)^x = 2\,2^{2x} = 2^{2x+1}$,

$8^{x-1} = (2^3)^{x-1} = 2^{3x-3}$,

$\therefore$ $2^{2x+1} = 2^{3x-3}$, or $2x + 1 = 3x - 3$, whence $x = 4$.

Solve the following equations

3 $m^{3x+1} = 1$ 4 $\frac{(27)^x}{3^{2x-1}} = 9^x$ 5. $\left(\frac{a}{c}\right)^{x-m} = \left(\frac{c}{a}\right)^{3n}$.

6. $2^{x-6}a^{x-3} = 4$. 7 $a^{3x-2m} = b^{3x-2m}$

8 $5^{1-x}(25)^3 = 3^{1-x}9^3$ 9 $3^{a+2x}a^{2x-3} = 3^{a+4}9a^3$

10 $e^x(e^{x+4} - e^{4-x}) = \frac{e^{a-4} - 1}{e^{-4}}$ 11 $1 + 4^x = 2^{x+1} + 9$.

12 $a^{x^3} = \{(\sqrt{a})^{3x}\}^a$ 13. $a^{2x^m} = \{(a^x)^m\}^x$

We shall conclude this Chapter by giving two very useful formulæ for the solution of equations

227. Formula I If $\frac{a}{b} = \frac{c}{d}$, whatever $a, b, c, d,$ may be, then

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

For since $\frac{a}{b} = \frac{c}{d}$ we have

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \text{ or } \frac{a+b}{b} = \frac{c+d}{d},$$

and

$$\frac{a}{b} - 1 = \frac{c}{d} - 1, \text{ or } \frac{a-b}{b} = \frac{c-d}{d}$$

$$\therefore \quad \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d},$$

or

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

Thus *if two fractions be equal, the sum of the numerator and denominator of one divided by their difference, is equal to the corresponding expression for the other.*

COROLLARY. Hence $\dfrac{a-b}{a+b}=\dfrac{c-d}{c+d}$.

REMARK. It is *generally* advantageous to use this formula or the corollary, when the variable occurs in *one side only*.

Examples

1. Solve $\dfrac{2x+3a}{2x-3a}=\dfrac{5}{3}$

By the formula,

$$\frac{(2x+3a)+(2x-3a)}{(2x+3a)-(2x-3a)}=\frac{5+3}{5-3},$$

or
$$\frac{4x}{6a}=\frac{8}{2}=4, \qquad x=6a$$

2 Solve $\dfrac{3c+5ax}{3c-5ax}=\dfrac{4a+5c}{4a-5c}$

By the COROLLARY,

$$\frac{(3c+5ax)-(3c-5ax)}{(3c+5ax)+(3c-5ax)}=\frac{(4a+5c)-(4a-5c)}{(4a+5c)+(4a-5c)},$$

$$\frac{5ax}{3c}=\frac{5c}{4a}, \qquad x=\frac{3c^2}{4a^2}$$

3 Solve $\dfrac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}-\sqrt{a-x}}=c$

$$\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}-\sqrt{a-x}}=c=\frac{c}{1} \text{ [§ 167, } Cor\ 2],$$

$$\therefore \quad \frac{\sqrt{a-x}}{\sqrt{x}}=\frac{c-1}{c+1},$$

$$\frac{a-x}{x}=\left(\frac{c-1}{c+1}\right)^2,$$

$$\frac{a}{x}-1=\left(\frac{c-1}{c+1}\right)^2,$$

$$\frac{a}{x}=1+\left(\frac{c-1}{c+1}\right)^2=\frac{2(c^2+1)}{(c+1)^2},$$

$$\frac{x}{a}=\frac{(c+1)^2}{2(c^2+1)}, \qquad \therefore\ x=\frac{a(c+1)^2}{2(c^2+1)}.$$

Solve the equations

4 $\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9$

5 $\dfrac{\sqrt{5x}-\sqrt{7-5x}}{\sqrt{5x}+\sqrt{7-5x}}=\dfrac{1}{4}$

6 $\dfrac{\sqrt{a}+c\sqrt{b-x}}{\sqrt{a}-c\sqrt{b-x}}=c$

7 $\dfrac{\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}-b}{\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}+b}=\dfrac{2}{3}$

8 $\dfrac{\sqrt{x}-\sqrt{5x+2a}}{\sqrt{x}+\sqrt{5x+2a}}=\dfrac{\sqrt{x}+\sqrt{a+b}}{\sqrt{x}-\sqrt{a+b}}$

9 $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{x+b}+\sqrt{x-b}}=\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{x+b}-\sqrt{x-b}}$

10 $\dfrac{ax+1+\sqrt{a^2x^2-1}}{ax+1-\sqrt{a^2x^2-1}}=\dfrac{bx}{2}$

11. $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}}=1$

12 Also the equations 38, 39 and 40 of § 225 by this method

The following equations demand greater skill.

13 $x-1=2\sqrt{x}+2$

14 $\left(\dfrac{a+x}{a-x}\right)^2=1+\dfrac{cx}{ab}$

15 $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=b$

16 $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x}=b$

17 $\dfrac{x}{\sqrt{1+x}+\sqrt{1-x}}=\dfrac{2a^3}{1-\sqrt{1-x^2}}$ [See *App*]

18 $1+\dfrac{x}{2}-\dfrac{x^2}{2(1+\sqrt{1+x})^2}=3$

19 $\sqrt{\dfrac{x+a}{x}}+2\sqrt{\dfrac{a}{x+a}}=b\sqrt{\dfrac{x}{x+a}}$

20 $\dfrac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}}=\dfrac{\sqrt{x^2-1}-1}{x^2-2}$

21 $\sqrt[4]{x^4-1}+x\sqrt{x^4-1}=x^3$.

22 $\sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2}$ [See *App*]

23 $\sqrt{\dfrac{x^2}{a^2}+1}-\sqrt[4]{\dfrac{x^2}{a^2}+1}=\dfrac{9a^2-1}{4}$.

Solve the equations

24. $(a+x)\sqrt{1+a}+(a-x)\sqrt{1-a}=2\sqrt{a^2+x^2}$. [See *App.*]

25. $\dfrac{1+x-\sqrt{2x+x^2}}{1+x+\sqrt{2x+x^2}}=a^3\,\dfrac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}$. [See *App.*]

26. $\dfrac{(1+a^2)(1+x^2)}{(1+ax)^2}=\dfrac{1}{4}\left\{\dfrac{x}{a}+\dfrac{a}{x}+2\right\}$.

228. Formula II. If $\dfrac{a}{b}=\dfrac{c}{d}$, where a, b, c, d are any quantities whatever, then each $=\dfrac{a+c}{b+d}$ or $=\dfrac{a-c}{b-d}$.

Let $\qquad \dfrac{a}{b}=\dfrac{c}{d}=k$,

then $\qquad a=kb,\ c=kd, \qquad (1)$,

or $\qquad a+c=k(b+d)$,

$$\therefore \frac{a+c}{b+d}=k=\frac{a}{b}=\frac{c}{d} \qquad (2).$$

Also from (1), $\qquad a-c=k(b-d)$,

$$\therefore \frac{a-c}{b-d}=k=\frac{a}{b}=\frac{c}{d} \qquad (3).$$

Thus *if two fractions be equal, each is equal to the sum or difference of the numerators divided respectively by the sum or difference of the denominators.*

COROLLARY. Hence $\dfrac{a}{b}=\dfrac{a+mc}{a+md}=\dfrac{a-mc}{a-md}$, $\because \dfrac{c}{d}=\dfrac{mc}{md}$, [§ 168].

Examples

1. Solve $\dfrac{x-a+b-c}{s-p+q-r}=\dfrac{a-b+c}{p-q+r}$.

By the formula,

$$\frac{a-b+c}{p-q+r}=\frac{x-a+b-c+a-b+c}{s-p+q-r+p-q+r}=\frac{x}{s},$$

$$\therefore x=s.\frac{a-b+c}{p-q+r}.$$

2 Solve $\frac{mx-(a+b)}{nx-(c+d)}=\frac{mx-(a+c)}{nx-(b+d)}$ [§ 224, Ex 117]

By the formula,

$$\frac{mx-(a+b)}{nx-(c+d)}=\frac{mx-(a+b)-mx+(a+c)}{nx-(c+d)-nx+(b+d)}=\frac{c-b}{b-c}=-1,$$

or $$mx-(a+b)=c+d-nx,$$

$$\therefore (m+n)x=a+b+c+d, \text{ or } x=\frac{a+b+c+d}{m+n}.$$

3 Solve $\frac{ax^2+bx+c}{px^2+qx+r}=\frac{ax+b}{px+q}$ [§ 224, Ex. 66]

Multiply numerator and denominator of second member by x, thus

$$\frac{ax^2+bx+c}{px^2+qx+r}=\frac{ax^2+bx}{px^2+qx},$$

by Cor, $$\frac{ax+b}{px+q}=\frac{ax^2+bx+c-(ax^2+bx)}{px^2+qx+r-(px^2+qx)}=\frac{c}{r},$$

$$r(ax+b)=c(px+q),$$

whence $$x=\frac{cq-br}{ar-cp}$$

Solve the equations

4 $\frac{4x-9}{3x+8}=\frac{8x+9}{9x-8}$

5 $\frac{6x+a}{4x+b}-\frac{3x-b}{2x-a}=0$

6 $\frac{\sqrt{ax}+b}{\sqrt{bx}-a}=\frac{\sqrt{ax}-c}{\sqrt{bx}+c}$

7 $\frac{x^2-ax+b}{x^2-px+q}=\frac{x^2-ax-q}{x^2-px-b}$

8 $\frac{x^3+ax^2-bx+c}{x^3-ax^2+bx+c}=\frac{x^2+ax-b}{x^2-ax+b}$

9 Also the equations 64, 116, 118 and 119 of § 224, by this method

* 229 Examination upon Chapter XVIII

1. Define an *Equation* What is a *Simple Equation* ?

2. Shew that every simple equation in one variable can be reduced to the form $ax+b=0$

3 Prove that a simple equation has only *one root and no more*

4. Reduce the following equations to *rational and integral forms* and state their *degrees*, x and y being the variables :—

(1) $\sqrt{ax}+\sqrt{bx}-\sqrt{cx}=\sqrt{d}$ (2) $\sqrt{x}+\sqrt{1-x}+\sqrt{1+x}=1$.

(3) $\sqrt[3]{1-x^2}+\sqrt{x^3}=0$. (4) $\frac{\sqrt{x}}{y}+\frac{y}{\sqrt{x}}=\frac{\sqrt{x}}{a}$

(5) $\frac{x}{ay}+\frac{y}{bx}=ax$. (6) $\frac{x}{\sqrt[3]{a}}+\frac{x\sqrt{b}}{\sqrt[3]{b}}=\sqrt{c}$

(7) $x-\sqrt[3]{yx^4+1}=y$ (8) $\frac{\sqrt{a}}{x}+\frac{a}{bx^{-1}}+\frac{x^{-2}}{\sqrt{c}}=3$

5. For what value of x will the *equation*

$$\frac{x}{a-c}+\frac{c}{a+c}=\frac{a}{a-c}-\frac{x}{a+c}$$

become an *identity* ?

6. Find the values of a and b which make the *equation*

$2x^2-ax+15=(2x-5)(x+b)$ an *identity*.

7 If $a+b+c=0$, solve the equation

$$\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x}=0$$

CHAPTER XIX

Problems Leading to Simple Equations

Section II

230 The present Chapter is a continuation of Chapter XI, to which the student is referred back In this Chapter, the translated problems will, in most cases, produce equations involving fractions.

Examples

Ex 1. What number is that to which if 23 be added and the sum divided by 5, the quotient will be 13 diminished by half the number ?

Let x=required number ,

then $\frac{x+23}{5}$=quotient of the sum of x and 23 divided by 5 ;

and $13-\frac{1}{2}x$=this quotient, by the condition of the problem ;

$\therefore \frac{x+23}{5}=13-\frac{1}{2}x$, whence $x=12$

Ex 2 A post is $\frac{1}{4}$ in the earth, $\frac{2}{3}$ in the water, and 5 cubits above the water, what is its length ?

Let $\qquad x=$the length of the post, in cubits,

then $\qquad \frac{1}{4}x+\frac{2}{3}x+5=$whole length of post$=x$,

whence $x=60$

3 Three-fourths of a number diminished by one is divided by half that number increased by one, and the result is $\frac{5}{4}$, what is that number ?

4 What is that number a third, a fourth, and a fifth part of which taken together amount to 94 ?

5 The denominator of a fraction is greater than the numerator by 2, and if 5 be added to both, the resulting fraction is $\frac{20}{21}$, find the fraction

6 The numerator of a certain fraction is half the denominator, and if 7 be added to the numerator and 19 subtracted from the denominator, the result is 2, what is the fraction ?

7 Divide 200 into two such parts that their difference divided by the greater may be $\frac{2}{11}$

8 Divide Rs 62 among A, B, C, D, so that A shall have twice as much as B and 8 as more, C Rs 2 less than A, and D Rs 3 more than B

9 A and B have together Rs 57, B and C Rs 50, A and C Rs 53, what has each ?

10 In a sea fight, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk, 21 escaped, and the fleet consisted of 7 times the number burnt Of how many ships did the fleet consist ?

11 Divide Rs 800 among 3 persons A, B, C, so that B may have $\frac{3}{4}$ths of what A will get $+40$ rupees, and C $\frac{1}{3}$ of what A and B will get together

12 A and B play together, first A loses Rs 10 and then he has $1\frac{1}{2}$ times as much as B, next B loses $\frac{1}{5}$th of what he had at first and one rupee more, and he has now half as much as A; what had each at first ?

13 Two shepherds owning a flock of sheep agree to divide its value; A takes 72 sheep, and B takes 92 sheep and pays A £25 What is the value of a sheep ?

14 How much water must a wine merchant mix with 50 gallons of wine at 12*s* a gallon, so that by selling the mixture at 10*s* he may gain £1 ?

15 Find two consecutive numbers such that the 4th and 11th parts of the less together exceed by 1 the 5th and 9th parts of the greater.

16 The difference between the squares of two consecutive numbers is 513; find the numbers

17. Find the number whose fourth root is equal to $\frac{1}{2}$ its cube root.

18. A person being asked his age replies—If I should live as many years more, half as many years more, and 10 years besides, I should have lived 100 years What is his age?

19. A general having lost a battle found that he had only half his army +3600 men left fit for action, $\frac{1}{8}$ of his men +600 being wounded, and the rest, which were $\frac{1}{5}$th of the whole army, either slain, taken prisoners, or missing, find the total number of men in his army

20 A tradesman, after expending 100*l* a year, augments the remainder of his property by one third part of it, and at the end of 3 years his original property is doubled, what had he at first?

21 *A* and *B* began to pay off their debts with different sums; *A*'s money at first was $\frac{2}{3}$rds of *B*'s; but after *A* had paid £1 less than $\frac{3}{4}$ths of his money and *B* £1 more than $\frac{7}{8}$ths of his, it was found that *B* had only half as much as *A* What had each at first?

22 A gentleman, meeting with 3 beggars, gave to the first $\frac{1}{3}$ of what he had in his pocket and then 1 rupee more; to the second $\frac{1}{3}$ of what he had left and then 1 rupee more; to the third $\frac{1}{3}$ of what he had still left and then one rupee more; after which he had nothing left. How much had he at first?

23 About one-half of India is under the English, one-third under the Allied Princes, $\frac{209}{1260}$ under independent kings and Hill Tribes, and the rest, about 1200 square miles, is Foreign possessions, what is its area?

24 How much gold as Rs 20 a tolah, must be mixed with 14 tolahs of gold at Rs 15 a tolah, so that the compound may be worth Rs 18 a tolah?

25 Find three numbers, differing in order from one another by 5, such that 12 times the product of the greatest and least may be equal to the square of the sum of the three numbers

26 A garrison of 4000 men had provisions for 20 days, after 11 days it was reinforced and then the provisions were exhausted in 8 days; find the number of men in the reinforcement

27 A besieged garrison had provisions for 70 days, after 10 days a party of 1500 men made a sally and escaped, and the allowance per head being now reduced to $\frac{6}{7}$ths of what it was before, the garrison held out for 100 days more, what was the number of men in the garrison at first?

28 A detachment from an army was marching in a regular column, with 5 men more in depth than in front, but upon the enemy coming in sight, the front was increased by 845 men, and by this

movement, the detachment was drawn up in 5 lines Find the number of men in the detachment

29 A square silver plate is to be made into a rectangular one, of equal thickness, whose length is to be 3 inches longer, and breadth 2 inches shorter, than the side of the plate required its area.

30 A bag contains sixpences, shillings and half-crowns, the amount expressed by each kind is the same, if the total number of coins in the bag be 119, find the number of each

31 A flock of ewes of which $\frac{1}{10}$th were barren and $\frac{1}{4}$th brought twins, produces 23 lambs Required the number of ewes, none being supposed to produce more than two

32 A man wished to enclose a piece of ground with palisades, and found that if he set them a foot asunder, he should have too few by 150, but if he set them a yard asunder, he should have too many by 70 How many had he?

33 In a certain lake, the tip of a bud of lotus was seen a span above the surface of the water, forced by the wind it gradually advanced, and was submerged at a distance of two cubits tell me quickly, O mathematician, what is the depth of the water. (*Lilavati*)

***231. Problems relating to digits** The difficulty which the student finds in solving these problems, arises from the arbitrary manner in which a number, consisting of two or more digits, is represented in Arithmetic Thus for instance, *thirty eight* is represented by putting 3 and 8 *together*, whereas it should have been more properly represented by $30+8$, as thirty-eight means *thirty and eight*. Hence the digit 3 in this number has acquired, on account of its position, a new value, different from its *absolute* value This new value is called its *local* value, which in *this example* is *ten* times its absolute value, as 3 is in the *tens'* place Hence if a stand for 3 and b for 8, 38 would be represented, not by ab, but by $10a+b$ [see § 27, Ex 9] Similarly if a, b, c stand respectively for 5, 6, and 7, 567 would be represented by $100a+10b+c$, and so on

In problems of this kind therefore the distinction between *absolute* and *local* values of a digit should be carefully remembered Thus if the digits in 38 be inverted, we have 83 which would be represented by $10b+a$ again if 5 and 7 in 567 interchange places, the new number would be represented by $100c+10b+a$, and so on

Examples.

Ex 1. A number consists of two digits, that in the tens' place being greater that the other by 3, if 13 be added to the number, the sum is 65. Find the number

Let $\qquad x =$ digit in the units' place,

then $\qquad x+3 = \ldots\ldots\ldots$ tens' $\ldots\ldots$,

$$\therefore \; 10(x+3)+x = \text{the required number},$$

$$\therefore \; 10(x+3)+x+13=65,$$

whence $\qquad x=2$,

$$\therefore \; \text{required number} = 10(x+3)+x=52$$

Ex 2 The digit in the tens' place of a number of two digits is $\frac{1}{3}$ of the digit in the units' place, and if the sum of the number and 5 be divided by the number formed by inverting the digits, the quotient will be $\frac{1}{2}$; find the number.

Let $\qquad x =$ digit in the tens' place,

then $\qquad 3x = \ldots\ldots\ldots$ units' $\ldots\ldots$,

$$\therefore \; 10x+3x = \text{required number},$$

and $\qquad 30x+x =$ number formed by inverting the digits;

$$\therefore \; \frac{10x+3x+5}{30x+x} = \frac{1}{2},$$

or $\qquad \dfrac{13x+5}{31x} = \dfrac{1}{2}$, whence $x=2$,

$$\therefore \; \text{required number} = 13x = 26.$$

3. The sum of the two digits of a number is 14, and the quotient of the one divided by the other is $\frac{4}{3}$; find the number,

4 If from the sum of the digits of a number 10 be subtracted the remainder is 2, and the number formed by reversing the digits is equal to $\frac{1}{3}$ of the number, increased by 56 · what is the number ?

5 The difference between the digits of a number is 2, and if 3 times the units' digit, which is the greater of the two, be added to the number, the digits are inverted , find the number

6 The tens' digit of a number, is twice the other, and if 3 times the greater digit together with 12 be taken from the number, the remainder is the number which is formed by interchanging the digits what is the number ?

7. A number consists of 3 digits of which the middle one is 0 and the sum 8 , the number formed by interchanging the extreme digits is greater than the number itself by 198 . what is the number ?

***232 Problems relating to work and agent** If A performs a piece of work in 1 day, it is clear that in a days he will perform a times as much work , that is, *the capacity of an agent*

multiplied by the time he is employed on a work, is equal to the work Hence if

w = a *unit of work* done in a *unit of time*,

W = total work, done in t units of time

then $$W = wt \quad (1).$$

And from (1), $$w = \frac{W}{t} \quad (2),$$

$$t = \frac{W}{w} \quad (3),$$

that is, *capacity of agent = work ÷ time*,

and *time = work ÷ capacity of agent*

Now suppose A and B, whose capacities for work are w_1 and w_2 (*i e, the units of work which they respectively do in a unit of time*), can together perform the work W, required the time Here w is the sum of w_1 and w_2, and from (3)

$$\text{required time} = \frac{W}{w_1 + w_2}$$

Examples

Ex. 1. A can do a piece of work in 5 hours and B in 6 hours, in what time can they together do it ?

Let W = whole work,

and x = required time, i e, number of hours, here *one hour being the unit of time*

Now A does the whole work W in 5 hours, in 1 hour he does $\frac{1}{5}$th of W, i e, here $w_1 = \frac{W}{5}$, similarly $w_2 = \frac{W}{6}$ Therefore from (1)

$$(w_1 + w_2)x = W,$$

$$\therefore \left(\frac{W}{5} + \frac{W}{6}\right)x = W,$$

or $$\frac{x}{5} + \frac{x}{6} = 1,$$

whence $$x = \tfrac{30}{11} = 2\tfrac{8}{11}$$

Ex 2 A and B can finish a piece of work in 10 days, B and C in 15 days, and A and C in 18 days ; how long would they take to finish the work together

Let $\quad W=$ total amount of work,

$x=$ required time (*i.e.*, number of days),

and $\quad w_1, w_2, w_3=$ the capacities of A, B, C respectively ;

$$\therefore\ (w_1+w_2)10=W, \text{ or } w_1+w_2=\frac{W}{10} \qquad (a),$$

$$(w_2+w_3)15=W, \text{ or } w_2+w_3=\frac{W}{15} \qquad (b),$$

$$(w_3+w_1)18=W, \text{ or } w_3+w_1=\frac{W}{18} \qquad (c) ;$$

whence by addition, $2(w_1+w_2+w_3)=(\frac{1}{10}+\frac{1}{15}+\frac{1}{18})W$,

or
$$w_1+w_2+w_3=\frac{W}{9} \qquad (d) ;$$

subtracting (*b*), (*c*) and (*a*) respectively from (*d*), we have

$$w_1=\frac{W}{9}-\frac{W}{15}=\frac{2W}{45},$$

$$w_2=\frac{W}{9}-\frac{W}{18}=\frac{W}{18},$$

$$w_3=\frac{W}{9}-\frac{W}{10}=\frac{W}{90}.$$

But here $w=w_1+w_2+w_3$, therefore from (1) or (3), we get

$$x=\frac{W}{w_1+w_2+w_3}=\frac{W}{\frac{2W}{45}+\frac{W}{18}+\frac{W}{90}}=9 \text{ days.}$$

Ex. 3 A cistern can be filled by 2 pipes in 5 and 6 hours respectively, and emptied by a third in 10 hours ; if all the three be opened simultaneously, when will the cistern be filled ?

Let $\quad x=$ time required, in hours.

Now the capacities of the pipes are respectively $\frac{V}{5}$, $\frac{V}{6}$ and $\frac{V}{10}$, where V represents the cubical content of the cistern It is evident therefore that $\frac{V}{5}+\frac{V}{6}-\frac{V}{10}$ is the *joint capacity of the three pipes* for filling the cistern.

Hence
$$x\left(\frac{V}{5}+\frac{V}{6}-\frac{V}{10}\right)=V,$$

solving which $\quad x=3\frac{3}{4}$ hrs. $=3$ hrs. 45 min.

4 A man alone can reap a field in 10 days, and with the assistance of his son, in 6 days, how long will it take the son alone to reap it?

5 A can do as much work in 6 hours, as B can do in 8 hours, or as C can do in 10 hours, in what time will A and B together complete a piece of work $\frac{1}{4}$ of which has been done by C in 25 hours?

6 A man can drink a cask of beer in 15 days, after he has been drinking for five days, he is joined by his wife and they together finish it in $6\frac{1}{4}$ days more, how long could the cask last the wife alone?

7 A rectangular bath can be filled by 3 spouts in 3, 4 and 5 hours respectively, if 65 cubic feet of water be first thrown in and the 3 spouts be then opened together, the rest can be filled in 1 hour, find the volume of the bath

8 A leaky cistern is filled by 2 pipes in 15 and 20 hours respectively, if the leak be plugged, but if the leak as well as the pipes run together, the cistern can be filled in 12 hours, in what time can the leak empty the cistern when full, if the supply pipes be stopped?

9 After A has done $\frac{3}{8}$ths of a piece of work in 15 hours, B joins him and the two together finish it in 8 hours, when could they separately do it?

10 Two pipes A and B together fill a tank in 20 hours, A runs alone for 4 hours, when B is opened and in 15 hours more $\frac{7}{8}$ths of the tank is filled, in what time would each pipe have filled the tank separately?

11 A can do a piece of work in 15 days and B in 18 days; they work together for 3 days, when B leaves but A continues, and after 3 days is joined by C, and they together finish it in 4 days, how long would it take C to do the piece of work alone?

***233 Problems relating to Motion** We have seen [§ 147] that A walking a miles per hour, walks ab miles in b hours, that is, if d represent the distance passed over by a body in the time t at the rate of r per unit of time, then

$$d = rt \qquad (1);$$

hence $$\textit{distance} = \textit{rate} \times \textit{time}.$$

From (1) $$r = \frac{d}{t} \qquad (2);$$

and $$t = \frac{d}{r} \qquad (3),$$

that is, $\textit{rate} = \textit{distance} \div \textit{time}$; and $\textit{time} = \textit{distance} \div \textit{rate}$.

Again if A and B describe respectively the distances d, d' in the times t, t' at the rates r, r'; then

$$d = rt \text{ and } d' = r't',$$

whence
$$\frac{d}{d'} = \frac{rt}{r't'} \qquad (4);$$

that is, *the distances are proportional to the product of the rates into the times.*

If $d = d'$, from (4)
$$\frac{rt}{r't'} = 1 \text{ or } \frac{r}{r'} = \frac{t'}{t} \qquad (5);$$

that is, *the distance being the same, the rates are inversely proportional to the times.*

If $r = r'$, from (4)
$$\frac{d}{d'} = \frac{rt}{r't'} = \frac{t}{t'} \qquad (6);$$

that is, *the rate being the same, the distances are proportional to the times.*

If $t = t'$, from (4)
$$\frac{d}{d'} = \frac{rt}{r't'} = \frac{r}{r'} \qquad (7);$$

that is, *the time being the same, the distances are proportional to the rates.*

REMARK. In all algebraical problems relating to motion, *motion is always supposed to be uniform.*

Examples

Ex 1 A boat goes 57 miles down a river in 6 hours; if the river flows at the rate of 4 miles an hour, find the rate of the boat in still water.

Here $(x+4)$ miles is the *actual* rate of the boat when it goes down stream, if we represent by x the required rate of the boat,

∴ from (2), $\qquad x + 4 = \frac{57}{6}$ miles,

whence $\qquad x = 5\frac{1}{2}$ miles.

Ex. 2. A messenger is sent to a town which is distant 80 miles: after passing the midway station, he doubles his speed; if the whole time taken be 6 hours, what was his rate at first?

Let $\qquad x$ = required rate in miles per hour,

∴ from (3), $\qquad \frac{40}{x}$ = time taken to travel the first half,

and $\qquad \frac{40}{2x}$ = time taken to travel the second half,

and these two times taken together = 6 hours; therefore

$$\frac{40}{x} + \frac{40}{2x} = 6,$$

whence $\qquad x = 10$

Ex. 3 If the number of the crew be doubled in coming up a river, a boat takes the same time to come up as to go down, the river flowing at the rate of $3\frac{1}{2}$ miles per hour Find the rate of the boat.

In going down, the crew is *assisted* by the river, hence the actual rate of the boat in this case = its own rate + rate of the river.

Again, in coming up, the crew is *obstructed* by the river, hence the actual rate of the boat in this case = its own rate − rate of the river

Let x = required rate of the boat in miles per hour,

$\therefore$ $x+3\frac{1}{2}$ = the rate of the boat in going down,

and $2x-3\frac{1}{2}$ = ...coming up

Therefore, the distance and the time both being the *same*, we get from (4)

$$\frac{x+3\frac{1}{2}}{2x-3\frac{1}{2}}=1,$$

whence $x=7$,

$\therefore$ the boat goes 7 miles per hour.

4 *A* and *B* start at the same time to go to a certain place, *A*, who walks $4\frac{1}{2}$ miles an hour, takes $\frac{5}{6}$ths of the time which *B* takes, to reach the place, find the rate of *B*

5 A man can walk a certain distance in 4 hours if he were to reduce his rate by one sixteenth, he could walk one mile less in that time, what is his rate?

6 A person after walking the first half of his journey, quickens his pace by one-fifth, and thus arrives at his destination 35 minutes earlier, how long does he take to walk the whole distance?

7 A crew, which can pull at the rate of 9 miles an hour, finds that it takes twice as long to come up a river as to go down; at what rate does the river flow?

8. *A* and *B* walk over the same ground, going out one way and coming home the other, they start at the same time in opposite directions, *A* walking $3\frac{3}{4}$ miles and *B* 4 miles per hour, *A* wants $\frac{1}{4}$ of a mile of being half-way when he meets *B* Required the length of the walk and the time each was out

9 A person walks to Baitsbite at the rate of 3 miles an hour, runs part of the way back at the rate of $8\frac{1}{2}$ miles an hour, and walks the remainder in 1 hour and 5 minutes, he is out 2 hours 44 minutes, find the distance he ran and that to Baitsbite.

10 A student is allowed just 3 hours and 20 minutes for exercise, how far can he walk at the rate of $1\frac{3}{4}$ miles an hour so as to come home in time, riding back the distance at the rate of $10\frac{1}{2}$ miles an hour?

11. A railway train after running for sometime meets with an accident after which it proceeds with $\frac{3}{4}$ths of its former rate, had the accident occurred 30 miles further on, it would have arrived at the terminus 25 minutes sooner Required the rate of the train

12 A and B start to run to a flagstaff 450 yards off, and back. A returning meets B 30 yards from the flagstaff, and arrives at the starting point half a minute before B How long did A take to run the whole distance?

***234 Problems relating to Relative Motion.** When two bodies are moving (for instance two persons are walking), it is clear that if their rates of moving be not the same, the interval between them will vary at the end of each unit of time Thus in Ex 43 [p. 179], the original distance between P and Q is diminished by $(4+5)$ miles every hour Hence if the original distance between two objects, which are moving towards each other (*i e, in opposite directions*) at the rates of m and n miles an hour, be d miles, then

$$d-(m+n)h$$

will represent *the distance between them at the end of* h *hours*; and if they meet after x hours, this distance vanishes, therefore

$$d-(m+n)x=0,$$

whence
$$x=\frac{d}{m+n} \qquad (1).$$

Thus when the motions of two bodies are in *opposite directions, the time of meeting = distance ÷* sum *of the rates*

Again if P and Q start at the same time from A and O respectively to proceed towards B, and if the distance $AO=d$, then evidently d is diminished every hour by $(m-n)$ miles; hence

$$d-(m-n)h$$

will represent *the distance between* P *and* Q *at the end of* h *hours*; and at the instant when P overtakes Q, this distance will vanish Therefore if x represent the time of meeting of P and Q,

$$d-(m-n)x=0,$$

or
$$x=\frac{d}{m-n} \text{ hours} \qquad (2).$$

Thus when the motions of two bodies are in the *same directions, the time of meeting = distance ÷* difference *of the rates*

Examples

Ex. 1 A Goods-train which runs 20 miles per hour, starts 6 hours before a Mail-train which runs 28 miles per hour; when and at what distance from the station will the Goods-train be overtaken?

Here $d=20\times 6=120$ miles, therefore from (2),

$$\text{time of meeting}=\frac{120}{28-20}\text{ hrs}=15\text{ hours},$$

and $\therefore$ required distance $=(28\times 15)$ miles $=420$ miles

It is always well, however, to solve these problems from the principles explained above, thus —

Let A represent the station, B the position of the Goods train at the end of 6 hours, and C the place where the Goods-train is overtaken.

A B C

Let $x=$ time when the M will overtake the G ;

then $28x=$ distance run by the M in x hours $=AC$,

$20x=$ .... G .. .. $=BC$,

and $AB=20\times 6=120$ miles,

hence, since $AC=AB+BC$, we have

$$28x=120+20x,$$

whence

$$x=15\text{ hours},$$

and therefore required distance $=28\times 15=420$ miles

2 A steamer running 14 miles an hour, descries another 21 miles off, going at the rate of 11 miles an hour, how many miles will the latter have run before she is overtaken?

3 A and B set out at the same time from P and Q, after 5 hours A meets B who walks 4 miles an hour, and reaches Q in 4 more hours, find the rate of A and the distance between P and Q

4. A person starts from Ely to walk to Cambridge, which is distant 16 miles, at the rate of $4\frac{4}{9}$ miles per hour, at the same time that another person leaves Cambridge for Ely, walking at the rate of a mile in 18 minutes, find where they meet

5 A is sent on an errand and travels at the rate of 5 miles an hour; 1 hour 24 minutes after, B is despatched to call him back; if after the first hour, B increases his pace by $\frac{1}{12}$ and overtakes A in 4 more hours, find the rate of B and the distance where he meets A

6 A constable in pursuit of a thief at a uniform pace finds by inquiry that the thief is travelling $1\frac{1}{2}$ miles per hour quicker than himself, he therefore doubles his speed after the first 4 hours, and takes the thief at the end of 6 hours 20 minutes from the time of his starting Given that the thief had a start of 1 hour, and never varied his speed, find the rates of travelling of the parties, and the distance where the capture took place

7 Two persons, who walk at the rates of 3 and 4 miles per hour

respectively, started from two places 70 miles apart; after passing each other they continued their journey, reached the places, turned back immediately, and met again; when and where will the second meeting take place?

***235 Problems relating to watches and clocks** The chief peculiarity of these problems is that, unlike other problems, *one of their conditions is generally understood* In solving them therefore the student should carefully remember this condition, *viz.*, that *the minute hand moves* 12 *times faster than the hour-hand*, and if the second-hand be also on the same axis as the other two, that *the second-hand moves* 60 *times faster than the minute-hand and* 720 *times faster than the hour-hand*

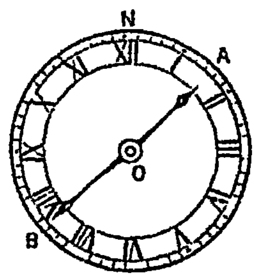
Two hands of a watch are said to be (1) *together* when there is *no interval* between them, (2) in the *same straight line*, when the interval between them is 30 *minute-divisions*, and (3) *at right angles*, when the interval between them is 15 *minute-divisions*, and this last happens *twice* during the same hour

Examples.

Ex 1. At what time between 1 and 2 o'clock will the hour-hand and minute-hand of a watch be in the same straight line?

Let x=number of minutes after One, when the two hands are in the same straight line.

Then if AOB represent their position, it is easy to see that OB representing the minute-hand, has moved for x minutes, *i e*, the arc NB described by it represents x minute-divisions, and OA, which represents the hour-hand, and which has moved also for x minutes, has described an arc of $\frac{1}{12}$ of x, *i e*, the arc IA represents $\frac{x}{12}$. But



$$\text{arc NB} = \text{arc NI} + \text{arc IA} + \text{arc AB};$$

and arc NI = 5 minute-divisions,

arc AB = 30 .... ;

$$\therefore\ x = 5 + \frac{x}{12} + 30,$$

whence $x = 38\frac{2}{11}$;

i e, the two hands will be in the same line after $38\frac{2}{11}$ min. past One.

2 At what time between 1 and 2 o'clock will the two hands of a watch be, (1) together, and (2) at right angles ?

3 At what time between 6 and 7 o'clock will the hands of a clock be, (1) together, (2) in the same straight line, and (3) at right angles ?

4 When between 2 and 3, will the minute-hand be exactly (1) one minute in advance of the hour-hand, and (2) 7 minutes behind it ?

5 It is between 2 and 3 o'clock, but a person looking at the clock and mistaking the hour hand for the minute-hand, fancies that the time of day is 55 minutes earlier than the reality, what is the true time ?

236 The following are additional problems for exercise

Ex (i) A person bought a picture at a certain price and paid the same price for the frame, if the frame had cost £1 less and the picture 15s more, the price of the frame would have been only half that of the picture Find the cost of the picture [*Cal*, 1860]

Let $x=$ price of the picture, in £,
then also $x=$ .... frame,
By supposition, $x-1=$ .... . frame, ...
$x+\frac{15}{20}=$ . . .. picture, ...

$$\therefore x-1=\tfrac{1}{2}(x+\tfrac{15}{20}),$$

whence $x=\frac{11}{4}£=£2\ 15s$

Ex (ii) A boy buys a certain number of oranges at 3 for 2*d.*; and one third of that number at 2 for 1*d*, at what price must he sell them to get 20 per cent profit ? If his profit be 5s 4*d*, find the number bought [*Cal*, 1885]

Let $x=$number of oranges bought at first,
$\therefore \frac{1}{3}x=$. . ... next ,
thus $\frac{2}{3}x=$price, in d of the first lot,
and $\frac{1}{2}\times\frac{1}{3}x$ or $\frac{1}{6}x=$... . ...second ,

$$\therefore \tfrac{2}{3}x+\tfrac{1}{6}x=\tfrac{5}{6}x=\text{total cost in } d$$

And since by the question 20 per cent is the profit, we have

$\frac{1}{5}\times\frac{5}{6}x$ or $\frac{1}{6}x=$profit in d, realised by selling the oranges (A),
and $\frac{5}{6}x+\frac{1}{6}x=x=$selling price, in d, of the whole lot,

$$\therefore \text{selling price of an orange} =\frac{x}{x+\frac{1}{3}x}d=\tfrac{3}{4}d,$$

that is, he must sell at the rate of 4 for 3*d*

Again, if 5*s*. 4*d* be the profit, we have from (A),

$$\tfrac{1}{6}x = 5s\ 4d = 64d\text{, whence } x = 384\text{,}$$

$$\therefore \text{ total number of oranges} = 384 + \tfrac{1}{3} \text{ of } 384 = 512$$

Ex (iii) A cask A contains 12 gallons of wine and 18 gallons of water, another cask B contains 9 gallons of wine and 3 gallons of water; what quantity of liquid must be taken from A and B respectively, so that their mixture may contain 7 gallons of wine and 7 gallons of water?

The quantity in $A = (12+18)$ gallons $= 30$ gallons; therefore $\frac{12}{30}$ or $\frac{2}{5}$ of the mixture in A is *wine*, similarly $\frac{9}{12}$ or $\frac{3}{4}$ of the mixture in B is *wine* Now the quantity of new mixture is to be $(7+7)$ gallons $= 14$ gallons Therefore if

x = quantity of mixture, in gallons, to be taken from A,

then $14 - x$ = ... B,

therefore in the new mixture,

$\frac{2}{5}x$ = quantity of *wine* in gallons drawn from A,

and $\frac{3}{4}(14-x)$ = ... *wine* . . .. B,

$$\therefore \tfrac{2}{5}x + \tfrac{3}{4}(14-x) = 7\text{, whence } x = 10$$

Thus 10 gallons are to be taken from A and $14 - 10$ or 4 gallons from B

Note The student will do well to solve this problem by equating the quantities of *water*

Ex. (iv) A hare is 80 of her own leaps before a greyhound; she takes 4 leaps for every 3 that he takes, but he covers as much ground in one leap as she does in two How many leaps will the hare have taken before she is caught?

Let x = required number of leaps,

and a = length of ground covered by one leap of the hare;

thus, since the hare takes 4 leaps for every 3 leaps that the greyhound takes,

$\frac{3}{4}x$ = number of leaps, the greyhound takes in the same time that the hare takes x leaps,

and $2a$ = length of ground covered by one leap of the greyhound;

$\therefore$ $80a + xa$ = distance of the place of capture from the place where the dog first was,

and $\frac{3}{4}x \times 2a$ = the distance the dog had to run to overtake the hare; now these two distances are the same, [see § 234, Ex. 1], therefore

$$80a + xa = \tfrac{3}{4}x \times 2a,$$

divide by a; thus $\quad 80 + x = \frac{3}{2}x$, when $x = 160$.

Ex (v) A wall costs Rs 300 when brick sells at Rs 8 a thousand and it costs Rs 365 when brick sells at Rs 10 a thousand; find the cost of labour in building the wall.

Let $\quad x$ = required cost in rupees,

then $\quad 300-x$ = price of brick, in rupees, by the first supposition,

and $\quad 365-x$ = ... second supposition;

$\therefore \quad \dfrac{300-x}{8}\times 1000$ = number of bricks required for the wall;

also $\quad \dfrac{365-x}{10}\times 1000$ = ...;

$$\therefore \quad \frac{300-x}{8}\times 1000=\frac{365-x}{10}\times 1000,$$

whence $x=40$

Ex (vi) A man rides one third of the distance from A to B at the rate of a miles per hour and the remainder at the rate of $2b$ miles per hour. If he had travelled at a uniform rate of $3c$ miles per hour, he could have ridden from A to B and back again in the same time. Prove that

$$\frac{2}{c}=\frac{1}{a}+\frac{1}{b} \quad [Cal, 1889]$$

Let $3x$ = distance, in miles, from A to B

then since he rides x miles at the rate of a miles, $2x$ miles at the rate of $2b$ miles and $6x$ miles at the rate of $3c$ miles per hour, we have by § 233 (3),

$$\frac{x}{a}=\text{time in hours, taken to ride } \frac{1}{3}AB,$$

$$\frac{2x}{2b} \text{ or } \frac{x}{b}= \text{............} \frac{2}{3}AB,$$

and $\quad \dfrac{6x}{3c}$ or $\dfrac{2x}{c}$ = ... $2AB$,

and the first two times are together equal to the last; therefore

$$\frac{2x}{c}=\frac{x}{a}+\frac{x}{b},$$

divide by x, thus $\quad \dfrac{2}{c}=\dfrac{1}{a}+\dfrac{1}{b}$

Ex (vii) AB is a railway 220 miles long; three trains (P, Q, R) travel upon it at the rate of 25, 20 and 30 miles per hour respectively; P and Q leave A at 7 A.M. and 8-15 A.M. respectively, and R leaves

B at 10 30 A M When and where will P be equidistant from Q and R? [*Cal*, 1870]

Let x = time after R leaves B, when P is equidistant from Q and R.

Let the figure represent the positions of the trains at the required time.

A ———————————————— B
Q P R

Now to come to their respective positions, P takes $(3\frac{1}{2}+x)$ hours, Q takes $(2\frac{1}{4}+x)$ hours and R takes x hours, thus

$AQ=(2\frac{1}{4}+x)\times 20$ miles, $AP=(3\frac{1}{2}+x)\times 25$ miles, and $BR=30x$ miles,

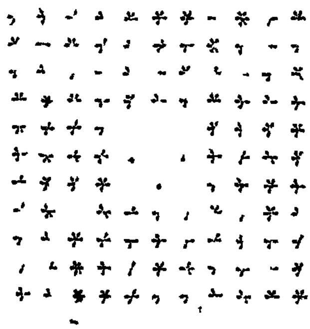
and therefore $AR=AB-BR=(220-30x)$ miles

And from the geometry of the figure, we have $2AP=AQ+AR$; therefore $2(3\frac{1}{2}+x)25=(2\frac{1}{4}+x)20+(220-30x)$; whence $x=1\frac{1}{2}$ hours Hence P will be at its present position $1\frac{1}{2}$ hours after R starts, *i.e.*, at 12 o'clock, and at a place whose distance from A

$$=AP=(3\frac{1}{2}+1\frac{1}{2})25 \text{ miles}=125 \text{ miles}.$$

Ex. (viii) An officer can form his men into a hollow square 5 deep, and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former; find the number of men [*Cal*, 1887]

[Note Problems relating to "*hollow squares*" will be best understood by the following illustration Suppose we want to know the number of men in a hollow square, 4 deep, having 11 men in the front In the annexed scheme, the asterisks mark the position of men in the hollow square, while the dots indicate the position of men necessary to fill the hollow at the centre, the asterisks and dots together forming a solid square equal in area to the hollow one It is clear that the number of men in the hollow square = the number of men in the solid square *minus* the number filling the square hollow at the centre Now this square, as the dots shew, has a side containing 3 men $=(11-2\times 4)$ men Therefore the number of men in the hollow square $=(11)^2-(11-2\times 4)^2=112$ Reasoning similarly we see that *the total number of men in a hollow square*



$$=a^2-(a-2b)^2,$$

where a represents the number of men in the front, and b the number of men forming the depth]

Let x = number of men in the front of the first square,

then $x-4$ = ... second . . . . ,

$x^2-(x-2\times5)^2$ = total number of men required,

also $(x-4)^2-(x-4-2\times6)^2$ = total number of men required,

$$\therefore\ x^2-(x-10)^2=(x-4)^2-(x-16)^2\ ;$$

whence $x=35$,

required number of men $=(35)^2-(25)^2=60\times10=600$

---

1. What number is that to which if you add 1, then multiply the sum by 2, next from the product subtract 3, and lastly divide the remainder by 4, the quotient will be the sum of 1, 2, 3 and 4

2 Find a number such that whether it be divided into 4 or 5 equal parts, the continued product of the parts shall be the same.

3 A market woman bought a certain number of eggs at 2 a penny and as many at 3 a penny, and sold them at the rate of 5 for 2*d* , shew that she will lose by the bargain If she loses 7*d*, find the number of eggs she bought

4 *A* and *B* being at play severally cut packs of cards so as to take off more than they left Now it so happened that *A* cut off twice as many as *B* left, and *B* cut off 7 times as many as *A* left How many cards did each cut ?

5 Three men *A*, *B* and *C* entered into partnership , *A* paid in as much as *B* and $\frac{1}{3}$ of *C* , *B* paid in as much as *C* and $\frac{1}{4}$ of *A* ; and *C* paid in £10 and $\frac{1}{5}$ of *A* , what did each contribute to the stock ?

6 A debt which might have been paid exactly with $5x$ half-sovereigns and x half-crowns, was paid out of a ten-pound note, and the change was to be equal to $15x$ half crowns and x half sovereigns. Find x and the amount of the debt

7 From each of 16 coins an artist filed the worth of half a crown, and then offered them in payment for their original value , but being detected the pieces were found to be really worth no more than 8 guineas What was the original value of each coin ?

8 A sum of 119*l* is to be divided omong 10 men, 32 women and 48 children if each man's share is to be equal to the shares of 2 women, and if 32 women are to have twice as much as the 48 children, how much will the several individuals receive ?

9 Divide a given sum of money between *A* and *B*, so that as often as *A* gets Rs 8, *B* will get Rs 5, and *A* will altogether get Rs 24 more than *B* ; what did each receive ?

10 A fish was caught, whose tail weighed 9 lbs , his head weighed as much as his tail and half his body , and his body weighed as much as his head and tail , what did the fish weigh ?

11. A farmer bought equal numbers of two kinds of sheep, one at £3 each, the other at £4 each, if he had expended his money equally in the two kinds, he would have had 2 sheep more than he did. Find how many of each kind he bought

12. A hundred gallons of liquid contains 70 per cent. of wine and the rest water How much wine should be added to the mixture to raise the proportion of wine to 80 per cent ?

13 Her Majesty Empress Victoria was born May 24, A D x, and Prince Albert was born August 26th in the same year, their united ages on the 26th August, 1849, amounted to 3 times the age of Prince Albert on the birthday next preceding his marriage, which took place February 10, 1840 In what year was each born ?

14 Find a number such that if you divide it by 13 or by 14, the remainder would be 1 and the difference between the quotients 1

15 A ship 40 miles from the shore springs a leak which admits $3\frac{3}{4}$ tons of water in 12 minutes, 60 tons would suffice to sink her, but the ship's pumps can throw out 12 tons of water in an hour, find the average rate of sailing so that she may reach the share just as she begins to sink.

16 A certain number of sovereigns, shillings and sixpences together amount to £8 6*s* 6*d*, and the amount of the shillings is a guinea less than that of the sovereigns and a guinea and a half more than that of the sixpences Find the number of each coin.

17 One cask contains a mixture of 12 gallons of wine and 18 gallons of water, another cask contains a mixture of 9 gallons of wine and 6 gallons of water, how many gallons must be drawn from each cask so as to produce a mixture containing 7 gallons of wine and 8 gallons of water ?

18. Wishing to buy a certain number of railway shares, I found that if I bought them in the E B Railway at Rs 400 a share, I should invest all my money, but if I bought them in the E I Railway at Rs 450 a share, I should not have money enough by Rs 2400 How much money had I to invest ?

19. The fore-wheel of a carriage makes 12 revolutions more than the hind-wheel in 130 yards, if the diameter of the latter be $\frac{3}{10}$ths as much again as the diameter of the former, find the circumference of each wheel

20 A fraudulent trader has two weights, one as much over one seer as the other is under it; by using the heavier weight, he bought 6 maunds of an article at 8 annas a seer, which he sold at the same rate by using the lighter weight, thus gaining Rs. 16 What were the weights ?

21. *A* and *B* are two railway stations 60 miles apart twenty minutes after a Passenger train has left *A* for *B* at the rate of 24 miles per hour, the station-master of *B* sends a Pilot Engine to

repair a disabled bridge, 18 miles off, the Pilot takes 1 hour 4 minutes to repair the bridge, and then starts back at $1\frac{1}{5}$ of its former rate, and reaches B just in time to avert a collision What was the original rate of the Pilot, and how far was the Passenger train from the bridge at the time when the Pilot first reached the bridge?

22 A watch has the second hand on the same axis as the other two, and the hands are all together at 12 o'clock, find when the minute hand and the second hand are next together

23 A train leaves B at 9 A M and runs to C at the rate of 15 miles an hour, and another train leaves A at noon, and running through B to C at 25 miles an hour, arrives half an hour later than the train from B, if the distance AB be 15 miles, find the distance from A to C

24 A carrier charges 3*d* each, for all parcels not exceeding a certain weight, and on heavier parcels he makes an additional charge for every 7 lbs above that weight The charge for half a cwt is 1*s* 3*d*, and the charge for 9 stones is 5 times that for 1 quarter. What is the scale of charges? (1 stone = 14 lbs)

25 An officer can form the men in his battalion, numbering 1152, into a hollow square 12 deep, of how many men does the front consist?

26 When flour costs 7*s* a bushel, the baker sold a loaf for 16*d*., when it rose to 10*s*. 6*d*, he sold a loaf of the same weight for 21*d*, the price of baking being the same in each case, find that price.

27. The duty on salt being raised 8 annas per maund, it is found that the consumption has fallen one-eighth, but the revenue, instead of increasing, has remained stationary, what was the duty at first?

28 The annual rent of a paddy-field consists of Rs 55 and a corn-rent, when paddy sells at Re 1 12 as a maund, the landlord gets at the rate of Rs 3 per bigha, when it sells at Rs 2 per maund, he gets at the rate of Rs $3\frac{1}{6}$ per bigha Required the amount of the corn rent and the area of the field

29 The expenses of a tram car company are fixed, and when it only sells three penny tickets for the whole journey, it loses 10 per cent, it then divides the route into 2 parts and sells two penny tickets for each part, thereby gaining 4 per cent and selling 3300 more tickets every week How many persons used the cars weekly under the old system?

30 If 18 tolahs of gold weigh 17 tolahs in water, and 8 tolahs of silver weigh 7 tolahs in water: find the quantities of gold and silver in a compound of gold and silver, which weighs 100 tolahs in air and 90 tolahs in water.

31 A vessel is filled with a mixture of spirit and water in which there is 70 per cent of spirit, 19 gallons are taken out, and the

vessel is filled up again with water; the proportion of spirit is now found to be 56 7 per cent, find how much the vessel contains.

32 A regiment can be drawn up into a hollow square 6 deep; and with the addition of 140 men to its ranks, it can be drawn up into the same square, the depth being now increased by 1; find the number of men in the regiment.

33 At a review of an army, the troops were drawn up in a solid mass 40 deep, when there were just one-fourth as many men in front as there were spectators, had the depth, however, been increased by 5 and the spectators were drawn up in a mass with the army, the number of men in front would have been 100 fewer than before. Determine the number of men of which the army consisted

34 Find a number such that if it be divided by a or by b, the remainder will be c, and the sum of the quotients will be c.

35 There are two kinds of coin, of which a and b pieces respectively are equivalent to £1, how many pieces of each kind must be taken so that c pieces together may be equivalent to £1?

36 A labourer is engaged for n days, on condition that he receives a pence for every day he works and pays b pence for every day he is idle, at the end of the time he receives m pence; how many days does he work and how many days is he idle?

37. A person has just a hours at his disposal; how far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

38 Two persons set out at the same time to meet each other, the one walking a miles per hour meets the other, who walks b miles per hour, at a place which is c miles from the mid-station of the road Required the length of the road

39. A merchant buys an article, subject to a duty of a per cent, and sells it at a loss of b per cent, but if he had sold it for Rs c more, he would have gained d per cent on his bargain What was the price of the article?

40. If a men or b boys can dig m acres in n days, shew that the number of boys whose assistance will be required to enable $(a-p)$ men to dig $(m+p)$ acres in $(n-p)$ days is

$$\frac{pb}{a}\left\{1+\frac{a}{m}\cdot\frac{m+n}{n-p}\right\}.$$

CHAPTER XX

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE

237 Simultaneous Equations If the equation

$$x+y=5 \tag{1}$$

be considered as an equation in *one* variable *viz*, x, its solution is

$$x=5-y \tag{a}$$

and we have seen that this is the *only solution* [§ 223] But if we consider it as an equation in *two* variables, *viz*, x and y, then it has an *infinite number of solutions*, for giving to y in (*a*) any value we please, we can get as many values of x as we like Thus corresponding to the values 1, 2, 3, 4, of y, we get for x the values 4, 3, 2, 1,

The same remark holds good in the case of any other *single* equation in *two* variables, as for instance

$$x-y=1 \tag{2}$$

But if it is known that the x and the y of (1) are the same as the x and the y of (2), then the two equations (1) and (2) are said to *hold together*, or in other words they are then called SIMULTANEOUS EQUATIONS, and in this case they shall have *definite solutions*, as we shall presently shew

The *general form* of all such equations in two variables, is

$$ax+by+c=0 \tag{3}$$

This can be very easily shewn as in § 222

238 Solution of Simultaneous Equations in two variables Let us solve equations (1) and (2) of § 237 From (1), $x=5-y$ and from (2), $x=1+y$, and since these two xs *are by definition equal*, we have $5-y=1+y$, an equation *in one variable* only Solving this we get $y=2$ and from $x=5-y$ or from $x=1+y$, we get, by substituting the value of y, $x=3$

Hence it is clear that to solve linear equations in two variables, all that we have to do is to *eliminate* (*i e*, cause to disappear) *one of the variables between the given equations, thus finding a single, equation in one variable*, for we can solve this equation [Chapter XVIII], and by substituting the value of this variable in either of the proposed equations, reduce it also to one in *one variable* and thus solve it

We shew in the next four articles how the *elimination* may be effected

239 First Method of Elimination—Cross Multiplication The principle to be followed is to *make the coefficients of the variable to be eliminated the same in both the equations*, if they are not already so This is in general done by multiplying the first equation by the co-efficient of the variable in the second equation, and the second equation by the coefficient of the *same* variable in the first equation Then elimination is effected by *adding* or *subtracting* these equations according as the variable appears with *different* signs or with the *same* sign.

Examples.

1 **Solve** $$x+y=a \qquad (1),$$
$$x-y=b \qquad (2)$$

Here the coefficients are the same in both equations, hence by addition and subtraction, we have

$$2x=a+b \text{ or } x=\tfrac{1}{2}(a+b), \text{ and } 2y=a-b \text{ or } y=\tfrac{1}{2}(a-b)$$

2 **Solve** $$5x+3y=78 \qquad (1),$$
$$x-13y=2 \qquad (2).$$

Here the coefficient of x in (2) is made equal to that of x in (1) by simply multiplying (2) by 5, thus

$$5x-65y=10 \qquad (3),$$

subtract (3) from (1), thus

$$3y+65y=68, \text{ whence } y=1$$

Substitute this value of y in (1) or (2), say in (2), thus

$$x-13\times 1=2, \text{ whence } x=15$$

3 **Solve** $$5x+2y=16 \qquad (1),$$
$$2x+9y=31 \qquad (2)$$

Multiply (1) by 2, the coefficient of x in (2), thus

$$10x+4y=32 \qquad (3),$$

multiply (2) by 5, the coefficient of x in (1), thus

$$10x+45y=155 \qquad (4);$$

now subtract (3) from (4), and x is eliminated, and we get

$$45y-4y=123, \text{ whence } y=3$$

Substituting this value of y in (1) or (2), say in (1), we get

$$5x+6=16, \text{ whence } x=2$$

Note The same result of course would have been arrived at by eliminating y first instead of x.

4 Solve
$$ax+by=c \quad (1),$$
$$a'x-b'y=c' \quad (2).$$

Multiply (1) by b', thus
$$ab'x+bb'y=b'c \quad (3);$$
multiply (2) by b, thus
$$a'bx-bb'y=bc' \quad (4),$$
add (3) and (4) together, thus
$$(ab'+a'b)x=b'c+bc',$$
whence
$$x=\frac{b'c+bc'}{ab'+a'b}.$$
Substitute in (1),
$$by=c-ax=c-\frac{a(b'c+bc')}{ab'+a'b}=\frac{b(a'c-ac')}{ab'+a'b},$$
whence
$$y=\frac{a'c-ac'}{ab'+a'b}$$

5 Solve
$$8x-13y=66 \quad (1),$$
$$12x+3y=54 \quad (2)$$
Multiply (1) by 3, $\quad 24x-39y=198 \quad (3),$

" (2) by 2, $\quad 24x+6y=108 \quad (4),$

subtract (3) from (4), $\quad 39y+6y=-90$, or $y=-2$,

. from (1), $\quad 8x=66+13y=40$, or $x=5$

6 Solve
$$\frac{x}{3}+\frac{y}{5}=7 \quad (1),$$
$$\frac{x}{4}+\frac{2y}{3}=13 \quad (2)$$
Multiplying (1) by $\frac{1}{4}$, $\quad \frac{1}{12}x+\frac{1}{20}y=\frac{7}{4}$,

" (2) by $\frac{1}{3}$, $\quad \frac{1}{12}x+\frac{2}{9}y=\frac{13}{3}$,

whence by subtraction, $\quad \frac{2}{9}y-\frac{1}{20}y=\frac{13}{3}-\frac{7}{4}$,

multiplying by 9×20, $\quad 40y-9y=780-315$,

whence $\quad y=15$,

. from (1), $\quad \frac{x}{3}=7-\frac{15}{5}=4$ or $x=12$

We might have solved these equations otherwise, thus —

from (1), $\quad 5x+3y=105$,

from (2), $\quad 3x+8y=156$,

now proceed as in other examples given above

Solve the equations

7. $x+y=59,\quad x-y=29.$

8. $40x+y=43,\quad 25x-8y=1$

9. $3x=4y,\quad 10x-8y=48.$

10. $6x+5y=31,\quad 15x+2y=25$

11. $5x+13y=41,\quad 12x+2y=40$

12. $24x-7y=51,\quad 30x-5y=75.$

13. $x+2y=24,\quad y-2x=2$

14. $5x-3y=1,\quad 5y-3x=9$

15. $3x=26-2y,\quad 3y=5+4x$

16. $\frac{1}{3}x+\frac{1}{4}y=9,\quad \frac{1}{5}y+\frac{1}{6}x=6$

17. $\frac{x}{5}+5y=51,\quad \frac{y}{5}+5x=27$

18. $\frac{x}{4}+\frac{y}{3}=10,\quad \frac{x}{3}+\frac{y}{4}=11$

19. $x+ay=b,\quad ax-by=c$

20. $bx+ay=b,\quad ax-by=a$

21. $\frac{x}{a}=\frac{y}{b},\quad \frac{x}{b}+\frac{y}{a}=c.$

22. $x+y=a+b,\quad \frac{x+a}{y+b}=\frac{b}{a}.$

23. $\frac{x}{a}+\frac{y}{b}=1,\quad \frac{y}{a}+\frac{x}{b}=1.$

240 Second Method of Elimination—Substitution. From one of the given equations find one of the variables in terms of the other, and substitute this value in the other equation

Example. Solve $5x+2y=16$ (1),

$2x+9y=31$ (2), [Ex 3, § 239].

From (1), $x=\frac{16-2y}{5}$ (3),

substitute this value of x in (2), thus

$$2\times\frac{16-2y}{5}+9y=31,$$

$$32-4y+45y=155, \text{ whence } y=3,$$

and from (3), $x=\frac{16-2\times 3}{5}=2.$

[Solve the equations of § 239 by this method.]

241 Third Method of Elimination—Comparison Express one of the variables in terms of the other from each of the equations, and then *equate* these values.

Examples

1 Solve $5x+2y=16$ (1),

$2x+9y=31$ (2) [Ex 3, § 239]

From (1), $$x=\frac{16-2y}{5} \quad (3),$$

from (2), $$x=\frac{31-9y}{2} \quad (4),$$

from (3) and (4), $$\frac{16-2y}{5}=\frac{31-9y}{2},$$

or $32-4y=155-45y$, whence $y=3$,

either from (3) or (4), $x=2$

[Solve the equations of § 239 by this method]

In the following examples, most of the equations are given in fractional forms To solve them, proceed, as a general rule, by first reducing them to the *general form* [§ 237, (3)]

Solve the equations

2. $2x-9y=11,\quad 3x-12y=15$

3. $10x-9y=1,\quad 11y-12x=1$

4. $x=42-8y,\quad y=84-8x$

5. $10x+13y=210,\quad 13x+10y=204$

6. $\frac{2x}{3}=10-\frac{y}{2},\quad \frac{19y}{4}=5x-7$

7. $12x+\frac{y}{5}=98,\quad 12y-\frac{x}{4}=118$

8. $\frac{x}{3}=5-2y,\quad \frac{2x-1}{5}-y+1=0$

9. $\frac{x+10}{3}=4y-3,\quad \frac{y-2}{7}=x-5$

10. $2x-\frac{y-3}{5}=4,\quad 3y+\frac{x-2}{3}=9$

11. $\frac{2x+3y}{6}+\frac{x}{3}=8,\quad \frac{7y-3x}{2}-y=11$

12. $\frac{x}{8}+\frac{y}{9}=42,\quad \frac{x}{9}+\frac{y}{8}=43$

13. $\frac{x}{4}+\frac{y}{3}=4+\frac{x+y}{6},\quad \frac{x}{3}-\frac{y}{4}=3+\frac{x-y}{6}$

Solve the equations

14. $4x-\frac{15-x}{2}=\frac{30y}{12}$, $15x-8y=35-\frac{2x+5y}{5}$

15. $\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{2}$, $\frac{2y+4}{3}-\frac{2x+y}{8}=\frac{x+13}{4}$

16. $2-\frac{\frac{5x}{y}-3}{4}=7-\frac{4x+3}{2y}$, $11y+\frac{6-\frac{y}{3}}{5}=35-\frac{\frac{x}{5}+4}{7}$.

17. $ax+by+c=0$, $a'x+b'y+c'=0$

18. $(a+b)x+(a-b)y=m$, $(a-b)x-(a+b)y=n$

19. $a(x-y)-b(x+y)=c(x-y)-d(x+y)=2$

20. $\frac{x}{a}-\frac{y}{b}=m$, $\frac{x}{c}+\frac{y}{d}=n$.

21. $\frac{x}{a+b}+\frac{y-1}{a-b}=0$, $\frac{x}{a-b}-\frac{y+1}{a+b}=0$.

22. $\frac{2}{y}+\frac{9}{x}=\frac{11}{xy}$, $\frac{4}{y}+\frac{1}{x}=\frac{5}{xy}$

23. $\frac{x}{a}+\frac{y}{b}=1$, $\frac{b}{x}+\frac{a}{y}=\frac{a^3}{bxy}$

24. $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$, $\frac{1}{x}-\frac{1}{y}=\frac{1}{b}$.

25. $\frac{4}{5x}+\frac{5}{6y}=5\frac{11}{15}$, $\frac{5}{4x}-\frac{4}{5y}=\frac{11}{20}$

26. $\frac{a}{x}+\frac{1}{y}=m$, $\frac{1}{x}+\frac{b}{y}=n$

27. $\frac{a}{x}+\frac{b}{y}=m$, $\frac{c}{x}+\frac{d}{y}=n$

28. $2x+3y=8xy$, $5x-y=3xy$

29. $\frac{x}{2}+\frac{y}{3}=\frac{3}{20}xy$, $\frac{y}{2}-\frac{x}{3}=\frac{2}{45}xy$.

30. $\frac{x+y}{5}+\frac{x-y}{4}=6$, $\frac{x+y}{4}+\frac{x-y}{5}=6\frac{8}{20}$

31. $\frac{57}{x+y}+\frac{6}{x-y}=5$, $\frac{38}{x+y}+\frac{21}{x-y}=9$

Solve the equations:

32. $\dfrac{5x}{3}+\dfrac{2}{5y}=7,\quad \dfrac{7x}{6}-\dfrac{1}{10y}=3.$

33. $\dfrac{b}{ax}+\dfrac{ay}{b}=a+b,\quad \dfrac{a}{x}+by=a^2+b^2.$

34. $\dfrac{x+1}{y}=\dfrac{1}{2},\quad \dfrac{x}{y+3}=\dfrac{1}{3}.$

35. $\dfrac{x+y}{x-y}=9,\quad \dfrac{2x-y}{3x+8}=\dfrac{6}{23}.$

36. $\dfrac{a}{b+y}=\dfrac{b}{a-x},\quad \dfrac{c}{d-x}=\dfrac{d}{c+y}.$

37. $(y+5)(x+4)=116+xy,\quad (y+4)(x+5)=113+xy.$

38. $(x+3)(y+8)-135=(x-2)(y-3),\quad \dfrac{5y}{3x+9}=\dfrac{4}{3}.$

39. $\dfrac{x+3}{x-8}=\dfrac{y+1}{y-4},\quad 10x-13y=4.$

40. $\dfrac{(m+n)x}{5}-\dfrac{(m-n)y}{3}=4mn,\quad 3x+\dfrac{5(m+n)y}{m-n}=30(m+n).$

41. $\dfrac{x}{a}+\dfrac{y}{b}=1-\dfrac{x}{c},\quad \dfrac{x}{b}+\dfrac{y}{a}=1+\dfrac{y}{c}.$

42. $axy=c(bx+ay),\quad bxy=c(ax-by).$

43. $2y+\dfrac{3}{x}-4=5y+\dfrac{12}{x}+2=y-\dfrac{2}{x}+4.$

44. $\dfrac{x-a}{y-a}=\dfrac{a-b}{a+b},\quad \dfrac{x}{y}=\dfrac{a^3-b^3}{a^3+b^3}.$

45. $\dfrac{x}{b+c}+\dfrac{y}{a+c}=2,\quad \dfrac{ax-by}{(a-b)c}=1.$

46. $x-\dfrac{2y-x}{23-x}=20-\dfrac{59-2x}{2},$

$y+\dfrac{y-3}{x-18}=30-\dfrac{73-3y}{3}.$ [See § 224, Ex. 52.]

47. $4x-\dfrac{11y-\dfrac{y}{2}}{17-3x}=20-\dfrac{103-8x}{2},$

$8y+\dfrac{3y-3}{5x-10}=$ [illegible] $-\dfrac{147-21y}{3}.$ [See § 224, Ex. 52.]

48. $3y+11=\dfrac{4x^2-\text{[illegible]}y)}{x-\text{[illegible]}}+31-4x,$

$(x+7)(y-2)+$ [illegible] $y-(x+1)(y-1).$

Solve the equations

49 $\frac{2}{3}\left(x-\frac{3}{5}y\right)+\frac{x+\frac{y}{5}}{6}=\frac{1}{3}-\frac{1}{2}\left\{\frac{\frac{4}{5}y-2}{6}-(x-y)\right\}$,

$x-2y-\frac{3y-5x}{2}=\frac{11}{2}(x+y)+3(x-y)$

50. $xy+3y=20$,
$5y-4 \;=2xy$

51 $\frac{x}{2+x}+\frac{y}{2+y}=\frac{x+y}{2y}$,
$\frac{x}{y}-\frac{x-y}{x}=\frac{y}{x}$

52 $\frac{1}{x+\frac{1}{y-\frac{5}{x}}}=\frac{1}{x-\frac{1}{y-\frac{7}{x}}}$, $\frac{1}{y}\left(1-\frac{1}{x}\right)=1$

53 $2x+\cdot 4y=1\cdot 2$,
$3\cdot 4x-\cdot 02y=\cdot 01$.

54 $\frac{\cdot 2}{x}+\frac{3}{y}=\frac{1}{4}$,
$\frac{5}{x}-\frac{\cdot 1}{y}=\frac{1\cdot 2}{3}$

55 $2\cdot 4x+\cdot 32y-\frac{\cdot 36x-\cdot 05}{5}=\cdot 8x+\frac{2\cdot 6+\cdot 005y}{\cdot 25}$, $\frac{\cdot 04y+\cdot 1}{\cdot 3}=\frac{\cdot 07x-\cdot 1}{6}$

56 $3^x 9^y=27$,
$4^x 8^y=32$

57 $2^x 4^y=32$,
$\frac{3^x}{9^y}=3$

58 $x^y=y^x$,
$x^3=y^2$.

***242** **Fourth method of Elimination—Indeterminate Multipliers** This method is best illustrated by an example

Example Solve $5x+2y=16$ (1),
$2x+9y=31$ (2) [Ex 3, § 239].

Multiply (2) by λ, and add the product to (1), thus

$$5x+2y+(2x+9y)\lambda=16+31\lambda,$$

or bracketing the co-efficients of x and y,

$$(5+2\lambda)x+(2+9\lambda)y=16+31\lambda \qquad (3).$$

Now, to find x, let such value be given to λ as will cause y to disappear from (3), which will be done by putting

$$2+9\lambda=0, \text{ or taking } \lambda=-\tfrac{2}{9} \qquad (4)$$

Thus (3) reduces to

$$(5+2\lambda)x=16+31\lambda$$

whence

$$x=\frac{16+31\lambda}{5+2\lambda},$$

or from (4),

$$=\frac{16-\frac{31\times 2}{9}}{5-\frac{2\times 2}{9}}=\frac{82}{41}=2$$

To find y, we put the coefficient of $x=0$ in (3), and thus find

$$\lambda=-\frac{5}{2}, \text{ and } \quad y=\frac{16-\frac{31\times 5}{2}}{2-\frac{9\times 5}{2}}=\frac{-123}{-41}=3$$

REMARK This method is very useful and requires careful study Its advantage consists in finding *a particular variable without finding the other variables* of the equations

243 Solution of equations in more than two variables. In solving these equations the principle of § 238 is to be followed viz, that of eliminating all the variables *except one* by the several methods given, thus finding a single equation in one variable, &c.

Examples

1 Solve

$$3x+5y-z=26 \qquad (1),$$

$$x-3y+4z=20 \qquad (2),$$

$$4y+3z-12x=8 \qquad (3)$$

Multiply (1) by 4, and add the product to (2), thus

$$13x+17y=124 \qquad (4)$$

Multiply (1) by 3, and add the product to (3), thus

$$-3x+19y=86 \qquad (5).$$

Now (4) and (5) are two equations in two variables, solving which [§ 238], we get $x=3$ and $y=5$ Substitute these values in *any one* of the given equations, say (1), and we have

$$3\times 3+5\times 5-z=26, \text{ whence } z=8$$

Solve the equations

2 $2x+3y+4z=19,$
$8x+5y-3z=12,$
$5x-6y+3z=13$

3 $x+\frac{1}{2}(y+z)=27,$
$y+\frac{1}{2}(z+x)=29,$
$z+\frac{1}{2}(x+y)=30$

Solve the equations

4. $4(y-x)+22=5z,$
$3z+4x-2=6y.$
$3y-z+14=10x$

5 $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=9,$
$\frac{y}{2}+\frac{z}{3}-\frac{x}{4}=5,$
$\frac{x}{2}+\frac{z}{3}=8$

6. $x+2y=21,$
$x+2z=32,$
$z+2x=22$

7 $x+y=c,$
$y+z=a,$
$z+x=b$

8 $3x+4y=63,$
$8x+5z=60,$
$3z-\ y=0$

9 $\frac12 x+y+\frac12 z=162,\ \frac14 x+\frac18 y=26,\ 5y=4z$

10 $x+y+z=46,\ 13x-10y=65,\ 11y-5z=53$

11. $5x-3y=4,\ 8y-9z=2,\ 15z-16x=10$

12. $y+\frac13 z=\frac15 x+5,$
$\frac14(x-1)-\frac15(y-2)=\frac1{10}(z+3),$
$x-\frac14(2y-5)=1\frac34-\frac1{12}z$

13. $bz+cy=a,$
$az+cx=b,$
$ay+bx=c$

Note 1 When all the variables, with or without coefficients, occur in the denominators only, it is well to find them by finding their **reciprocals**, as shewn below

14 **Solve** $\frac{2}{x}+\frac{3}{y}=1,\ \frac{2}{y}+\frac{3}{z}=\frac{17}{24},\ \frac{2}{z}+\frac{3}{x}=1$

Assume $\frac{1}{x}=u,\ \frac{1}{y}=v,\ \frac{1}{z}=w,$ then the proposed equations are

$$2u+3v=1,\ 2v+3w=\tfrac{17}{24},\ 2w+3u=1$$

Solving these we get $u=\frac14,\ v=\frac16,\ w=\frac18$,

whence $x=4,\ y=6,\ z=8$

Solve the equations

15. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1,$
$\frac{1}{2x}+\frac{1}{4y}+\frac{4}{z}=1,$
$\frac{5}{3x}+\frac{3}{4y}-\frac{1}{2z}=1$

16 $\frac{a}{x}+\frac{b}{y}=m,$
$\frac{b}{y}+\frac{c}{z}=n,$
$\frac{c}{z}+\frac{a}{x}=p.$

Note 2 Sometimes equations may be transformed into others in which their reciprocals are easily found out

17 Solve $\dfrac{xy}{x+y}=1,\ \dfrac{xz}{x+z}=2,\ \dfrac{yz}{y+z}=3$

Invert the given equations, thus

$$\frac{x+y}{xy}=1,\ \frac{x+z}{xz}=\frac{1}{2},\ \frac{y+z}{yz}=\frac{1}{3},$$

or $$\frac{1}{x}+\frac{1}{y}=1,\ \frac{1}{x}+\frac{1}{z}=\frac{1}{2},\ \frac{1}{y}+\frac{1}{z}=\frac{1}{3} \qquad (a),$$

add together, thus

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{11}{6} \text{ or } \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{11}{12} \qquad (b)$$

From (b) subtracting each of the equations (a), we get

$$\frac{1}{x}=\frac{7}{12},\ \frac{1}{y}=\frac{5}{12},\ \frac{1}{z}=-\frac{1}{12}, \text{ whence } x=1\tfrac{5}{7},\ y=2\tfrac{2}{5},\ z=-12$$

Solve the equations

18 $xy=x+y,\ yz=y+z,\ zx=z+x$

19 $\dfrac{xy}{x+y}=\dfrac{1}{5},\ \dfrac{xz}{x+z}=\dfrac{1}{6},\ \dfrac{yz}{y+z}=\dfrac{1}{7}$

20 $\dfrac{4xy}{2x-3y}+2=0,\ \dfrac{5yz}{4y+6z}=\dfrac{15}{13}\ \ \dfrac{2zx}{4z-3x}=\dfrac{2}{3}$

21 $15(x+y)=8xy,\ 40(y+z)=13yz,\ 24(z+x)=11zx$

22 $a(y+z)=lyz,\ b(z+x)=mzx,\ c(x+y)=nxy$

23 $xyz=a(zx+xy-yz)=b(xy+yz-zx)=c(yz+zx-xy)$

The theorem of the next article is sometimes very useful in the solution of simple equations

*244 Theorem *If*

$$ax+by+cz=0 \qquad (1),$$
$$a'x+b'y+c'z=0 \qquad (2),$$

then $$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b} \qquad (3)$$

Multiply (1) by c' and (2) by c, and subtract, thus

$$(ac'-a'c)x+(bc'-b'c)y=0,$$

transpose, thus $(bc'-b'c)y=-(ac'-a'c)x=(a'c-ac')x,$

i e, $\qquad (ca'-c'a)x=\ \ (bc'-b'c)y,$

or $$\frac{x}{bc'-b'c}=\ \ \frac{y}{ca'-c'a} \qquad (4)$$

Again, multiply (1) by a' and (2) by a, and subtract; thus

$$\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b} \qquad (5).$$

Therefore from (4) and (5),

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}$$

Another proof by the Method of Indeterminate Multipliers

Divide (1) and (2) by z, thus

$$a\frac{x}{z}+b\frac{y}{z}+c=0,\ a'\frac{x}{z}+b'\frac{y}{z}+c'=0,$$

or putting $\frac{x}{z}=u$ and $\frac{y}{z}=v$,

we have

$$au+bv+c=0 \qquad (\text{i}),$$

$$a'u+b'v+c'=0 \qquad (\text{ii}).$$

Multiply (i) by λ, and add the product to (ii); thus

$$(\lambda a+a')u+(\lambda b+b')v+(\lambda c+c')=0 \qquad (\text{iii}).$$

Now since λ is an arbitrary multiplier, we may so choose its value that the co-efficient of v may vanish, that is, we put

$$\lambda b+b'=0 \qquad (\text{iv});$$

and therefore (iii) reduces to

$$(\lambda a+a')u+(\lambda c+c')=0,$$

whence $u=-\dfrac{\lambda c+c'}{\lambda a+a'}$;

from (iv),

$$u=-\frac{-\frac{b'}{b}c+c'}{-\frac{b'}{b}a+a'}=\frac{bc'-b'c}{ab'-a'b}.$$

And by putting $\lambda a+a'=0$, we get as above

$$v=-\frac{\lambda c+c'}{\lambda b+b'}=\frac{ca'-c'a}{ab'-a'b}.$$

Now replacing u and v by their values, we have

$$\frac{x}{z}=\frac{bc'-b'c}{ab'-a'b},\ \frac{y}{z}=\frac{ca'-c'a}{ab'-a'b};$$

or

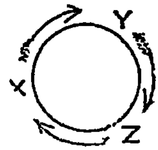
$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}.$$

This Result is sometimes called the **Formula of Cross Multiplication**

REMARK The following device will help the student in remembering the relation (3) :—It will be seen that

x has for denominator a combination of coefficients of y and z,
y ,, ,, ,, ,, z ,, x,
z ,, ,, ,, ,, x ,, y,

we have said in the second line z and x, and not x and z, for in forming the denominators out of the coefficients of x, y and z, these *viz*, x, y and z are to be taken as following one another in *cyclic order* *i e*, x being followed by y, y by z, z by x, etc, as shewn in the annexed diagram [See § 113]



The combination is formed thus —

$$\begin{array}{ccccc} x & y & z & x & y \\ a & b & c & a & b \\ a' & b' & c' & a' & b' \end{array}$$

For x—Begin at b *under* y in the second column, proceed *diagonally from left to right to* c' *under* z, in the second row, and take the product bc', then go from c' to b', *i e*, go in a direction from *right to left*, and let this be a hint of affecting the next product with the sign $-$, to obtain this product proceed from b' to c, again diagonally from left to right and take the product $b'c$ and put the negative sign, as we have said, before it Similarly for y and z

Examples

1. **Solve** $2x+3y+9=0,$
$4x+5y+13=0$

We have $2x+3y+9\cdot1=0,$
$4x+5y+13\cdot1=0,$

$$\frac{x}{3\times13-5\times9}=\frac{y}{9\times4-13\times2}=\frac{1}{2\times5-4\times3},$$

or $$\frac{x}{-6}=\frac{y}{10}=\frac{1}{-2},$$

whence $$x=\frac{-6}{-2}=3,\ y=\frac{10}{-2}=-5$$

2. Solve
$$ax+by+c=0,$$
$$a'x+b'y+c'=0 \quad [\S 241, \text{ Ex } 17]$$
Since $c=c\cdot 1$ and $c'=c'\cdot 1$, we have here $z=1$,
$$\therefore \quad \frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b},$$
whence
$$x=\frac{bc'-b'c}{ab'-a'b}, \quad y=\frac{ca'-c'a}{ab'-a'b}.$$

REMARK. The student is recommended to solve some of the equations of §§ 239 and 241 by applying this theorem.

3. Solve
$$3x+y-5z=0 \qquad (a),$$
$$7x-3y-9z=0 \qquad (b),$$
$$5x-3y+13z=12\tfrac{2}{3} \qquad (c).$$
From (a) and (b), we get
$$\frac{x}{1(-9)-(-3)(-5)}=\frac{y}{-5\times 7-(-9)\times 3}=\frac{z}{3(-3)-7\times 1},$$
or
$$\frac{x}{-24}=\frac{y}{-8}=\frac{z}{-16},$$
i.e.,
$$\frac{x}{3}=y=\frac{z}{2}=l \text{ say},$$
$$\therefore \quad x=3l,\ y=l,\ z=2l \qquad (d).$$
Substitute these values in (c), thus
$$15l-3l+26l=12\tfrac{2}{3},$$
whence
$$l=\tfrac{1}{3},$$
$\therefore$ from (d),
$$x=1,\ y=\tfrac{1}{3},\ z=\tfrac{2}{3}.$$

4. Solve
$$x+y+z=a+b+c \qquad (a),$$
$$ax+by+cz=bc+ca+ab \qquad (b),$$
$$(b-c)x+(c-a)y+(a-b)z=0 \qquad (c).$$
We cannot apply the theorem here, unless two of the proposed equations are reduced to the forms (1) and (2).

From (b),
$$(ax-ca)+(by-ab)+(cz-bc)=0,$$
or
$$a(x-c)+b(y-a)+c(z-b)=0 \qquad (d),$$
and from (a),
$$(x-c)+(y-a)+(z-b)=0 \qquad (e),$$
$\therefore$ from (d) and (e), we have
$$\frac{x-c}{b\times 1-1\times c}=\frac{y-a}{c\times 1-1\times a}=\frac{z-b}{a\times 1-1\times b},$$
or
$$\frac{x-c}{b-c}=\frac{y-a}{c-a}=\frac{z-b}{a-b}=k \text{ suppose},$$
whence
$$x=c+k(b-c),\ y=a+k(c-a),\ z=b+k(a-b) \qquad (f)$$

Substitute these values in (e), thus

$$(b-c)\{c+\lambda(b-c)\}+(c-a)\{a+\lambda(c-a)\}+(a-b)\{b+\lambda(a-b)\}=0,$$

or $\qquad \lambda\{(b-c)^2+(c-a)^2+(a-b)^2\}=a^2+b^2+c^2-bc-ca-ab,$

whence $\qquad \lambda=\frac{1}{2}$,

$\therefore$ from (f), $\qquad x=\frac{1}{2}(b+c),\ y=\frac{1}{2}(c+a),\ z=\frac{1}{2}(a+b)$

Solve the equations

5 $\quad 4x+3y+2z=0,$
$\quad 3x+5y+4z=0,$
$\quad 2x+y+3z=54$

6 $\quad 3x+4y-10z=0,$
$\quad 5x-8y+10z=0,$
$\quad 2x+6y+7z=52$

7. $\quad \frac{1}{3}x-\frac{2}{5}y+\frac{4}{9}z=0,$
$\quad \frac{3}{5}y-\frac{5}{4}x-\frac{9}{5}z=0,$
$\quad \frac{5}{12}x-\frac{3}{16}y+\frac{28}{9}z+1\frac{1}{15}=0$

8 $\quad x+y+z=0,$
$\quad (b+c)x+(c+a)y+(a+b)z=0,$
$\quad bcx+cay+abz=1$

9 $\quad x+y+z=0,$
$\quad ax+by+cz=0,$
$\quad bcx+cay+abz+(b^2-c^2)(c^2-a^2)(a^2-b^2)=0$

10 $\quad x+y+z=a+b+c,\ bx+cy+az=cx+ay+bz=a^2+b^2+c^2$

11 $\quad cx+ay+bz=bx+cy+az=0,$
$\quad ax+by+cz=(b-c)^2+(c-a)^2+(a-b)^2$

12 $\quad x+y+z=a+b+c,\ ax+by+cz=a^2+b^2+c^2,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$

13 $\quad ax+cy+bz=cx+by+az=bx+ay+cz=a^3+b^3+c^3-3abc$

14 $\quad (b-c)x+(c-a)y+(a-b)z=0$
$\quad a(b^2-c^2)x+b(c^2-a^2)y+c(a^2-b^2)z=0,$
$\quad a(b-c)x+b(c-a)y+c(a-b)z=(b-c)(c-a)(a-b)$

15 $\quad ax+by+cz=0\quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0,\quad \frac{x-a}{b-c}+\frac{y-b}{c-a}+\frac{z-c}{a-b}=0$

*245 **More Examples** We shall in this Article solve some equations by way of illustration, giving others for exercise, some of which will involve more than three variables *

* In this article, we have given some simple examples of Quadratics *reducible* to Simple Equations, as they are sometimes set at the Entrance Examination (*vide* Entrance Paper, 1883]

Examples.

1. Solve
$$a_1x+b_1y+c_1z=d_1 \qquad (1),$$
$$a_2x+b_2y+c_2z=d_2 \qquad (2),$$
$$a_3x+b_3y+c_3z=d_3 \qquad (3).$$

Multiply (2) by λ and (3) by μ, and add to (1); thus
$$(a_1+\lambda a_2+\mu a_3)x+(b_1+\lambda b_2+\mu b_3)y+(c_1+\lambda c_2+\mu c_3)z$$
$$=d_1+\lambda d_2+\mu d_3 \qquad (4)$$

To find x, give such values to λ and as μ will cause the coefficients of y and z to vanish, i.e. put
$$\left.\begin{array}{l} b_1+\lambda b_2+\mu b_3=0 \\ c_1+\lambda c_2+\mu c_3=0 \end{array}\right\} \qquad (5),$$
which reduces (4) to
$$(a_1+\lambda a_2+\mu a_3)x=d_1+\lambda d_2+\mu d_3,$$
$$\therefore\ x=\frac{d_1+\lambda d_2+\mu d_3}{a_1+\lambda a_2+\mu a_3} \qquad (6).$$

Thus x is found in terms of λ and μ whose values can be determined from (5). By § 244, we have
$$\frac{1}{b_2c_3-b_3c_2}=\frac{\lambda}{b_3c_1-b_1c_3}=\frac{\mu}{b_1c_2-b_2c_1};$$
whence
$$\lambda=\frac{b_3c_1-b_1c_3}{b_2c_3-b_3c_2},\ \mu=\frac{b_1c_2-b_2c_1}{b_2c_3-b_3c_2};$$
$\therefore$ from (6),
$$x=\frac{d_1+d_2\dfrac{b_3c_1-b_1c_3}{b_2c_3-b_3c_2}+d_3\dfrac{b_1c_2-b_2c_1}{b_2c_3-b_3c_2}}{a_1+a_2\dfrac{b_3c_1-b_1c_3}{b_2c_3-b_3c_2}+a_3\dfrac{b_1c_2-b_2c_1}{b_2c_3-b_3c_2}}$$
$$=\frac{d_1(b_2c_3-b_3c_2)+d_2(b_3c_1-b_1c_3)+d_3(b_1c_2-b_2c_1)}{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)}.$$

2. Given
$$ax+by+cz=m \qquad (1),$$
$$a^2x+b^2y+c^2z=m^2 \qquad (2),$$
$$a^3x+b^3y+c^3z=m^3 \qquad (3),$$
to find x.

Multiply (2) by λ and (3) by μ, and add to (1); thus
$$ax(1+\lambda a+\mu a^2)+by(1+\lambda b+\mu b^2)+cz(1+\lambda c+\mu c^2)$$
$$=m(1+\lambda m+\mu m^2) \qquad (4)$$

To find x, put
$$1+\lambda b+\mu b^2=0,\ 1+\lambda c+\mu c^2=0\ ,$$
whence by § 244,
$$\lambda=-\frac{b+c}{bc},\ \mu=\frac{1}{bc} \tag{5}$$
Therefore from (4) and (5), we get
$$x=\frac{m}{a}\,\frac{1+\lambda m+\mu m^2}{1+\lambda a+\mu a^2}$$
$$=\frac{m}{a}\,\frac{bc-(b+c)m+m^2}{bc-(b+c)a+a^2}=\frac{m(m-b)(m-c)}{a(a-b)(a-c)}.$$

REMARK. From the *symmetry* of the equations
$$y=\frac{m(m-c)(m-a)}{b(b-c)(b-a)},\ z=\frac{m(m-a)(m-b)}{c(c-a)(c-b)}.$$

3. **Solve** $yz=a^2,\ zx=b^2,\ xy=c^2$

Multiply the equations together, thus
$$x^2y^2z^2=a^2b^2c^2, \text{ whence } xyz=abc\ ,$$
divide this equation by each of the proposed equations, thus
$$x=\frac{bc}{a},\ y=\frac{ca}{b},\ z=\frac{ab}{c}$$

4. **Solve** $x+y=a,\ xy=b^2$

Square the first equation, and subtract 4 times the second, thus
$$x^2+y^2+2xy-4xy=a^2-4b^2,$$
or $(x-y)^2=a^2-4b^2,\qquad x-y=\sqrt{a^2-4b^2}$

hence from this equation and the first equation
$$x=\tfrac{1}{2}(a+\sqrt{a^2-4b^2}),\ y=\tfrac{1}{2}(a-\sqrt{a^2-4b^2})$$

4 (1) **Solve**
$$x+2y+3z=14 \tag{1}$$
$$3x+4y+12z=47 \tag{2}$$
$$2x+7y+5z=31 \tag{3}$$
Eliminate x between (1) and (2) *i e*, multiply (1) by 3 and subtract (2), thus
$$2y-3z=-5, \text{ or } 2y-3z+5=0 \tag{4}$$
Again, eliminate x between (1) and (3), *i e*, multiply (1) by 2, and subtract from (3), thus
$$3y-z=3, \text{ or } 3y-z-3=0 \tag{5}$$
Hence from (4) and (5) by *Cross Multiplication* [§ 244],
$$\frac{y}{(-3)(-3)-(-1)\times 5}=\frac{z}{5\times 3-(-3)\times 2}=\frac{1}{2(-1)-3(-3)}\text{[see §244, Ex 2]},$$
or
$$\frac{y}{14}=\frac{z}{21}=\frac{1}{7},$$
whence
$$y=\tfrac{14}{7}=2, \text{ and } z=\tfrac{21}{7}=3.$$

Substitute y and z in (1), thus

$$x+2\times2+3\times3=14, \text{ whence } x=1$$

Note This example is sufficient to shew that we may apply the Formula of Cross Multiplication to solve the three linear equations [see Ex. 1]

$$a_1x+b_1y+c_1z=d_1,\ a_2x+b_2y+c_2z=d_2,\ a_3x+b_3y+c_3z=d_3,$$

(1) by eliminating *any one* of the variables x, y and z, or (2) by eliminating *all* the constants d_1, d_2 and d_3. The last process is left as an exercise to the student

Solve the equations

5. $3x+5y+7z+u=48$,
$7x+6y+5z+4u=53$,
$x+2y+3z+4u=27$,
$5x+8y+10z-2u=65$

6. $2u+5x=8$,
$x+2y=9$,
$7z+u=5$,
$3y+4z=14$

7. $19x-3u=4z-7y=7x-3y=11z-7u=1$

8. $3x-4y=7$,
$4y+u=12$,
$3z-5v=4$,
$5v-2x+5=0$,
$7v+3u-4z=7$

9. $9x-2z+u=41$,
$4y-3x+2u=5$,
$3y-4u+3v=7$,
$7y-5z-v=12$,
$7z-5u=11$

10. $x-ay+a^2z=a^3$,
$x-by+b^2z=b^3$,
$x-cy+c^2z=c^3$.

11. $cx+(a+b)y+(a-b)z=a$,
$ax+(b+c)y+(b-c)z=b$,
$bx+(c+a)y+(c-a)z=c$. [*App*]

12. $(b+c)x+by+cz=(c+a)y+cz+ax=(a+b)z+ax+by$,

$$xyz-(c+a-b)yz-(a+b-c)zx-(b+c-a)xy$$
$$=2xyz\frac{a^2+b^2+c^2}{(a+b+c)^2} \quad [App\ ]$$

13. $ax+by+cz=0$,
$a^2x+b^2y+c^2z+(a-b)(b-c)(c-a)=0$,
$a^3x+b^3y+c^3z+(a+b+c)(a-b)(b-c)(c-a)=0$

14. $u+ax+a^2y+a^3z+a^4=0$,
$u+bx+b^2y+b^3z+b^4=0$,
$u+cx+c^2y+c^3z+c^4=0$,
$u+dx+d^2y+d^3z+d^4=0$.

15. $(b+c)(y+z)=a(x+1)$,
$(c+a)(z+x)=b(y+1)$,
$(a+b)(x+y)=c(z+1)$ [*App*]

16. $xyz=24,\ yzu=72,\ zux=48,\ uxy=36$.

17. $x^2+y^2=a^2,\ xy=bc$

18. $x-y=5,\ xy=84$.

Solve the equations

19 $x(y+z)=a,\ y(z+x)=b,\ z(x+y)=c$

20 $xy+xz=a^2+yz,\ yz+yx=b^2+zx,\ zx+zy=c^2+xy.$

21 $xyz=a(y+z-x)=b(z+x-y)=c(x+y-z)$

22 $x(x+y+z)=96,\ y(x+y+z)=216,\ z(x+y+z)=264$

246 Problems These problems do not essentially differ from those of Chapters XI and XIX, the only difference being that the variables here are *more than one in number*.

Examples.

Ex 1 What fraction is that which becomes $\frac{1}{3}$ when 1 is added to its numerator and becomes $\frac{1}{4}$ when 1 is added to its denominator ?

Let x = the numerator of the required fraction,

and y = the denominator ,

$$\therefore \frac{x}{y}=\text{required fraction}$$

By the conditions of the problem, therefore

$$\frac{x+1}{y}=\frac{1}{3} \text{ and } \frac{x}{y+1}=\frac{1}{4},$$

whence $x=4$, $y=15$, and required fraction $=\frac{4}{15}$.

Ex 2 A farmer bought 30 oxen and 19 sheep for Rs 597, and 25 oxen and 18 sheep for Rs 504 , what is the price of each ?

Let x = price of each ox, in rupees,

and y = ... sheep, ... ,

by the conditions of the problem,

$$30x+19y=597 \text{ and } 25x+18y=504,$$

solving these we have $x=18$ and $y=3$.

Ex 3. A rectangular bowling green having been measured, it was observed that, if it were 5 feet broader and 4 feet longer, it would contain 116 square feet more , but if it were 4 feet broader and 5 feet longer, it would contain 113 square feet more Required its area

Let x = its length, in feet,

and y = breadth, ,

xy = its area, in square feet.

By the conditions of the problem, therefore,

$$(x+4)(y+5)=xy+116\ ;$$
$$(x+5)(y+4)=xy+113\ ;$$

solving which we get $x=12$ and $y=9$ Therefore the required area

$$=(12\times 9)\text{ sq feet }=108\text{ sq ft}$$

Ex. 4. A grocer bought for £8 two kinds of tea, the one at 4*s*. and the other at 3*s* 4*d* a lb, after mixing them, he sold them at 4*s* a lb and thus gained 10 per cent, how many lbs of each did he buy ?

Let x=required number of lbs. of tea at 4*s*.,
and y= ....3*s* 4*d* ;
then $4x+\frac{10}{3}y$=cost price $=£8=160s.$,
and $(x+y)4$=selling price $=(160+\frac{10}{100}\times 160)s=176s.$;
solving these we have $x=30$ and $y=24$

Ex 5 Fifteen boxes and 8 trunks will exactly fill a room ; but if the room were to be half as large again as it is, them 20 boxes and 14 trunks would be necessary to fill it up ; how many of each will separately fill the room ?

Let V=content of the room,
x= ... . each box,
y= .... trunk ,

$$\therefore 15x+8y=V \qquad (1),$$

and

$$20x+14y=\tfrac{3}{2}V \qquad (2)$$

Divide (2) by (1), thus

$$\frac{20x+14y}{15x+8y}=\frac{\frac{3}{2}V}{V}=\frac{3}{2},$$

whence $x=\frac{4}{5}y$ (3);
and $y=\frac{5}{4}x$ (4).

$\therefore$ From (1) and (4), $V=15x+8\times\frac{5}{4}x=25x$, i e, 25 boxes will fill the room ; and from (1) and (3), $V=15\times\frac{4}{5}y+8y=20y$, i e, 20 trunks will fill the room

Ex 6. A number, consisting of two digits, is equal to 4 times the sum of the digits, and if 18 be added to the number, the digits will be inverted Find the number [See § 231]

Let x=the digit in the tens' place,
y= .... units'. .,
then $10x+y$=the required number,
and $10y+x$=the number formed by the inverted digits ;

$$10x+y=4(x+y),$$

and

$$10x+y+18=10y+x,$$

whence $x=2, y=4$, therefore the required number=24

Ex 7 A boat goes up stream 30 miles and down stream 44 miles in 10 hours, it also goes up stream 40 miles and down stream 55 miles in 13 hours, find the rate of the stream and of the boat [*Cal*, 1880]

Let x = the rate of the boat, in miles per hour,

y = ... stream, ... ;

then $\dfrac{30}{x-y}$ = time of coming up the stream [§ 233],

$\dfrac{44}{x+y}$ = ... going down. . . . . . . [§ 233];

and since these two times taken together = 10 hours, we get

$$\frac{30}{x-y}+\frac{44}{x+y}=10 \qquad (1),$$

similarly

$$\frac{40}{x-y}+\frac{55}{x+y}=13 \qquad (2)$$

Multiply (1) by 4 and (2) by 3, and subtract, thus

$$\frac{11}{x+y}=1, \text{ or } x+y=11 \qquad (3)$$

Again multiply (1) by 5 and (2) by 4, and subtract, thus

$$\frac{10}{x-y}=2, \text{ or } x-y=5 \qquad (4)$$

Add and subtract (3) and (4), thus $x=8$ and $y=3$.

Ex 7 (1) *A* challenged *B* to ride a bicycle race of 1040 yards. He first gave *B* 120 yards start, but lost by 5 seconds, he then gave *B* 5 seconds start and won by 120 feet How long does each take to ride the distance? [*Cal*, 1881]

Let x = *A*'s rate and y = *B*'s rate, in yards per second,

$\dfrac{1040}{x}$ = time (in seconds) which *A* takes to run 1040 yards,

$\dfrac{920}{y}$ = . . . *B* . . . . 920. . . . . . .,

and $\dfrac{1000}{y}$ = . *B* . . . . .1000. . . . .

[§ 233 (3)]

Now by the first condition of the problem the time which *A* takes to run 1040 yards is the same as *B* takes to run 1040 − 120 (or 920) yards + 5 seconds, therefore

$$\frac{1040}{x}=\frac{920}{y}+5 \qquad (1)$$

Again the time which A takes to run 1040 yards is, by the second condition, the same as B takes to run 1040 yards less 120 feet (i.e., 1000 yards) − 5 seconds ; therefore

$$\frac{1040}{x}=\frac{1000}{y}-5 \qquad (2)$$

Subtract (1) from (2), thus

$$\frac{1000}{y}-\frac{920}{y}-10=0, \text{ whence } y=8$$

Substitute y in (1), thus

$$\frac{1040}{x}=\frac{920}{8}+5, \text{ whence } x=\tfrac{26}{3}.$$

Hence the time A takes to ride 1040 yards $=\dfrac{1040}{\frac{26}{3}}$ sec. $=120$ sec.,

and............ .. .. B . ... $=\dfrac{1040}{8}$ sec $=130$ sec.

Ex 7 (2) The price of a passenger's ticket on a French railway is proportional to the distance he travels, he is allowed 25 kilogrammes of luggage free, but on every kilogramme beyond this amount, he is charged a sum proportional to the distance he goes. If a journey of 200 miles with 50 kilos of luggage cost 25 francs, and a journey of 150 miles with 35 kilos cost $16\frac{1}{2}$ francs, what will a journey of 100 miles with 100 kilos of luggage cost?

Let x francs = price of a ticket for each mile,

and y francs = fare charged per mile on every kilo. of excess luggage,

thus $200x+200\times(50-25)y=25$, or $x+25y=\frac{1}{8}$ (1),

$150x+150\times(35-25)y=16\frac{1}{2}$, or $x+10y=\frac{11}{100}$ (2),

and $100x+100\times(100-25)y=$ required cost in francs (3).

Subtract (2) from (1), thus

$$25y-10y=\tfrac{1}{8}-\tfrac{11}{100}, \text{ or } y=\tfrac{1}{1000}.$$

Substitute y in (1), thus

$$x+25\times\tfrac{1}{1000}=\tfrac{1}{8}, \text{ or } x=\tfrac{1}{10}.$$

Substitute x and y in (3), thus

$$\text{cost required } =(100\times\tfrac{1}{10}+100\times 75\times\tfrac{1}{1000}) \text{ francs}=17\tfrac{1}{2} \text{ francs}$$

8. Find two numbers, such that their sum may be 183 and their difference 13 [§ 148, Ex 12]

9. What two numbers are those whose difference is 42 and the quotient of the greater by the less is 3?

10 Find two numbers such that the sum of the first and half the second shall be 18, and the sum of the second and half the first 21.

11 The sum of a certain number and $\frac{1}{6}$th of another is 30, and if $\frac{1}{8}$ of the first be subtracted from $\frac{2}{3}$ of the second the remainder is 21, what are the numbers?

12. What fraction is that which becomes $\frac{1}{3}$ if 3 be added to its numerator, and $\frac{1}{5}$, if 5 be subtracted from its denominator?

13 The sum of the numerator and denominator of a proper fraction, divided by their difference is $4\frac{1}{5}$, what is that fraction?

14. The difference between the digits of a number is 3, and the sum of the number and the number formed by inverting the digits is 143, find the number

15 Says *A* to *B*—If you give me 10 guineas of your money, I shall then have twice as much as you will have left, but says *B* to *A*—Give me 10 of your guineas, and then I shall have three times as many as you How many had each?

16 Divide 300 into 3 parts such that one-third of the first, one-fourth of the second, and one-fifth of the third, shall all be equal to one another?

17 If 16 ducks and 5 geese cost as much as 11 ducks and 7 geese at the same rate, find how many geese are worth 20 ducks

18 A father being asked by his son how old he was, replied—When we were 7 years younger, I was 4 times as old as you, and if we both live till we are 7 years older, I shall be twice as old as you What was the age of each?

19. A person had two casks, the larger of which he filled with ale and the smaller with cider Ale being half a crown, and cider 11*s* per gallon, he paid £8 6*s*, but had he filled the larger with cider, and the smaller with ale, he would have paid £11 5*s* 6*d* How many gallons did each hold?

20 A bill of £26 5*s* was paid with half-guineas and crowns, and twice the number of half guineas exceeded three times the number of crowns by 17 How many were there of each?

21 The area of a room is 192 sq ft, had each of the sides been 2 feet longer, the area would have been increased by 60 sq ft Find the sides of the room

22 Divide Rs 908 among *A*, *B* and *C*, so that the shares of *A* and *B* together may exceed the share of *C* by Rs 300, and the shares of *B* and *C* together may exceed the share of *A* by Rs 278

23 A gentleman gave away in charity a certain sum of money, had there been 3 more beggars, each would have received 1*s* less than he did; but if there had been 2 fewer, each would have received

is more required the number of beggars and the amount each received.

24 A ceatain number when divided by another gives a quotient 4 and remainder 3 If twice the fiist number be divided by three times the second number, the quotient is 2 and the remainder 20. Find the numbers

25 A sum of money put out to simple interest amounts in 8 months to £297 12*s*, and in 1 year 3 months to £306; find the sum and the rate of interest

26 A grocer gains 20 per cent by selling at 2*s* a lb a mixture formed by mixing with 7 lbs of a common tea, 2 lbs. of a better kind But if he had mixed 7 lbs of the latter with 2 lbs of the former kind, he would have lost 20 per cent by selling the mixture at that price What did each kind of tea cost him per lb.?

27 A certain number consisting of 2 digits becomes 110 when the number obtained by reversing the digits is added to it also the first number exceeds unity by 5 times the excess of the second number over unity What is the number?

28 A number consists of 2 digits, whereof the second is greater than the first if the number be divided by the sum of its digits, the quotient is 4; but if the digits be inverted and the number thus formed divided by a number greater by 2 than the difference of the digits, the quotient is 14 Find the number

29 A number of 2 digits is multiplied by 4, and the product is less by 3 than the number formed by inverting its digits if it be multiplied by 5, the tens' digit in the product is grerter by 1 and the units' digit less by 2, than the units' digit in the original number. Find the number

30 A certain number consisting of 2 digits is multiplied by 4, and the tens' digit of the product is one less and the units' digit one more than the units' digit in the original number it is be multiplied by 5, the product is one less than the number formed by reversing the order of the digits of the former product Find the number

31. A number has 3 digits, the sum of which is 10, the first and third together exceed the second by 4, and the first and second together exceed the third by 8 Find the number

32 *A*, *B* and *C* are 3 villages connected by straight roads, from *A* to *C* through *B*, the distance is 64 miles, from *B* to *A* through *C*, the distance is 74 miles, and from *C* to *B* through *A*, the distance is 80 miles. Find the length of each road

33 Some smugglers discovered a cave, which would exactly hold the cargo of their boat, *viz*, 13 bales of cotton and 33 casks of rum, while unloading, a custom-house cutter came in sight, consequently

they sailed away with 5 of the bales and 9 of the casks, leaving the cave two thirds full How many bales or casks would it hold ?

34 £500 was lent at simple interest in 2 separate sums, the smaller sum at 2 per cent more than the other, the interest of the greater was afterwards increased and that of the smaller diminished 1 per cent by this alteration, the whole interest was augmented one fourth of its former value, but if the interest of the greater had been so increased without any diminution in that of the less, the interest of the whole would have been increased a third Required the different sums and the rate of the interest of each

35 The united ages of a man and his wife are six times the united ages of their children Two years ago their united ages were ten times the united ages of their children, and six years hence their united ages will be three times the united ages of their children How many children have they ?

36 *A*, *B* and *C* sit down to play in the first game, *A* loses to each of *B* and *C* as much as each of them has, in the second, *B* loses similarly to *A* and *C*, and in the third, *C* loses similarly to *A* and *B*, and now they have each 24*s* What had they each at first ?

37 A cistern is filled by 3 cocks, two of which are exactly of the same dimensions when they are all open $\frac{5}{12}$ths of the cistern is filled in 4 hours, and if one of the equal cocks be stopped, $\frac{7}{8}$ths of the cistern is filled in 10 hours and 40 minutes In how many hours would each cock fill the cistern ?

38 A man rowing against a stream meets a log of wood which is being carried down by the current He continues rowing in the same direction for a quarter of an hour longer and then turns and rows down the stream overtaking the log $1\frac{1}{2}$ miles lower down than the point where he first met it Find the rate at which the current flows.

39 A person walks a certain distance in a certain time, had his rate been half a mile an hour faster, he would have performed his journey in $\frac{4}{5}$ths of the time, and had it been half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road Required the distance and his rate of travelling

40 A Rhine steamer sails up the stream 40 miles and down the stream 48 miles in 8 hours, on another occasion she sails up the steam 56 miles and down 96 miles in 13 hours Find the rate of the streamer and that of the river

41 *A* and *B* start to run a race to a certain post and back again, *A* returning meets *B* at 90 yards from the post and arrives at the starting place 3 minutes before him; if he had returned immediately to meet *B*, he would have met him at one sixth of the distance between the post and the starting place Required the length of the course and the duration of the race

42 A railway train after running for half an hour meets with an accident, after which it proceeds with $\frac{3}{5}$ths of its former rate, and is thus 1 hour 10 minutes behind time; had the accident occurred 30 miles further on, it would have arrived 25 minutes earlier Required the rate of the train and the length of the line [See Ex 11, § 233]

43 The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in 120 yards, but only 4 revolutions more when the circumference of the fore wheel is increased one-fourth and that of the hind-wheel one-fifth Find the circumference of each wheel.

44 A and B run a mile race In the first heat B receives 12 seconds start and is beaten by 44 yards. In the second heat B receives 165 yards start, and arrives at the winning post 10 seconds before A Find the time in which each can run a mile

45. Two trains, one a feet and the other b feet long, move with uniform speed on parallel rails when they move in opposite directions they pass each other in m seconds but when they move in the same direction, the faster train passes the other in n seconds Find the rates of the two trains [See §§ 233 and 234]

Ex $a=125$, $b=115$, $m=3$, $n=30$

46 A waterman rows a distance of a miles on a river and back again in t hours, and finds that he can row b miles with the stream in the same time that he can row c miles against it, determine the times of going and returning, and the velocity of the stream.

Ex $a=39$, $t=16$, $b=26$, $c=6$

47 A boy at a fair spends his money in oranges, if he had received 5 more for his money, they would have averaged a half-penny each less, but if 3 less a half-penny each more; how much did he spend and what was the number of oranges he bought?

48 A railway train travels from A to C passing through B where it stops 7 minutes; 2 minutes after leaving B, it meets an express train which started from C when the former was 28 miles on the other side of B. the express travels at double the rate of the other, and performs the journey from C to B in $1\frac{1}{3}$ hours, if on reaching A, it returned at once to C, it would arrive 3 minutes after the first train Find the distances between A, B and C, and the speed of each train.

49 If 22 oxen and 28 cows eat 24 acres of grass in 18 weeks, and 20 oxen and 38 cows eat 30 acres of grass in 27 weeks, and 41 oxen and 26 cows eat 50 acres of grass in 60 weeks; how long will 40 acres of the same grass last 35 oxen and 14 cows, the grass being supposed in all cases to grow uniformly [See Ex 7 (2)]

50 To complete a certain work A requires m times as long a time as B and C together, B requires n times as long as A and C together, and C requires p times as long as A and B together Prove that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1.$$

CHAPTER XXI

Ratio and Proportion

247 Ratio, Commensurable and Incommensurable Quantities Ratio is the relation which one quantity bears to another in respect of magnitude, the relation being ascertained by considering what multiple, part or parts, the one is of the other Thus there may be a ratio between two *abstract* numbers, or between two *concrete* quantities In the latter case, the quantities must be of the *same kind*, for we are unable to make a comparison between two quantities of different kinds, as for instance a line and an area, a horse and a cart, &c Hence when we say that a square has a certain ratio to a circle (as they are both areas), we mean that the former is a certain number of times or a certain number of parts, of the latter It is clear, therefore, that the ratio of two *concrete* quantities must necessarily be an *abstract* number Hence *the ratio between two quantities, whether abstract or concrete, is an* Abstract *number*

A ratio is expressed by placing two dots between the quantities, thus $a : b$ expresses the ratio between a and b The quantities a and b are called the Terms of the ratio, and the first, *viz*, a, is called the Antecedent and the second, *viz*, b, the Consequent of the ratio

When the ratio of two quantities *is expressed by the ratio of two integers*, the quantities are called Commensurable Quantities Thus $8\frac{1}{3}$ miles and 10 miles are commensurable quantities for their ratio is $5 : 6$

When the ratio of two quantities *cannot be expressed by the ratio of two integers*, the quantities are termed Incommensurable Quantities Thus the *diagonal* of a square and its *side* are incommensurable quantities, for their ratio is $\sqrt{2} : 1$, the *height* of an equilateral triangle and its *side* are incommensurable, for their ratio is $\sqrt{3} : 2$, &c But although in these cases we cannot find the *exact ratio*, yet we can find it to a *sufficient degree of approximation* that may be necessary

248 A ratio is expressed by a Fraction Suppose there are two lines AB and CD, the former containing 4 units and the latter 5 units of length The ratio between them is clearly $4 : 5$, and we have seen in the chapter on Fractions, that AB is $\frac{4}{5}$ths of CD Thus $4 : 5$ is the same as $\frac{4}{5}$, hence *a ratio is expressed by a fraction, whereof the numerator is the antecedent and the denominator is the consequent of the ratio* Conversely a fraction denotes a ratio whose antecedent is the numerator, and whose consequent is the denominator

We thus see that all the propositions regarding Fractions will apply to Ratios Thus for instance, since

$$\frac{a}{b}=\frac{am}{bm}=\frac{a\div m}{b\div m} \quad §[168],$$

we have the theorem—*If the terms of a ratio be both multiplied or both divided by the same quantity, the ratio remains unaltered.*

DEFINITIONS When the antecedent of a ratio is *equal* to the consequent, the ratio is termed a RATIO OF EQUALITY; when it is *greater*, a ratio of GREATER INEQUALITY or of MAJORITY; and when *less*, a ratio of LESS INEQUALITY or of MINORITY These terms therefore correspond to the terms Improper and Proper fractions [§ 166] $\frac{5}{2}$, $\frac{2}{5}$

249 Change of Ratios THEOREM—*A ratio of majority is diminished, and that of minority is increased, by adding a positive quantity to both terms of the ratio*

Let a positive quantity x be added to the terms of the ratio $\frac{a}{b}$; thus the new ratio is $\frac{a+x}{b+x}$

We have
$$\frac{a}{b}-\frac{a+x}{b+x}=\frac{ax-bx}{b(b+x)}=\frac{x(a-b)}{b(b+x)},$$

now a, b, x being all supposed positive, $\frac{a}{b}-\frac{a+x}{b+x}$ is positive or negative, if $\frac{x(a-b)}{b(b+x)}$ is positive or negative, *i. e*, if $a >$ or $< b$

Therefore $\frac{a}{b} >$ or $< \frac{a+x}{b+x}$ according as $a >$ or $< b$ [§ 45]

COROLLARY From this theorem, it is easy to see that as x increases, the ratio $\frac{a+x}{b+x}$ continually approaches to unity; and by taking x sufficiently large, the difference between $\frac{a+x}{b+x}$ and unity may be made *smaller than any assignable magnitude* Therefore when x is *infinitely great*, the *ultimate value* of the ratio is said to be *unity* Hence we obtain the following important

Definition of "Equal to" *Two quantities are said to be equal when their ratio is unity* Thus if $\frac{a+x}{b+x}=1$ when x is infinitely large $a+x=b+x$, where a and b may be *unequal.*

250 Definitions. If the antecedents of any number of ratios be multiplied together as also the consequents, the ratio of the two products is said to be a ratio *compounded* of the given ratios Thus $ace : bdf$ is the ratio which is obtained by compounding the ratios $a : b$, $c : d$, and $e : f$

When *two equal* ratios are compounded, the resulting ratio is called the DUPLICATE RATIO of the given ratios Thus $a^2 : b^2$ is the duplicate ratio of $a : b$

When *three equal* ratios are compounded, the new ratio is called the TRIPLICATE RATIO of the given ratios Thus $a^3 : b^3$ is the triplicate ratio of $a : b$

The ratio of the square roots of the terms of a ratio is called its SUB DUPLICATE ratio Thus $\sqrt{a} : \sqrt{b}$ is the sub-duplicate ratio of $a : b$ Similarly $\sqrt[3]{a} : \sqrt[3]{b}$ is the SUB TRIPLICATE ratio of $a : b$, and so on

The duplicate, triplicate, sub duplicate, &c, ratios are sometimes called respectively the *double, triple, half*, &c, ratios; hence $a^{\frac{1}{n}} : b^{\frac{1}{n}}$ is sometimes termed the $\frac{1}{n}$th of the ratio $a : b$ The ratio $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is called the SESQUIPLICATE RATIO of $a : b$

Examples

1. If $17x - 3y = 24y - x$, find the ratio $x : y$.
2. If $ax + by = cx + dy$, find the ratio $x : y$
3. If $\dfrac{4a-3}{4(b-1)} = \dfrac{a+9}{b+12}$, find the ratio of a to b.
4. If $(x+y)^2 = a(x-y)^2$, find the ratio of x to y
5. Write down the duplicate ratio of $4 : 5$, the triplicate ratio of $1 : 2$, and the sub duplicate ratio of $250 : 361$, and of $507 : 588$

251 Proportion Four quantities are said to be proportionals, when the first has the same ratio to the second, that the third has to the fourth in other words PROPORTION *is the equality of two ratios* Thus a, b, c, d are said to be proportionals when $a : b = c : d$ It was usual formerly to use the sign $::$ for the sign of equality The proportion $a : b :: c : d$, or its equivalent $a : b = c : d$, is read thus "a is to b as c is to d" The two terms a and d are called the EXTREMES, and the two terms b and c, the MEANS, also the term d is called a FOURTH PROPORTIONAL to a, b and c

Four quantities are said to be INVERSELY PROPORTIONAL, when the first and second are Proportional to the *reciprocals* of the third and fourth. Thus $a : b = \frac{1}{c} : \frac{1}{d}$. This is equivalent to $a : b = d : c$; hence four quantities are in inverse proportion when the *first* has to the *second* the same ratio as the *fourth* has to the *third*.

252. **Proportion I.** *If four quantities, a, b, c, d be proportionals, the product of the extremes is equal to the product of the means.*

We have $a : b = c : d$, that is, $\frac{a}{b} = \frac{c}{d}$. Let $\frac{a}{b} = \frac{c}{d} = k$.

$$\therefore a = kb \text{ and } c = kd, \text{ whence } kad = kbc \text{ or } ad = bc.$$

Corollary. Hence it is evident, that *any three* of the four quantities a, b, c, d being given, the *fourth* can be found.

Examples

Ex. 1. The last 3 terms of a proportion are 4, 5 and 6; find the first term.

Denoting this term by x, we have

$x : 4 = 5 : 6$, whence $6x = 4 \times 5$, or $x = 3\frac{1}{3}$.

2. Find the fourth proportional to 8, 10 and 12.

3. The ratio $11 : 18$ becomes $2 : 3$ when a certain number is added to its terms; find the number.

4. Two numbers are in the ratio of m to n, and if c be subtracted from each, the ratio is $a : b$; find the numbers.

5. If $ax + by : ax - by = m : 1$, find the ratio $x : y$.

6. Solve $\frac{10+x}{5} : \frac{4x-9}{7} = 14 : 5$.

7. If $2x + 3 : 5x - 2$ in the duplicate ratio of $2 : 3$, find x.

253. **Proposition II.** *If the product of two quantities be equal to the product of two others, the four are proportionals, the extremes being the factors of one of the products, and the means those of the other.*

The Proposition is the *converse* of the Proposition of § 252.

Let $ad = bc$; divide by bd, thus $\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d$.

Similarly by dividing by ac, cd and ab respectively, we shall obtain three other proportions. Thus if $ad=bc$, we get the following four proportions

$$a : b=c : d \quad (1),$$

$$b : a=d : c \quad (2),$$

$$a : c=b : d \quad (3),$$

and

$$c : a=d : b \quad (4).$$

Hence if any one of the proportions be true, all the others are true, for any one of them will give $ad=bc$.

254. From (2) of § 253, we have **Proportion III**. *If four quantities a, b, c, d, be proportionals, they are proportional when taken inversely.* [Invertendo.]

255. From (3) of § 253, we have **Proportion IV**. *If four quantities a, b, c, d be proportionals, they are proportionals when taken alternately.* [Alternando.]

REMARK. From the definition of ratio [§ 247], it may seem that if the four quantities a, b, c, d, be concrete this alternation cannot take place *unless they be of the same kind*. But when once the ratios are obtained, the numbers representing them become *abstract numbers*, and then alternation may take place.

256. **Proposition V.** *If four quantities a, b, c, d be proportionals, then $a+b : b=c+d : d$.* [Componendo.]

We have $\quad a : b=c : d$, that is, $\dfrac{a}{b}=\dfrac{c}{d}$,

therefore $\quad \dfrac{a}{b}+1=\dfrac{c}{d}+1$, or $\dfrac{a+b}{b}=\dfrac{c+d}{d}$,

that is $\quad a+b : b=c+d : d$.

257. **Proposition VI.** *If four quantities a, b, c, d be proportionals, then $a-b : b=c-d : d$.* [Dividendo.]

We have $\quad a : b=c : d$, that is, $\dfrac{a}{b}=\dfrac{c}{d}$;

therefore $\quad \dfrac{a}{b}-1=\dfrac{c}{d}-1$, or $\dfrac{a-b}{b}=\dfrac{c-d}{d}$,

that is $\quad a-b : b=c-d : d$.

257. **Proposition VII.** *If four quantities a, b, c, d be proportionals,* then $a+b : a-b=c+d : c-d$. [Componendo and Dividendo.]

(For proof see § 227, see also § 262, Ex. 1).

259. Proposition VIII. *If the terms of one proportion be compounded with the corresponding terms of another proportion, the products will be proportionals.*

If $a : b = c : d$ and $m : n = p : q$, then will $am : bn = cp : dq$

Now by definition, $\frac{a}{b} = \frac{c}{d}$ and $\frac{m}{n} = \frac{p}{q}$; therefore $\frac{a}{b} \times \frac{m}{n} = \frac{c}{d} \times \frac{p}{q}$;

that is $$am : bn = cp : dq.$$

Similarly it may be shewn that

if $$a : b = c : d$$
$$e : f = g : h,$$
$$\cdots\cdots\cdots$$

and $$w : x = y : z,$$

then $$ae\ldots w : bf\ldots x = cg\ldots y : dh\ldots z$$

Corollary. If $a : b = c : d$ and $b : x = d : y$, then $a : x = c : y$. [*Ex æquali.*]

260 In a similar way, many other Propositions in Proportion may be proved. We give below some of them which the student will have no difficulty in proving.

(1) If $a : b = c : d$, then $a : a+b = c : c+d$

(2) If $a : b = c : d$, then $a : a-b = c : c-d$ [*Convertendo*].

(3) If $a : b = c : d$, then $ma : mb = nc : nd$.

(4) If $a : b = c : d$, then $ma : nb = mc : nd$

(5) If $a : b = c : d$, then $a^n : b^n = c^n : d^n$.

(6) If $a : b = c : d$, then $ma+nb : pa+qb = mc+nd : pc+qd$

261 Continued Proportion. Quantities are said to be in *continued proportion* when the first : the second = the second : the third = the third : the fourth = &c. Thus if $a : b = b : c = c : d = d : e$ = &c., then $a, b, c,$ &c are in continued proportion

If three quantities a, b, c, be in continued proportion, b is said to be a Mean Proportional between a and c; and c is said to be a Third Proportional to a and b.

If a, b, c are in continued proportion, we get

$$\text{(i)}\quad b^2 = ac; \quad \text{(ii)}\quad a : c = a^2 : b^2 = b^2 : c^2.$$

For $$a : b = b : c \text{ gives } b^2 = ac\ [\S\ 252];$$

and taking $\frac{a}{b} = \frac{b}{c} = k$, we get $\frac{a}{b} \times \frac{b}{c} = k^2$, or $\frac{a}{c} = k^2 = \frac{a^2}{b^2} = \frac{b^2}{c^2}$.

The last result may evidently be generalised, for if a number of quantities, $a, b, c, \ldots \ldots x$ be in continued proportion, giving n ratios, then

$$\frac{a}{x}=\left(\frac{a}{b}\right)^n=\left(\frac{b}{c}\right)^n=\left(\frac{c}{d}\right)^n=\&c$$

For let $\quad \frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\ldots=k,$

$$\therefore \frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\ldots\times\frac{v}{x}=kk\ldots \text{ to } n \text{ factors}=k^n,$$

$$\therefore \frac{a}{x}=k^n=\left(\frac{a}{b}\right)^n=\left(\frac{b}{c}\right)^n=\left(\frac{c}{d}\right)^n=\&c$$

Examples

1. Find a mean proportional between 12 and 48, also between x^3y and xy^3

2. Find a third proportional to 14 and 35

3. If $x-1$, x and $x+3$ be in continued proportion, find x.

4. Given that a, b, c, d are in continued proportion, prove that

$$\frac{a}{d}=\frac{b^3}{c^3} \text{ and } a^{\frac{2}{3}}=c^2d^{-\frac{4}{3}}.$$

5. If a, b, c be in continued proportion, then $\frac{a+b}{b+c}=\frac{\sqrt{a}}{\sqrt{c}}$

6. If a, b, c, d are in continued proportion, then also $a+b$, $b+c$, and $c+d$ are in continued proportion

262. Theorem If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$, then each of these ratios $=\left\{\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right\}^{\frac{1}{n}}$, where p, q, r, n are any quantities whatever

Let $\quad \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=k,$

$$\therefore a=kb,\ c=kd,\ e=kf,\ \ldots$$

$$\therefore a^n=(kb)^n,\ c^n=(kd)^n,\ e^n=(kf)^n,\ \ldots\ \ldots;$$

$$\therefore pa^n=p(kb)^n,\ qc^n=q(kd)^n,\ re^n=r(kf)^n,\ldots\ \ldots,$$

$$\therefore pa^n+qc^n+re^n+\ldots=k^n(pb^n+qd^n+rf^n+\ldots),$$

$$\therefore \left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots\ldots$$

The following three deductions from this theorem are very useful.

Cor. 1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then each $=\frac{a+c+e}{b+d+f}$. Here $n=1$, $p=q=r$.

Cor. 2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then each $=\frac{a+c-e}{b+d-f}$. Here $n=1$ and $p=q=-r$.

Cor. 3. If $n=1$ and $p=q=r=\dots$, it is easy to see that

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots=\frac{a+c+e+\dots}{b+d+f+\dots}.$$

That is, *if any number of quantities be proportionals, as any one antecedent is to its consequent so is the sum of all the antecedents to the sum of all the consequents*

These results can of course be obtained immediately by the method employed to prove the theorem

Examples

Ex. 1. If $a:b=c:d$, then $a+b:a-b=c+d:c-d$ [§ 258]

Let $\frac{a}{b}=\frac{c}{d}=k$, $\therefore a=kb,\ c=kd$,

$$\frac{a+b}{a-b}=\frac{kb+b}{kb-b}=\frac{b(k+1)}{b(k-1)}=\frac{k+1}{k-1} \quad (1),$$

and $$\frac{c+d}{c-d}=\frac{kd+d}{kd-d}=\frac{d(k+1)}{d(k-1)}=\frac{k+1}{k-1} \quad (2),$$

from (1) and (2), $\frac{a+b}{a-b}=\frac{c+d}{c-d}$

Q. E. D.

Ex 2 If $a:b=c:d$, prove that $ma+nc:pa+qc=mb+nd:pb+qd$

Let $\frac{a}{b}=\frac{c}{d}=k$, thus $a=kb,\ c=kd$ (1).

Now $$\frac{ma+nc}{pa+qc}=\frac{mkb+nkd}{pkb+qkd} \text{ from (1)}=\frac{k(mb+nd)}{k(pb+qd)}=\frac{mb+nd}{pb+qd}$$

Ex 3 If $a:b=c:d$, prove that $a:c=\sqrt[3]{ma^3+nb^3}:\sqrt[3]{mc^3+nd^3}$

Let $\frac{a}{b}=\frac{c}{d}=k$, then $a=kb,\ c=kd$,

$$\therefore \frac{\sqrt[3]{ma^3+nb^3}}{\sqrt[3]{mc^3+nd^3}}=\frac{\sqrt[3]{mk^3b^3+nb^3}}{\sqrt[3]{mk^3d^3+nd^3}}=\frac{b\sqrt[3]{mk^3+n}}{d\sqrt[3]{mk^3+n}}=\frac{b}{d}=\frac{a}{c} \text{ [§ 255]}$$

Ex 3 (1) If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{(a+c)^3}{(b+d)^3}=\dfrac{a(a-c)^2}{b(b-d)^2}$ [*Cal*, 1888]

Assume $\dfrac{a}{b}=\dfrac{c}{d}=k$, then $a=kb$, $c=kd$,

thus $$\frac{(a+c)^3}{(b+d)^3}=\frac{(kb+kd)^3}{(b+d)^3}=\frac{k^3(b+d)^3}{(b+d)^3}=k^3,$$

and $$\frac{a(a-c)^2}{b(b-d)^2}=\frac{kb(kb-kd)^2}{b(b-d)^2}=\frac{k^3b(b-d)^2}{b(b-d)^2}=k^3,$$

$$\frac{(a+c)^3}{(b+d)^3}=\frac{a(a-c)^2}{b(b-d)^2}$$

Ex. 4. If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{2a^2+ab+3b^2}{2a^2-ab+3b^2}=\dfrac{2c^2+cd+3d^2}{2c^2-cd+3d^2}$

Put $\dfrac{a}{b}=\dfrac{c}{d}=k$, then $a=kb$, $c=kd$,

$$\therefore \frac{2a^2+ab+3b^2}{2a^2-ab+3b^2}=\frac{2k^2b^2+kb^2+3b^2}{2k^2b^2-kb^2+3b^2}=\frac{2k^2+k+3}{2k^2-k+3},$$

and $$\frac{2c^2+cd+3d^2}{2c^2-cd+3d^2}=\frac{2k^2d^2+kd^2+3d^2}{2k^2d^2-kd^2+3d^2}=\frac{2k^2+k+3}{2k^2-k+3},$$

$$\frac{2a^2+ab+3b^2}{2a^2-ab+3b^2}=\frac{2c^2+cd+3d^2}{2c^2-cd+3d^2}$$

If $a : b=c : d$, shew that

5 $a : a-c=b : b-d$ 6 $a : c=a+b : c+d$

7 $3a+4b : 3c+4d=5a+6b : 5c+6d$

8 $14a+19b : 14c+19d=b^2c : ad^2$

9 $7a-5b : 7a+5b=7c-5d : 7c+5d$

10 $a^2+b^2 : a^2-b^2=ac+bd : ac-bd$

11 $a^2+b^2 : c^2+d^2=(a-b)^2 : (c-d)^2$ 12 $a^2+c^2 : b^2+d^2=ac : bd$.

13 $a^2+ab : c^2+cd=ab-b^2 : cd-d^2$

14 $\sqrt{ac} : \sqrt{bd}=a-c : b-d$

15 $ma-nb : ma+nb=mc-nd : mc+nd$

16. $ma+nb : mc+nd=pa+qb : pc+qd$

17. $ma+nc : mb+nd=\sqrt{a^2+c^2} : \sqrt{b^2+d^2}=a : b$

18 $a+\sqrt{a^2+b^2} : a-\sqrt{a^2+b^2}=c+\sqrt{c^2+d^2} : c-\sqrt{c^2+d^2}$.

19 $(a^3+a^2b+b^3)(a+b) : (c^3+c^2d+d^3)(c+d)=a^4+b^4 : c^4+d^4$

If $a : b = c : d = e : f$, shew that

20 $a : b = mc - ne : md - nf$

21 $a + 3c + 2e : a - c = b + 3d + 2f : b - d$

22 $a : b = ma + nc + pe : mb + nd + pf$

23 $c : d = ab + cd + ef : b^2 + d^2 + f^2$

24 $e : f = \sqrt[4]{m^4a^4 + n^4c^4 + p^4e^4} : \sqrt[4]{m^4b^4 + n^4d^4 + p^4f^4}$

25 $a^2 + c^2 + e^2 : ab + cd + ef = ab + cd + ef : b^2 + d^2 + f^2$

26 $(a^2 + ac + e\sqrt{ac})^3 : (ac^2 + ce\sqrt{ae} + e^3)^2$

$$= (b^2 + bd + f\sqrt{bd})^3 : (bd^2 + df\sqrt{bf} + f^3)^2.$$

27 $pa^2 + qc^2 + re^2 : pab + qcd + ref = lab + mcd + nef : lb^2 + md^2 + nf^2.$

28 If $x : a = y : b = z : c = \ldots\ldots$, shew that each of the ratios

$$= \left\{\frac{x^m + y^m + z^m + \ldots}{a^m + b^m + c^m + \ldots}\right\}^{\frac{1}{m}}.$$

263 Examination upon Chapter XXI

1 Define the terms *Ratio* and *Proportion* What are *Incommensurable Quantities*? Illustrate your answer by examples

2 When can concrete quantities have ratios to one another? What kind of numbers are ratios between concrete quantities?

3 When can four concrete quantities be proportionals?

4 If $a : b = x : y$ and $b : c = y : z$, then $a : c = x : z$

5 If a and b be inversely proportional to c and a fourth quantity, find this quantity

6. What is a *duplicate ratio*? Give examples

7 State how the ratios $3 : 5$ and $14 : 11$ will be affected by adding 3 and -3 respectively to the terms of each ratio.

8 Compare the ratios $3 : 4$, $5 : 6$ and $2 : 3$

9 Shew that the ratio $a + c : b + d$ lies between the ratios $a : b$ and $c : d$

10 If $7(x - y) = 3(x + y)$, what is the ratio of x to y?

11. Having given $b - a : b + a = 4a - b : 6a - b$, find the ratio $a : b$.

12 If the ratio $a : b$ remain unchanged when x and $2x$ are added to a and b respectively, find the value of $a : b$

13. When are quantities said to be in *continued proportion*?

14. If a, b, c, be in continued proportion, then a has to c the duplicate ratio of $a + b$ to $b + c$

15 What is a *mean proportional*? If y be a mean proportional between x and z, then $x+y : x-y = y+z : y-z$.

16 When is a ratio said to be compounded of two or more ratios? Find the ratio compounded of $2 : 3$, $5 : 8$, and $12 : 25$; of $3ax : 4by$, $a^2-x^2 : c^2-x^2$, $bc+bx : a^2+ax$ and $c-x : a-x$; and of $1-x^2 : 1+y$, $1-y^2 : x+x^2$, and $1+\frac{y}{1-y} : 1$.

17 What is a *third proportional*? If $a : b = c : d$ and x be a third proportional to a and b, and y to b and c, prove that the mean proportional between x and y is equal to that between c and d.

18 If $\frac{b+c}{a}=\frac{c+a}{b}=\frac{a+b}{c}$, shew that $a=b=c$.

19 If $a-d=m$ and $\frac{1}{a}+\frac{1}{d}=\frac{m}{b}$, shew that $a : b = c : d$, and find the value of m in terms of a, b and c.

20 If $x^2+y^2 : xy = 85 : 42$, find the ratio $x : y$.

21 The first and fourth terms of a proportion are 5 and 54; the sum of the second and third is 51; find them.

22 If $\frac{a}{b}=\frac{c}{d}$ then $\frac{ax+b}{cx+d}$ has always the same value, whatever value be given to x.

23 What is the least integer that must be added to the terms of the ratio $9 : 23$, so as to make it greater than the ratio $7 : 11$?

24 A number of two figures is altered in the ratio of n to m, if its digits be interchanged; shew that the digits are to each other in the ratio of $10m-n$ to $10n-m$.

25 A's age is 25 years and B's age is 6 years; find the least number of years after which the ratio of their ages will be less than the ratio $7 : 3$.

26 If $\frac{b^2-a^2-c^2}{b-c}=\frac{b^2-a^2+c^2}{a+c}$, find the value of each ratio in terms of c.

27. If $\frac{a+c}{b}=\frac{c}{a}=\frac{a}{c-b}$, determine the ratios $a : b : c$.

Miscellaneous Examples VIII.

Ex 1 If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$,

shew that $(b-c)x+(c-a)y+(a-b)z=0$. [*Cal*., 1878].

Let each of the given ratios $=k$, thus we have

$$x=k(b+c-a) \quad (1),$$

$$y=k(c+a-b) \quad (2),$$

$$z=k(a+b-c) \quad (3),$$

multiplying (1) by $b-c$, (2) by $c-a$ and (3) by $a-b$, and adding, we get $(b-c)x+(c-a)y+(a-b)z$

$$=k\{b^2-c^2-a(b-c)+c^2-a^2-b(c-a)+a^2-b^2-c(a-b)\}=k\times 0=0.$$

Ex 2 Shew that if $\dfrac{ay-bx}{c}=\dfrac{cx-az}{b}=\dfrac{bz-cy}{a}$, then $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

Let $$\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}=k,$$

$$\therefore \frac{acy-bcx}{c^2}=\frac{bcx-abz}{b^2}=\frac{abz-acy}{a^2}=k \text{ [§ 168]},$$

by Cor. 1, § 262, $k=\dfrac{\text{sum of numerators}}{\text{sum of denominators}}=0$,

hence each of the given ratios $=0$, therefore

$$ay-bx=0,\ cx-az=0,\ bz-cy=0,$$

whence $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

Ex 3 If $m:n=p:q$, prove that $\dfrac{(m-n)(m-p)}{m}=(m+q)-(n+p)$ [*Cal*, 1859]

$$\frac{(m-n)(m-p)}{m}=\frac{m^2-m(n+p)+np}{m}=m-(n+p)+\frac{np}{m}$$

$$=m-(n+p)+\frac{mq}{m} \text{ [}\because np=mq, \text{ § 252]}$$

$$=m-(n+p)+q=(m+q)-(n+p) \text{ [§ 46]}.$$

Ex 3 (1) If $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$, prove that

$$(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2) \quad \text{[}Cal., 1887\text{]}$$

Let each of the given ratios $=k$, thus

$$k=\frac{ab}{b^2}=\frac{bc}{c^2}=\frac{cd}{d^2} \text{ [§ 168]}=\frac{ab+bc+cd}{b^2+c^2+d^2} \text{ [§ 262, Cor 1]} \quad (1);$$

also $$k=\frac{a^2}{ab}=\frac{b^2}{bc}=\frac{c^2}{cd} \text{ [§ 168]}=\frac{a^2+b^2+c^2}{ab+bc+cd} \text{ [§ 262, Cor. 1]} \quad (2)$$

$\therefore$ from (1) and (2), $\dfrac{ab+bc+cd}{b^2+c^2+d^2}=\dfrac{a^2+b^2+c^2}{ab+bc+cd}$,

or $$(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2).$$

Ex 3 (2) If $a : b = c : d$, shew that

$$4(a+b)(c+d)=bd\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2 \text{ [Cal, 1874]}$$

Let each of the ratios $=k$, thus $a=kb$, $c=kd$,

$$4(a+b)(c+d)=4(kb+b)(kd+d)=4bd(k+1)^2$$
$$=bd\{2(k+1)\}^2=bd\{(k+1)+(k+1)\}^2$$
$$=bd\left\{\left(\frac{a}{b}+1\right)+\left(\frac{c}{d}+1\right)\right\}^2=bd\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2$$

Ex 4 Solve $\dfrac{ax+by}{cz}=\dfrac{cz+ax}{by}=\dfrac{by+cz}{ax}=x+y+z$

By § 262, Cor 1, $x+y+z=\dfrac{\text{sum of numerators}}{\text{sum of denominators}}=2$ (1)

And $$\frac{ax+by}{cz}+1=\frac{cz+ax}{by}+1=\frac{by+cz}{ax}+1,$$

$$\therefore\ \frac{ax+by+cz}{cz}=\frac{cz+ax+by}{by}=\frac{by+cz+ax}{ax},$$

or $$ax=by=cz=k \text{ suppose} \quad (2);$$

$\therefore$ from (1), $k\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=2$, whence $k=\dfrac{2abc}{bc+ca+ab}$;

$\therefore$ from (2), $$ax=by=cz=\frac{2abc}{bc+ca+ab}$$

Ex 4 (1) A and B compared their incomes and found that A's income was to that of B as 7 : 9, and that the third of A's income was Rs 30 greater than the difference of their incomes. Find what each received. [*Cal*, 1871]

Let x rupees $=A$'s income, and y rupees $=B$'s income,

thus $$\frac{x}{y}=\frac{7}{9} \quad (1),$$

and $$\tfrac{1}{3}x=(y-x)+30, \quad y>x \text{ by hypothesis} \quad (2)$$

From (1), $x=\frac{7}{9}y$, therefore from (2),

$$\tfrac{1}{3}\times\tfrac{7}{9}y=(y-\tfrac{7}{9}y)+30, \text{ whence } y=810$$

And from (1), $$x=\tfrac{7}{9}y=\tfrac{7}{9}\times 810=630$$

Ex 5. Two globes of gold whose radii are r, r', are melted and formed into a single globe, find its radius, having given that the volume of a sphere is proportional to the cube of its radius

Let v, v' denote the volumes of the given globes, and V and R, the volume and radius of the required globe

$$\therefore \frac{V}{R^3}=\frac{v}{r^3}=\frac{v'}{r'^3}=\frac{v+v'}{r^3+r'^3} \quad [\S\ 262, \text{Cor. 1}],$$

and $\therefore V=v+v'$, we get $R^3=r^3+r'^3$, or $R=\sqrt[3]{r^3+r'^3}$.

Ex 6 If from a cask of wine containing a gallons, b gallons be drawn off and the vessel filled up with water, and this operation be repeated n times successively, find the quantity of wine then remaining

Let $a_1, a_2, a_3, \ldots, a_n$ denote the quantities of wine remaining after the *first, second, third, ..., n^{th}*, operation respectively, then it is easily seen that

$$a : a_1 = a : a-b \tag{1}$$

but since the wine in the cask decreases every time in the ratio of a to $a-b$, we have

$$\left.\begin{aligned} a_1 &: a_2 = a : a-b, \\ a_2 &: a_3 = a : a-b, \\ &\ldots\ldots\ldots\ldots \\ a_{n-1} &: a_n = a : a-b \end{aligned}\right\} \tag{2}$$

Therefore compounding the n proportions in (1) and (2), we get

$$a : a_n = a^n : (a-b)^n,$$

whence $$a_n=\frac{(a-b)^n}{a^{n-1}}$$

7 If $a : b=c : d$, prove that

(1) $ac : bd=(a+c)(a-c) : (b+d)(b-d)$

(2) $a+b+c+d : a+b-c-d=a-b+c-d : a-b-c+d$

(3) $$\frac{(a^2-c^2)b^2(c^2-d^2)}{(a^2-b^2)c^2(b^2-d^2)}=1$$

(4) $a(a+b+c+d)=(a+b)(a+c)$

(5) $$\frac{1}{ma}+\frac{1}{nb}+\frac{1}{pc}+\frac{1}{qd}=\frac{1}{bc}\left\{\frac{a}{q}+\frac{b}{p}+\frac{c}{n}+\frac{d}{m}\right\}$$

8 If a, b, c, d be in continued proportion, shew that

(1) $(b+c)(b+d)=(c+a)(c+d)$

(2) $(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$.

9. If $a : b=c : d=e : f$, shew that

$$a^2 : b^2=a(a+mc)+nec : b(b+md)+nfd.$$

10 If $a_1 : b_1 = a_2 : b_2 = a_3 : b_3$, prove that
$$\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \sqrt{a_3 b_3} = \sqrt{(a_1+a_2+a_3)(b_1+b_2+b_3)}.$$

11 If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, shew that a, b, c, d are proportionals

12 If $a+b : b+c = c+d : d+a$, it is required to prove that
$$a=c \text{ or } a+b+c+d=0$$

13 If $a+x : a-x$ equals the duplicate ratio of $a+b : a-b$, then
$$x-b : a-x = b(a+b) : a(a-b)$$

14 If x be to y in the duplicate ratio of m to n, and m to n in the subduplicate ratio of p^2+x^2 to p^2-y^2, shew that
$$p^2 : xy = x+y : x-y$$

15 If x has to y the triplicate ratio of a to b, and a to b, the subtriplicate ratio of $c+x$ to $d+y$, prove that $x : y = c : d$

16 If $\dfrac{a}{x}=\dfrac{b}{y}=\dfrac{c}{z}$, prove that $\dfrac{la^3+mb^2y+nc^2z}{lx^3+mby^2+nz^3}=\dfrac{pa+qb+rc}{px+qy+rz}$.

17 If $\dfrac{d-a}{x-u}=\dfrac{d-b}{y-u}=\dfrac{d-c}{z-u}=\dfrac{a+b+c+d}{3(u+x+y+z)}$,

prove that $$\frac{a}{u+y+z}=\frac{b}{u+x+z}=\frac{c}{u+x+y}=\frac{d}{x+y+z}$$

18 If $\dfrac{a-b}{d-e}=\dfrac{b-c}{e-f}$, then each of these ratios $=\dfrac{(a-b)f+(b-c)d}{(d-f)e}$

19 If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, shew that

(1) $\dfrac{ax+by}{a^2+b^2}=\dfrac{by+cz}{b^2+c^2}=\dfrac{cz+ax}{c^2+a^2}$,

(2) $(ax+by+cz)^2=(a^2+b^2+c^2)(x^2+y^2+z^2)$

20 If $\dfrac{bx-ay}{cy-az}=\dfrac{cx-az}{by-ax}=\dfrac{z+y}{z+x}$, then each of these ratios $=\dfrac{x}{y}$, unless $b+c=0$ [App]

21 If $$\frac{bz+cy}{b-c}=\frac{cx+az}{c-a}=\frac{ay+bx}{a-b},$$

shew that $$(a+b+c)(x+y+z)=ax+by+cz$$

22 If $\dfrac{b+c-a}{y+z-x}=\dfrac{c+a-b}{z+x-y}=\dfrac{a+b-c}{x+y-z}$, then each of these ratios is

equal to $$\frac{a}{x}=\frac{b}{y}=\frac{c}{z},$$

also to $$\frac{b+c-3a}{y+z-3x}=\frac{c+a-3b}{z+x-3y}=\frac{a+b-3c}{x+y-3z}$$

23. If $\frac{x'}{xy}=\frac{y'}{xz}=\frac{z'}{yz}$, prove that $\frac{x}{x'y'}=\frac{y}{x'z'}=\frac{z}{y'z'}$.

24. If $\frac{c-a}{x-z}=\frac{c-b}{y-z}=\frac{a+b+c}{2(x+y+z)}$, prove that $\frac{a}{y+z}=\frac{b}{z+x}=\frac{c}{x+y}$.

25. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, $x^2+y^2+z^2=d^2$, and $ax+by+cz=f^2$,

prove that
$$(a^2+b^2+c^2)d^2=f^4.$$

26. If
$$\frac{\alpha}{x}=\frac{\beta}{y}=\frac{\gamma}{z} \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

shew that
$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\frac{\alpha^2+\beta^2+\gamma^2}{x^2+y^2+z^2}.$$

27. If
$$\frac{la-mb}{l}=\frac{ma-lb}{m}=c,$$

shew that (1) $c^2=a^2+b^2$; (2) $ac=a^2+b^2$.

28. If
$$\frac{y+z}{3b-c}=\frac{z+x}{3c-a}=\frac{x+y}{3a-b},$$

prove that
$$\frac{x+y+z}{ax+by+cz}=\frac{a+b+c}{a^2+b^2+c^2}.$$

29. Solve $\frac{17-4x}{4} : \frac{15+2x}{3}-2x=5 : 4$.

30. Solve $\frac{ax+by-cz}{b^2+c^2}=\frac{by+cz-ax}{c^2+a^2}=\frac{cz+ax-by}{a^2+b^2}=a+b+c$.

31. Solve $\frac{1}{a+b+c}=\frac{\frac{b}{y}+\frac{c}{z}}{a}=\frac{\frac{c}{z}+\frac{a}{x}}{b}=\frac{\frac{a}{x}+\frac{b}{y}}{c}$.

32. Solve
$$\left.\begin{array}{l} y+z : z+x : x+y=a : b : c \\ (y+z)^2+(z+x)^2+(x+y)^2=1 \end{array}\right\}.$$

33. If
$$\frac{bx+ay-cz}{a^2-b^2}=\frac{cy+bz-ax}{b^2-c^2}=\frac{az+cx-by}{c^2-a^2},$$

shew that
$$\frac{x+y+z}{a+b+c}=\frac{ax+by+cz}{ab+bc+ca}.$$

34. If $\frac{x}{l(mb+nc-la)}=\frac{y}{m(nc+la-mb)}=\frac{z}{n(la+mb-nc)}$,

then $\frac{l}{x(by+cz-ax)}=\frac{m}{y(cz+ax-by)}=\frac{n}{z(ax+by-cz)}$. [*App.*]

35. If $\dfrac{x}{y+z}=a$, $\dfrac{y}{z+x}=b$, $\dfrac{z}{x+y}=c$, shew that

$$x^2 : y^2 : z^2 = a(1-bc) : b(1-ca) : c(1-ab)$$

36. Find two numbers such that their sum, difference and the sum of their squares, may be as the numbers 5, 3 and 51.

37. A person buys tea at 6s. a lb. and also some at 4s. a lb.; in what proportion must he mix them so that by selling his tea at 5s. 3d. a lb., he may gain 20 per cent. on each lb. sold?

38. The length of a certain rectangular field is to its breadth as 6 : 5. One-sixth part of the area being planted there remains for ploughing 625 square yards. What are the dimensions of the field.

39. A bill before Parliament was lost on a division, there being 600 votes recorded. Afterwards, there being the same voters, it was carried by twice as many votes as it was before lost by, and the new majority was to the former as 5 : 4. How many members changed their minds?

40. Two casks, A and B, contain mixtures of wine and water, A in the ratio of 8 : 3, and B in the ratio of 5 : 1. In what proportion must liquid be drawn from each cask to give a mixture containing wine and water in the ratio of 4 : 1?

41. A packet, sailing from Dover with a fair wind, arrives at Calais in two hours; on its return, the wind being contrary, it proceeds six miles an hour slower than it went. When it is half way over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done, had not the wind changed, in the ratio of 6 : 7. Find the distance between Dover and Calais.

42. Having given that the illumination from a source of light is inversely proportional to the square of the distance, how much nearer to a candle must an object, which is now 10 inches off, be placed so as to receive just 9 times as much light?

43. Given that the areas of plane triangles are proportional to the product of their bases into heights, compare the areas of two triangles whose bases are as 3 and 4 and heights are as 8 and 7.

44. Three globes, whose radii are 3, 4 and 5 inches, are melted and formed into one; find its radius, having given that the volume of a sphere is proportional to (radius)^3.

45. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings be as high as a pile of q sovereigns, compare the values of equal bulks of gold and silver, having given that the volume of a coin is proportional to the product of its thickness into (radius)^2.

CHAPTER XXII.

Miscellaneous Theorems and Examples.

[*This Chapter, though interesting to the student of Algebra, is not designed for average Entrance students and may be left out*]

264 Theorem. *If the sum of any number of real positive quantities be zero, each of the quantities is severally zero*

Let $A^2+B^2+C^2+\ldots\ldots=0$, where A, B, C, ... are all *real* quantities

Now whether the expression for which A stands be, after reduction, positive or negative, *its square must always be positive*; hence A^2 is essentially positive Similarly B^2, C^2,...... are all essentially positive Now the *sum of positive quantities cannot be zero, unless each of them be severally zero* Hence $A^2=0$ $B^2=0$, $C^2=0$, &c ; whence $A=0$, $B=0$, $C=0$, ...which proves the Theorem

Examples.

Ex 1 If $a^2+b^2+c^2=2ab$, shew that $a=b$ and $c=0$

By transposition, we have

$$(a-b)^2+c^2=0, \text{ whence } (a-b)^2=0 \text{ and } c^2=0\,,$$
$$a-b=0 \text{ or } a=b, \text{ and } c=0$$

2 If $x^2+y^2+z^2+a^2+b^2+c^2=2(ax+by+cz)$, shew that
$$x=a,\ y=b,\ z=c$$

3 If $$a^2+b^2+c^2=ax+by+cz=x^2+y^2+z^2=1,$$
prove that $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=1$$

4 If
$$(x-a)^2+(x-b)^2+(x-c)^2=(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)$$
find the relations between a, b, c

5 If $\left\{x-\dfrac{m^2}{l-n}\right\}^2+\left\{x-\dfrac{n^2}{l-m}\right\}^2=0$, shew that $x=m+n$.

6 Solve $(1+a^2+b^2)(1+x^2+y^2)=(1+ax+by)^2$

7 Solve $(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)=0$

265 Theorem If $XY=0$, then (1) *either* $X=0$ or $Y=0$ *or* (2) $X=0$ *and* $Y=0$

$$\left.\begin{array}{ll}\text{For if} & X=a,\quad Y=0\\ \text{or if} & X=0,\quad Y=b\end{array}\right\}\text{ then}\qquad XY=0\,;$$
and if $\qquad X=0,\quad Y=0$, then also $XY=0$.

Hence *when nothing is known as to the values of X and Y*, (1) *is the only legitimate conclusion*

Corollary Hence if $XYZU \ldots = 0$, *one at least of the factors must be zero*, and conversely, *the product $XYZU \ldots$ cannot be zero, unless one at least of its factors be zero*

Ex 1 If $x^3+y^3+z^3-3xyz=0$, then must $x+y+z=0$, supposing x, y, and z all unequal

Now $x^3+y^3+z^3-3xyz$

$$=\frac{1}{2}(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\} \quad [\S\ 130],$$

$$\frac{1}{2}(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\}=0$$

Thus either the first factor or the second factor is zero, if the second factor be zero, we have

$$(y-z)^2+(z-x)^2+(x-y)^2=0,$$

whence $\quad (y-z)^2=0,\ (z-x)^2=0,\ (x-y)^2=0 \ [\S\ 264],$

that is, $\quad y-z=0,\ z-x=0,\ x-y=0,$

or $\quad x=y=z,$

which is contrary to hypothesis, therefore $x+y+z=0$

Ex 2 Solve $\left(\dfrac{x-a}{x-b}\right)^3=\dfrac{x-2a+b}{x-2b+a}$ [§ 224, Ex 123]

Add 1 to both sides, thus

$$\frac{(x-a)^3+(x-b)^3}{(x-b)^3}=\frac{2x-a-b}{x-2b+a},$$

whence $(2x-a-b)\left\{\dfrac{(x-a)^2-(x-a)(x-b)+(x-b)^2}{(x-b)^3}-\dfrac{1}{x-2b+a}\right\}=0;$

or after slight reduction,

$$(2x-a-b)\frac{(a-b)^2}{(x-b)^3(x-2b+a)}=0,$$

therefore either the first factor $=0$, or the second factor $=0$, but the second factor cannot be 0, for then $(a-b)^2$ would be equal to 0, *i e*, a would be equal to b, which is contrary to supposition, therefore $2x-a-b=0$, whence

$$x=\tfrac{1}{2}(a+b)$$

Ex 3 Given $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{x+y+z}$,

shew that $\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)^{2n+1}=\dfrac{1}{x^{2n+1}+y^{2n+1}+z^{2n+1}}$

Multiplying out, we have from the given relation

$$(x+y+z)(yz+zx+xy)-xyz=0,$$

or $$(y+z)(z+x)(x+y)=0 \text{ [§ 132, Ex 7];}$$

hence one at least of the factors must be 0 Let $y+z=0$, whence $y=-z$ and therefore $y^{2n+1}=(-z)^{2n+1}=-z^{2n+1}$, since $2n+1$ is always odd, [see § 187, Ex 6]

Now $$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{2n+1}=\left(\frac{1}{x}-\frac{1}{z}+\frac{1}{z}\right)^{2n-1}=\left(\frac{1}{x}\right)^{2n-1}=\frac{1}{x^{2n+1}}$$

$$=\frac{1}{x^{2n+1}-z^{2n+1}+z^{2n+1}}=\frac{1}{x^{2n+1}+y^{2n+1}+z^{2n+1}}.$$

266 Meanings of o, $\frac{a}{o}$, $\frac{o}{o}$.

(i) Ordinarily 0 stands for the difference of two equal positive quantities which is evidently *nothing* There is, however, another meaning of 0 not incompatible with this meaning. Suppose h represents the difference of a and x where $a>x$, then

$$a-x=h$$

Now it is clear that as x increases and approaches to a; h continually diminishes, and by giving to x a value sufficiently near to that of a, h may be made *smaller than any assignable quantity* This is expressed by saying that *the limiting value of the difference* h *when* a *and* x *are made to differ as little as we please. is* 0

(ii) $\frac{a}{o}=\infty$, i e, an infinitely large quantity

By actual division we get [§ 96]

$$\frac{a}{a-x}=1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\dots\dots$$

Now from what has been said above, it is easy to see that x is equal to a, when the difference between them *ultimately vanishes*, but then also each of the ratios $\frac{x}{a}$, $\frac{x^2}{a^2}$ $\frac{x^3}{a^3}$, ... is equal to 1 [§ 249, Cor], therefore

$$\frac{a}{0}=1+1+1+\dots \quad ad\ infinitum=\infty$$

Illustration $\frac{5}{1-\frac{9999999999}{10000000000}}=50000000000$, a large quantity

We shall now examine the equation

$$ax+b=ax+b'$$

Clearly this equation cannot be satisfied by any *finite value* of x, but may be satisfied when x is *infinite* [§ 249, Cor], and as its solution shews that x is infinite, the equation *is* satisfied, for

$$x=\frac{b'-b}{a-a}=\frac{P}{0}=\infty.$$

(iii) $\frac{0}{0}$ is the symbol of an **indeterminate form** *i e*, when any expression assumes this form for a particular value of any symbol, the value of the expression cannot be definitely determined, unless we *transform it into another of equal value*, such that, for the proposed value, it may not take the form $\frac{0}{0}$

We shall illustrate the above by examining the solution of $ax+b=0$, when $a=0$ and $b=0$ From the equation, we get

$$x=\frac{-b}{a}=\frac{0}{0},$$

which shews that x may have *any value whatever*, indeed when $a=0$ and $b=0$, the equation takes the form

$$0x=0,$$

which is evidently satisfied by *any finite value* of x

Examples

Ex 1 Evaluate $\frac{x}{\sqrt{1+x}-\sqrt{1-x}}$, when $x=0$

If at the outset, we substitute 0 for x, in the given expression, it becomes

$$\frac{0}{\sqrt{1+0}-\sqrt{1-0}}=\frac{0}{0},$$

and we cannot find its value, therefore we rationalise the denominator, thus the expression becomes

$$\frac{x(\sqrt{1+x}+\sqrt{1-x})}{2x}=\frac{\sqrt{1+x}+\sqrt{1-x}}{2}$$

Now put $x=0$, thus its value is 1

2 Evaluate $\frac{x^3-a^3}{x-a}$, when $x=a$

3 Evaluate $\frac{x^3-1}{x^3-2x^2+2x-1}$, when $x=1$

4 Evaluate $\frac{x^2(y+1)-xy-1}{x^2(y-1)-x(y-2)-1}$, when $x=1$.

267 Number of terms of an Expression. By the term *expression* is generally meant a *rational integral expression* [§ 220]. We know that a *complete* or *natural* expression is one in which *occur all the powers* of the symbol of reference; thus x^3+ax^2+bx+c is a complete expression [§ 220] An *incomplete* expression is one in which *some* of the powers of the symbol of reference are *wanting*; thus ax^3+x+q is an incomplete expression

From the first example as well as those given in § 220, it is easy to see that *the number of terms of an expression is equal to the number denoting its degree increased by one* Thus the expression ax^3+bx^2+cx+d, which is of the *third* degree, consists of $3+1$, i.e., *four* terms, viz., $+ax^3$, $+bx^2$, $+cx$ and $+d$

The truth of this statement is clearly very plain in the case of expressions which are complete, but not so plain in the case of incomplete expressions But it applies likewise to them, if we only suppose the place of the absent power supplied by 0, or rather by the *power with the coefficient* 0. Thus x^3+1, which is of the *third* degree contains *four* terms, viz., $+x^3$, $+0x^2$, $+0x$ and $+1$, for

$$\begin{aligned} x^3+1&=(x+1)(x^2-x+1)\\ &=x^3+x^2-x^2+x-x+1\\ &=x^3+(1-1)x^2+(1-1)x+1\\ &=x^3+0x^2+0x+1. \end{aligned}$$

Hence to write down the full form of an incomplete expression, we have to *supply the place of absent powers by those powers with the zero-coefficient*

Corollary 1 Hence if an expression of the m^{th} degree be multiplied by another of the n^{th} degree, the product will consist of $(m+n+1)$ terms, for it will be of $(m+n)$ dimensions [§ 90]

Corollary 2 Hence also if an expression of the m^{th} degree be divided by another of the n^{th} degree, the quotient will consist of $(m-n+1)$ terms, for it will be of $(m-n)$ dimensions [§ 97]

Note It is well to remark here that the number of terms of an expression depends on the symbol of reference Hence *the same expression may contain different number of terms* according as it is considered with reference to one or other of the involved symbols Thus the expression

$$(a^2+1)x^3+(a+2)x^2+3x+1$$

consists of *four* terms, viz., $+(a^2+1)x^3$, $+(a+2)x^2$, $+3x$ and $+1$, when the symbol of reference is x, and of *three* terms when it is a, in which case it assumes the form

$$x^3a^2+x^2a+(x^3+2x^2+3x+1),$$

and its terms are $\quad +x^3a^2$, $+x^2a$ and $+(x^3+2x^2+3x+1)$

268. Method of Detached Coefficients Multiplications and Divisions may be conveniently performed by writing down the coefficients only, as illustrated below

REMARK. In applying this method the student must be careful always to supply with zeros the place of the absent powers in *incomplete expressions*

EXAMPLES

Ex 1 Multiply $3x^3+2x^2-4x+1$ by x^2+3x+2

Here the given expressions are *complete*, therefore put down the coefficients as they are given,

$$\begin{array}{r}
3+2-4+1 \\
1+3+2 \\
\hline
3+2-4+1 \\
+9+6-12+3 \\
+6+\ 4-8+2 \\
\hline
3+11+8-\ 7-5+2
\end{array}$$

required product $=3x^5+11x^4+8x^3-7x^2-5x+2$

For the highest power of x in the product is x^5 [§ 90] which here occurs in the first term, and the other powers follow in order, the last term evidently not containing any of these powers

Ex 2 Multiply $2x^3+x-3$ by $5x^2+2$

Here the expressions are *incomplete*, therefore the coefficients are as given below [§ 267]

$$\begin{array}{r}
2+0+1-3 \\
5+0+2 \\
\hline
10+0+5-15 \\
+0+0+\ 0-0 \\
+4+\ 0+2-6 \\
\hline
10+0+9-15+2-6
\end{array}$$

$\therefore$ required product $=10x^5+0x^4+9x^3-15x^2+2x-6$

$=10x^5+9x^3-15x^2+2x-6$

Ex 3 Divide $35a^3+47a^2+13a+1$ by $5a+1$

$$\begin{array}{r}
5+1\ )\ 35+47+13+1\ (\ 7+8+1 \\
35+\ 7 \\
\hline
40+13 \\
40+\ 8 \\
\hline
5+1 \\
5+1 \\
\hline
\end{array}$$

$\therefore$ quotient $=7a^2+8a+1$

[since it must be of the *second* degree and a^2 must occur in the *first* term as the expressions are arranged according to the *descending powers* of a]

Ex. 4. Divide x^6-6x+5 by x^2-2x+1.

Since the dividend is *incomplete*, the coefficients are to be put down as follows [267]

$$\begin{array}{l}
1-2+1\;)\;1+0+0+0+0-6+5\;(\;1+2+3+4+5 \\
\underline{1-2+1} \\
2-1+0 \\
\underline{2-4+2} \\
3-2+0 \\
\underline{3-6+3} \\
4-3-6 \\
\underline{4-8+4} \\
5-10+5 \\
\underline{5-10+5}
\end{array}$$

$\therefore$ quotient $=x^4+2x^3+3x^2+4x+5$

269 Symmetrical Expressions. An expression is said to be symmetrical with respect to a pair of symbols, when their interchange does not affect its value and with respect to all its symbols, when the interchange of *any pair* does not affect its value

Thus $x+y+mz$ is symmetrical with respect to x and y, but not with respect to x and z, or y and z, for the interchange of x and y gives $y+x+mz$ which is equivalent to the given expression [§ 47], whereas the interchange of, say, y and z gives $x+z+my$ which differs from the proposed expression.

Again the expression $bc+ca+ab$ is symmetrical with respect to all its symbols, for the interchange of any pair, say, b and c gives $cb+ba+ac$ which is the same in value as the given expression [§§ 67 and 47]

The following are other examples · $x+y+z$, $a^3+b^3+c^3-3abc$, $(b+c-2a)(c+a-2b)(a+b-2c)$, &c.

It is worthy of note that the expression $x+y+mz$ would be symmetrical with respect to x, y and z, if $m=1$, or if x and y have each the coefficient m Thus $mx+my+mz$ is the *only general form* of symmetrical expressions *of the first degree* in x, y, z, where m is independent of x, y and z

Similarly it is easy to see that the expression

$$ax^2+by^2+cz^2+dyz+ezx+fxy+gxyz$$

would be symmetrical if $a=b=c$, and $d=e=f$; or in other words, if the expression assumes the form

$$Ax^2+Ay^2+Az^2+Byz+Bzx+Bxy+gxyz$$

It is needless to multiply examples, the above being sufficient to shew that in symmetrical expressions all the terms of the *same type*

must have the *same coefficient* Thus in the second example x^2, y^2, z^2, which are of the type of x^2, all have the same coefficient A, and yz, zx, xy which are of the type of yz, all have the same coefficient B

Again from the nature of symmetrical expressions, it follows that the sum, product, or quotient of two symmetrical expressions is symmetrical Thus the sum, product and quotient of $x^2+y^2+z^2$ and $a+b+c$ are respectively

$$x^2+y^2+z^2+a+b+c,$$
$$(x^2+y^2+z^2)(a+b+c),$$
$$(x^2+y^2+z^2)\div(a+b+c),$$

which are all symmetrical, as the student can himself see We thus obtain the following **Laws of Symmetry** —

(i) *In a symmetrical expression, all the terms of the* SAME TYPE *must have the same coefficient*

(ii) *The* SUM, PRODUCT *or* QUOTIENT *of two symmetrical expressions must also be symmetrical*

REMARK The expression $(y-z)(z-x)(x-y)$ is also called *symmetrical* though by the interchange of a pair of symbols its *sign* is changed

270 Homogeneous Expressions In this article we shall give a more convenient definition than that of § 86 A homogeneous expression of the r^{th} degree is such that when each of its symbols is multiplied by r, the expression itself is multiplied by r, *the degree of* r^n, *viz*, n, *denoting the degree of the expression* Thus x^2+xy+y^2, $(x^2+y^2)\div(x+y)$, and $(x^2+y^2+z^2)\div(yz+zx+xy)$ are respectively of the *second*, *first* and *zeroth* degrees, for when each of the symbols of these expressions is multiplied by r, we obtain respectively

$$r^2(x^2+xy+y^2),$$

$$\frac{r^2(x^2+y^2)}{r(x+y)}=r\frac{x^2+y^2}{x+y}, \text{ and } \frac{r^2(x^2+y^2+z^2)}{r^2(yz+zx+xy)}=r^0\frac{x^2+y^2+z^2}{yz+zx+xy}$$

Thus though the quotients are not obvious in the second and third examples, yet their degrees are apparent We have thus the following **Laws of Homogeneity** —

(i) *If the homogeneous expression of the* m^{th} *degree be multiplied by another of the* n^{th} *degree the product will be a homogeneous expression of the* $(m+n)^{\text{th}}$ *degree*

(ii) *If a homogeneous expression of the* m^{th} *degree be divided by another of the* n^{th} *degree, the quotient will be a homogeneous expression of the* $(m-n)^{\text{th}}$ *degree*

271 Identity is satisfied by all values. We have seen [§ 137], that an Identity holds whatever value be given to *any* of the symbols in it Thus the identity $x^2-(a+b)x+ab=(x-a)(x-b)$ is

satisfied when we put x or a or b equal 0, 1, 2, 3, &c. Hence as a particular case it is clear that an *Identity is satisfied by any value whatever of the Symbol of Reference*

272 **Theorem.** *In an Identity the coefficients of the like powers of the Symbol of Reference are equal*

Suppose $A+Bx+Cx^2+Dx^3+\ldots\ldots=a+bx+cx^2+dx^3+\ldots\ldots$ to be an *identity*. Since this is satisfied by *any value* of x [§ 271], put $x=0$, thus $A=a$, and

$$Bx+Cx^2+Dx^3+\ldots\ldots=bx+cx^2+dx^3+\ldots\ldots;$$

now divide by x, and we have

$$B+Cx+Dx^2+\ldots\ldots=b+cx+dx^2+\ldots\ldots,$$

hence, as before, $B=b$ Similarly it may be shewn that $C=c$, $D=d$, &c

This theorem is called the PRINCIPLE OF INDETERMINATE COEFFICIENTS

REMARK 1 A and a may be considered as coefficients of x having the *zero* power, for they may be written respectively Ax^0 and ax^0.

REMARK 2 In *incomplete expressions*, the coefficient of the *absent power* is 0 Thus from the identity

$$Ax^3+Bx^2+Cx+D=ax^3+bx^2+d,$$

we get $\quad Ax^3+Bx^2+Cx+D=ax^3+bx^2+0x+d$ [see § 267];

therefore $\quad A=a, B=b, C=0, D=d.$

Examples

Ex 1 If $ax^3+bx^2+cx+d=(2x+1)(x^2-4)$ for all values of x, determine a, b, c and d

We have

$$ax^3+bx^2+cx+d=(2x+1)(x^2-4) \text{ identically } =2x^3+x^2-8x-4;$$

$$\therefore a=2, b=1, c=-8 \text{ and } d=-4$$

2 If $ax^4+bx^3+cx^2+dx+e=3x^3+2(x+1)(x-1)+7$ identically, find the values of a, b, c, d and e

3. If px^3+qx^2+rx+s be identically equal to $2(x-1)(x+2)(x+3)$, what are the values of p, q, r and s?

4 If $x^4+ax^3+bx^2+cx+d=(x^2+x-1)(x^2+2x+3)$, shew that

$$a+d=0 \text{ and } b-4c=0$$

5. Find the values of a, b, c, d, when

$$ax^3+bx^2+cx+d=(2x^2+4x+1)(x-2),$$

and shew that $a+d=0$.

6 Determine l, m, n in order that x^3+lx^2+mx+n may be equal to the cube of $x-2$

Ex 7 If one of the factors of x^2+5x+q be $x+8$, determine q and the other factor

Since the proposed expression is a quadratic, it can have but *two linear* factors therefore the other must be of the form $x+a$ Hence

$$x^2+5x+q=(x+8)(x+a)=x^2+(a+8)x+8a\ ,$$

$$a+8=5 \text{ and } 8a=q$$

whence $a=-3$ and $q=-24$, and therefore the required factor is $x-3$.

8. If $x-5$ be a factor of $2x^2-7x+a$, what is the other factor and what the value of a ?

Ex 9 If x^2+5x+a be the square of $x+b$, find a and b.
We have

$$x^2+5x+a=(x+b)^2=x^2+2bx+b^2\ ,$$

$$2b=5 \text{ and } b^2=a\ , \text{ whence } \quad b=\tfrac{5}{2} \text{ and } a=\tfrac{25}{4}$$

10 Find p and q in terms of a, when $x^2+px+q=(x+a)^2$, and hence shew that $p^2=4q$.

Ex 11 Determine a and b in order that $3x^3+22x^2+ax+b$ may have a quadratic factor x^2+7x+8

Since the extreme terms of the given factor are x^2 and 8, the required factor, which is linear, must be $3x+\dfrac{b}{8}$, whence multiplying out and equating coefficients, we get $a=31$, $b=8$

12 If $4x^3-9x^2+ax-12$ be exactly divisible by $4x+3$, find a and the other factor

13 If $2x^4+x^3+lx^2+mx+n$ be divisible without remainder by x^3+2x^2+3x+4, find l, m and n

Ex 14 If x^3+x^2+px+q be divisible by $(x-1)(x-2)$, determine p and q

Since the given expression is a *cubic*, we must have another factor of the form $x+a$, therefore multiplying out and equating coefficients, we have

$$a-3=1,\ 2-3a=p,\ 2a=q\ ,$$

whence $p=-10$, $q=8$, and therefore the other factor $=x+4$

15 If $2x^4+3x^3+ax^2+bx+c$ be divisible by $(x-3)(x^2-1)$, find the values of a, b and c

16 For what values of a and b will the expression
$x^4+4x^3+ax^2+bx+25$ be a complete square ?

17 If $x^4+ax^3+bx^2+cx+d$ be a perfect square, shew that

$$d=\frac{c^2}{a^2},\quad \frac{a^2}{4}+2\sqrt{d}=b$$

Ex. 18 If x^3+ax^2+bx+c be exactly divisible by x^2+px+q, shew that $$p(p-a)=q-b,\ q^2=bq-cp$$

The other factor must evidently be $x+\frac{c}{q}$; therefore multiplying out and equating coefficients, we have

$$p+\frac{c}{q}=a \ (1), \text{ and } q+p\frac{c}{q}=b \quad (2)$$

From (1), $\frac{c}{q}=a-p$; $\therefore$ from (2), $q+p(a-p)=b$ The second relation at once follows from (2).

273 Degree of Remainder From the definition of Remainder [§ 96], it is clear that if the Dividend be of m dimensions and the Divisor of n dimensions in the symbol of reference, the Remainder will be of dimensions lower than n, i e, it will be of $n-1$ or of even lower dimensions, and the quotient will be of $m-n$ dimensions Hence as a particular case, *if the divisor be of the first degree, say $x+a$, the Remainder cannot contain* x

Corollary From the Corollary, § 96, we have $D=dQ+R$, i e. $\frac{D}{d}=Q+\frac{R}{d}$ Now since Q is integral, D will be exactly divisible by d, when $\frac{R}{d}$ is either integral or 0 But the degree of R being lower than that of d, $\frac{R}{d}$ is *fractional*; therefore $\frac{R}{d}=0$, i e, $R=0$ Hence *the condition for exact divisibility of one expression by another is that the remainder shall vanish* If therefore the Divisor be of n dimensions and consequently the Remainder of $n-1$ dimensions, the Remainder will contain in general n terms [§ 267] and therefore n coefficients of the symbol of reference, *which must each vanish identically if the remainder vanishes.*

Examples

Ex 1 If one of the factors of x^2+5x+q be $x+8$, determine q and the other factor [Ex 7, § 272]

$$\begin{array}{r|l}
x+8\ )\ x^2+5x+q\ (\ x-3 \\
\underline{x^2+8x} \\
-3x+q \\
\underline{-3x-24} \\
q+24.
\end{array}$$

The required factor must be the quotient $x-3$, and since the remainder vanishes, $q+24=0$ or $q=-24$.

2 Work Ex 8 of § 272 by *this method*

3 Find for what value of x, the expression $x^3+4x^2+7x+21$ is divisible by x^2+3x+5

4 Find the value of x for which $21x^4-29x^3+16x^2+14x-236$ is divisible by $3x^2-5x+8$

5 Find the value of x for which $x^4+ax^3+bx^2+cx+d$ is divisible by x^2+px+q

Ex 6 If x^2+5x+a be the square of $x+b$, find a and b [Ex 9, § 272]

By actual division, the quotient is seen to be $x+(5-b)$, and the remainder to be $a-b(5-b)$ Hence, since the given expression is a square, the quotient=the divisor, thus $x+(5-b)=x+b$, whence $b=\frac{5}{2}$, and since the remainder vanishes, $a=b(5-b)=\frac{25}{4}$, substituting the value of b

7 Find the relation between the coefficients p and q in order that x^2+px+q may be a perfect square

Ex 8 Determine a and b in order that $3x^3+22x^2+ax+b$ may have a quadratic factor x^2+7x+8 [Ex 11, § 272]

By division, the remainder is found to be $(a-31)x+(b-8)$ Now this remainder vanishes, therefore each of the coefficients vanishes identically, hence $a-31=0$ and $b-8=0$, thus $a=31$ and $b=8$.

9 Work examples 12, 13, 14, 15, 16 and 18 of § 272, by *this method.*

274 A Convenient Notation An expression in x is sometimes called *a function of x* It is often very conveniently represented by $f(x)$, where the letter within the bracket stands for the symbol of reference Thus $f(x)$ may stand for x^2+px+q, or for ax^3+bx^2+cx+d, or for any other expression in x In accordance with this notation, therefore, $f(a)$, $f(b)$, &c, represent expressions in a, b, &c

Similarly an expression in x and y may be represented by $f(x, y)$, an expression in x, y and z by $f(x, y, z)$, and so on Thus $f(x, y)$ may denote $ax+by+c$, or $gx^2+hxy+fy^2$, or &c, $f(x, y, z)$ may denote $ax+by+cz+d$, or $lx^2+my^2+nz^2+pxyz$, or &c

If necessary, other letters may be used instead of f, thus $F(x)$, $\phi(x)$, &c, represent functions of x

275 Remainder Theorem *When an integral expression in x is divided by $x-a$, the remainder is obtained by putting a for x in that expression*

Let Q represent the Quotient and R the Remainder, when

$$x^n+px^{n-1}+qx^{n-2}+\ldots+l\equiv f(x)$$

is divided by $x-a$, then [§ 96]

$$f(x)=Q(x-a)+R,$$

where R cannot contain x [§ 273] Now this being an identity, is satisfied by *any value* of x [§ 271]; put therefore $x=a$; thus

$$f(a)=Q(a-a)+R=Q\times 0+R=R\ ,$$

i.e.,
$$R=f(a)=a^n+pa^{n-1}+qa^{n-2}+\ldots\ldots+l$$

For particular cases, see examples 2 and 3 of § 96.

Corollary If the divisor be $x+a$, since $x+a=x-(-a)$, the remainder is obtained by *putting* $-a$ *for* x *in the given expression.*

Examples

Ex. 1 Find the remainder when $2x^3-7x^2+8x-9$ is divided by $x-4$

$$R=2\times 4^3-7\times 4^2+8\times 4-9=39$$

Ex 2 Find the remainder when $27x^3-18x^2+3x-11$ is divided by $3x-2$

$$R=27(\tfrac{2}{3})^3-18(\tfrac{2}{3})^2+3\times\tfrac{2}{3}-11=-9$$

Ex 3. Find the remainder when $x^4+5x^3+8x^2+7x+6$ is divided by $x+2$.

$$R=(-2)^4+5(-2)^3+8(-2)^2+7(-2)+6=0$$

Note. Evidently *the given expression is divisible* by $x+2$.

4 Find the remainder when x^3+5x^2-16 is divided by $x-2$

5. Find the remainder when x^4+2x^3+5x-3 is divided by $x+3$.

6. Find the remainder when $x^6+3x^3+x^2-x+8$ is divided by $ax+b$.

Ex. 7 Investigate whether $2x^3-3x^2-2x+3$ is divisible by x^2-1.

The factors of x^2-1 are $x-1$ and $x+1$ If the divisor be $x-1$,

$$\text{Remainder}=2-3-2+3=0\ ;$$

$\therefore$ the given expression is divisible by $x-1$. If the divisor be $x+1$,

$$\text{Remainder}=-2-3+2+3=0\ ;$$

$\therefore$ the expression is divisible by $x+1$ Hence it is divisible by the product of $x-1$ and $x+1$, i e, by x^2-1.

8. Shew that $6x^3-5x^2-8x+3$ is divisible by $(2x-3)(x+1)(3x-1)$.

9. Shew that $x^4-2ax^3+(1-a)ax^2+(2x-1)a^3$ is divisible by x^2-a^2

Ex 10. If one of the factors of x^2+5x+q be $x+8$, determine q [Ex. 7, § 272]

$$R=(-8)^2-5\times 8+q=0 \text{ by supposition,}$$

$$\therefore\ q=-24$$

11 Work Ex 8 of § 272

12 Determine k so that $3x^4+8x^3-4kx+k$ may be divisible by $x-2$

13. Work Ex 14 and Ex 15 of § 272

Ex 14 Shew that $x^n-nx+n-1$ is divisible by $(x-1)^2$, if n be a positive integer

Given expression $=(x^n-1)-n(x-1)$

$$=(x-1)\{(x^{n-1}+x^{n-2}+x^{n-3}+\dots+1)-n\};$$

therefore $x-1$ is a factor of the proposed expression Now if $x-1$ be also a factor of the quotient, then the whole expression is divisible by $(x-1)^2$ Put $x=1$ in the quotient, thus the quotient

$$=(1+1+1+\dots\text{to } n \text{ terms})-n=n-n=0,$$

therefore $x-1$ is a factor of the quotient, therefore, &c

15 Shew that $(1-x)^2$ is a factor of $1-x-x^n+x^{n+1}$, n being any positive integer

16. Shew that $(x-1)^2$ is a factor of $nx^{n+1}-(n+1)x^n+1$, where n is a positive integer

276. Divisibility of $x^n \pm a^n$ by $x \pm a$ where n is a positive integer We have four cases to consider, which we shall do in the following order —

(i) When x^n-a^n is divisible by $x-a$,

(ii) When x^n+a^n is divisible by $x-a$,

(iii) When x^n-a^n is divisible by $x+a$,

(iv) When x^n+a^n is divisible by $x+a$

By the Remainder Theorem [§ 275], we have, when

(i) Dividend $=x^n-a^n$ and Divisor $=x-a$, Rem $=a^n-a^n=0$ always,

$\therefore$ x^n-a^n is *always* divisible by $x-a$,

(ii) Dividend $=x^n+a^n$ and Divisor $=x-a$, Rem $=a^n+a^n=2a^n$ always,

x^n+a^n is *never* divisible by $x-a$;

(iii) Dividend $=x^n-a^n$ and Divisor $=x+a$, Rem $=(-a)^n-a^n$ [§ 275, *Cor*] $=0$ or $-2a^n$ according as n is *even* or *odd*,

$\therefore$ x^n-a^n is divisible by $x+a$ only when n is *even*,

(iv) Dividend $=x^n+a^n$ and Divisor $=x+a$, Rem $=(-a)^n+a^n$ [§ 275, *Cor*] $=0$ or $2a^n$ according as n is *odd* or even,

x^n+a^n is divisible by $x+a$ only when n is *odd*.

The following is *another proof* of the above theorem by the method of Mathematical Induction

(i) To investigate when $x^n - a^n$ is divisible by $x - a$.

We have
$$x^n - a^n = x^n - ax^{n-1} + ax^{n-1} - a^n$$
$$= x^{n-1}(x-a) + a(x^{n-1} - a^{n-1});$$
$$\therefore \frac{x^n - a^n}{x-a} = x^{n-1} + a\frac{x^{n-1} - a^{n-1}}{x-a}$$

Now since x^{n-1} is *integral*, $x^n - a^n$ will be divisible by $x - a$, if $\frac{x^{n-1} - a^{n-1}}{x-a}$ be *integral*, or in other words, if $x^{n-1} - a^{n-1}$ be divisible by $x - a$; that is, *if* x−a *divide a quantity which is the difference between any the same power of* x *and of* a, *it will also divide a quantity which is the difference between the next higher power of* x *and of* a. But we know that $x - a$ divides $x^2 - a^2$, therefore it will divide $x^3 - a^3$; and since it divides $x^3 - a^3$, it will divide $x^4 - a^4$; and so on. Hence

$x^n - a^n$ is always divisible by $x - a$.

(ii) To investigate when $x^n + a^n$ is divisible by $x - a$.

We have
$$x^n + a^n = x^n - a^n + 2a^n,$$
$$\therefore \frac{x^n + a^n}{x-a} = \frac{x^n - a^n}{x-a} + \frac{2a^n}{x-a}.$$

Now since $x^n - a^n$ is divisible by $x - a$, $x^n + a^n$ will be divisible by $x - a$, if $2a^n$ is so; but $2a^n$ is never divisible by $x - a$,

$\therefore$ $x^n + a^n$ is never divisible by $x + a$.

(iii) To investigate when $x^n - a^n$ is divisible by $x + a$.

From § 186, we know that $(-a)^n = +a^n$ only when n is even; therefore $x^n - a^n = x^n - (-a)^n$ only when n is *even*: and $x + a = x - (-a)$ Hence from (i),

$x^n - a^n$ is divisible by $x + a$ only when n is even.

(iv) To investigate when $x^n + a^n$ is divisible by $x + a$

As proved above, $(-a)^n = -a^n$ only when n is *odd*; therefore $x^n + a^n = x^n - (-a)^n$ only when n is odd; and $x + a = x - (-a)$. Hence from (i),

$x^n + a^n$ is divisible by $x + a$ only when n is odd.

Examples. (1).

1. Shew that $8^n - 1$ is always divisible by 7.
2. Shew that $(23)^n + 1$ is divisible by 24, if n be odd.

3 Shew that $(21)^n - 1$ is divisible by 20 or 22, if n be even

4 Shew that $(24)^n - (19)^n$ is divisible by 5 or 43, if n be even.

5 Shew that $7^{2n} - 1$ is always divisible by 48

6 Shew that $8^{2n} - 3^{2n}$ is always divisible by 55.

Note It is important to remember the form of the quotient in each of the above cases By actual division, we have

(1) $(x^n - a^n) \div (x - a) = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1}$.

(2) $(x^n - a^n) \div (x + a) = x^{n-1} - ax^{n-2} + a^2x^{n-3} - \ldots - a^{n-1}$

(3) $(x^n + a^n) \div (x + a) = x^{n-1} - ax^{n-2} + a^2x^{n-3} - \ldots + a^{n-1}$.

[Mark that in (1) *all* the signs in the quotient are +, and in (2) and (3) the signs are *alternately* + and −]

Examples (11)

Write down the following quotients

1 $(a^4 - b^4) \div (a \pm b)$ 2 $(a^5 + b^5) \div (a + b)$ 3 $(a^6 - b^6) \div (a \pm b)$

4 $(x^7 + y^7) \div (x + y)$ 5 $(x^8 - y^8) \div (x \pm y)$

6 $(x^9 + y^9) \div (x + y)$ 7 $(a^{10} - x^{10}) \div (a \pm x)$

8. $(a^{11} + x^{11}) \div (a + x)$ 9 $(x^{16} - a^{16}) \div (x \pm a)$

277 Theorem. *If an integral expression in x vanish when $x = a$, then $x - a$ is a factor of that expression*

Let $$f(x) = x^n + px^{n-1} + qx^{n-2} + \ldots + k$$

Suppose $$f(x) = Q(x - a) + R,$$

where Q is the quotient and R the remainder (if *any*), when $f(x)$ is divided by $x - a$

By § 275, $R = f(a)$, and when $x = a$, $f(x) = 0$, that is, $f(a) = 0$, hence $R = 0$,

$$f(x) = Q(x - a),$$

i e, $x - a$ is a factor of $f(x)$

This consequence of the Remainder Theorem is very useful in factorising **Homogeneous Symmetrical Expressions**

Examples

Ex. 1. Prove that $(b - c)^3 + (c - a)^3 + (a - b)^3 = 3(b - c)(c - a)(a - b)$. [§ 130, Ex 5].

Put $b = c$, i e, suppose $b - c = 0$, then

$$(b - c)^3 + (c - a)^3 + (a - b)^3 = 0 + (c - a)^3 + (a - c)^3 = 0,$$

$\therefore$ $b-c$ is a factor of $(b-c)^3+(c-a)^3+(a-b)^3$. Similarly it may be shewn that $c-a$ and $a-b$ are each factors of the expression.

Therefore $(b-c)^3+(c-a)^3+(a-b)^3=l(b-c)(c-a)(a-b)$ (A).

Now since the proposed expression is of the *third* degree in a, b and c, l cannot contain a, b, c and therefore remains constant for *any values* that may be assigned to these symbols And since (A) is an *identity*, we can give to a, b, c any value we please. Suppose then $a=0$, $b=1$, $c=2$, thus from (A)

$$-1+8-1=l\times -1\times 2\times -1\ ;\ \text{whence } l=3\ ,$$

$$\therefore\ (b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$$

Ex 2 Factorise $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

Denoting the given expression by $f(a, b, c)$ [§ 274], we have, by putting $b=-c$,

$$f(a, b, c)=c^2(c+a)+c^2(a-c)-2ac^2=0\ ;$$

thus $b+c$ is a factor of $f(a, b, c)$ Therefore from the symmetry of the given expression, we see that $c+a$ and $a+b$ are likewise its factors

Now $f(a, b, c)$ is of the *third* degree; therefore it has *three linear* factors, and consequently if it has any other factor, that factor must not involve a, b and c. Hence

$$f(a, b, c)=k(b+c)(c+a)(a+b),$$

where k is some constant. To determine k, proceed as in Ex 1; or more simply, *compare the coefficients of some one term on both sides* [§ 274] Thus by comparing the coefficients of a^2b, we have $k=1$;

$$\therefore\ \text{given expression}=(b+c)(c+a)(a+b)$$

Ex 3 Factorise $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

Denote for shortness the given expression by $f(a, b, c)$ [§ 274]. Put $b=c$ in $f(a, b, c)$; thus

$$f(a, b, c)=0+c^3(c-a)+c^3(a-c)=0\ ;$$

therefore $b-c$ is a factor of $f(a, b, c)$ [§ 277] Similarly it may be shewn that $c-a$ and $a-b$ are factors. Now $f(a, b, c)$ is of the *fourth* degree, therefore besides the 3 factors $b-c$, $c-a$, and $a-b$, it has another *linear* factor, and $f(a, b, c)$ being symmetrical, this factor must also be symmetrical in a, b and c, and therefore it is $la+lb+lc$ [§ 269] Hence

$$\begin{aligned} f(a, b, c)&=(b-c)(c-a)(a-b)(la+lb+lc)\\ &=l(b-c)(c-a)(a-b)(a+b+c). \end{aligned}$$

Compare the coefficients of a^3b; thus $l=-1$ Therefore

$$\text{given expression} = -(b-c)(c-a)(a-b)(a+b+c).$$

Ex 4 Factorise $a(b-c)^3+b(c-a)^3+c(a-b)^3=f(a, b, c)$

As in Ex 3, $b-c$, $c-a$, $a-b$ are factors of $f(a, b, c)$ But $f(a, b, c)$, being of the fourth degree has another linear factor, and as $f(a, b, c)$ is symmetrical in a, b and c this factor must also be symmetrical, whence it is $\lambda a+\lambda b+\lambda c$ [§ 269] Therefore

$$f(a, b, c)=(b-c)(c-a)(a-b)(\lambda a+\lambda b+\lambda c)$$
$$=\lambda(b-c)(c-a)(a-b)(a+b+c);$$

whence comparing the coefficients of ab^3, we get $\lambda=1$,

∴ given expression $=(b-c)(c-a)(a-b)(a+b+c)$

Ex 5. Factorise $(x+y+z)^3-(y+z-x)^3-(z+x-y)^3-(x+y-z)^3$
$$=f(x,y,z)$$

Put $x=0$, thus $f(x,y,z)=(y+z)^3-(y+z)^3-(z-y)^3-(y-z)^3=0$, and from the symmetry of $f(x,y,z)$, it is clear that $f(x,y,z)$ becomes zero, when $y=0$, $z=0$ thus x, y, z are factors of $f(x,y,z)$ Therefore since $f(x,y,z)$ is of the third degree, we have

$$f(x,y,z)=\lambda xyz,$$

where λ is some constant Comparing the coefficients of xyz, we obtain $\lambda=24$ We might obtain λ as in Ex 1 above, Put $x=y=z=1$, thus

$$(1+1+1)^3-(1+1-1)^3-(1+1-1)^3-(1+1-1)^3=\lambda.1\,1\,1,$$

whence $\lambda=24$

6 Prove that

$$(bc+ca+ab)^3-(bc)^3-(ca)^3-(ab)^3=3abc(b+c)(c+a)(a+b)$$

7 Prove that identity $x(x+y-z)(x+z-y)+y(y+z-x)(y+x-z)$
$$+z(z+x-y)(z+y-x)+(y+z-x)(z+x-y)(x+y-z)=4xyz$$

8 If n be a positive integer, shew that $(ab)^n-(bc)^n+(cd)^n-(da)^n$ is divisible by $ab-bc+cd-da$.

278 To investigate when $x-1$, $x+1$ and x^2-1 are factors Let

$$f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\dots+p_n$$

be an integral expression Divide $f(x)$ by $x-1$, thus by the Remainder Theorem,

$$R=p_0+p_1+p_2+\dots+p_n$$

Now if $x-1$ be a factor of $f(x)$, R must vanish, therefore

$$p_0+p_1+p_2+\dots+p_n=0 \qquad (1)$$

when $x-1$ is a factor of $f(x)$ Hence *if the algebraic sum of the coefficients of an integral expression vanish, the expression has a factor* $x-1$

Again divide $f(x)$ by $x+1$, thus by the Remainder Theorem

$$R=(-1)^np_0+(-1)^{n-1}p_1+(-1)^{n-2}p_2+\dots+p_n$$

Now R must vanish if $x+1$ be a factor of $f(x)$; therefore

$$(-1)^n p_0+(-1)^{n-1}p_1+(-1)^{n-2}p_2+\ldots\ldots+p_n=0 \qquad (2)$$

when $x+1$ is a factor of $f(x)$. Hence *if the algebraic sum of the coefficients of the odd powers of* f(x) *be equal to that of the coefficients of the even powers, the expression has a factor* x+1.

Lastly from (1) whether n be even or odd, we have

$$p_0+p_1+p_2+\ldots\ldots+p_n=0 \qquad (\alpha);$$

and from (2), supposing n to be even, we have

$$p_0+p_2+p_4+\ldots\ldots+p_n=p_1+p_3+p_5+\ldots\ldots+p_{n-1} \qquad (\beta)$$

Therefore from (α) and (β), we have

$$2(p_0+p_2+p_4+\ldots\ldots+p_n)=0;$$

which shews that the left side cannot vanish [§ 265] unless

$$p_0+p_2+p_4+\ldots\ldots+p_n=0,$$

thus if $\quad p_0+p_2+p_4+\ldots\ldots+p_n=0$

and $\therefore$ from (β), $\quad p_1+p_3+p_5+\ldots\ldots+p_{n-1}=0 \qquad (3),$

$f(x)$ has a factor $(x-1)(x+1)$ or x^2-1.

The same result will similarly follow by supposing n to be odd. Hence *if the algebraic sum of the coefficients of the odd powers and that of the even powers severally vanish, the expression has a factor* x$^2-1$.

Examples.

Ex. 1 Factorise $x^4+6x^3-12x^2+2x+3$.

Here the *algebraic sum* of the coefficients of the several powers of x is zero. Therefore the expression has $x-1$ for a factor. Hence

$$\text{given expression}=x^3(x-1)+7x^2(x-1)-5x(x-1)-3(x-1)$$
$$=(x-1)(x^3+7x^2-5x-3).$$

The algebraical sum of the coefficients of the second factor is also zero; therefore it has a factor $x-1$. Hence

$$x^3+7x^2-5x-3=x^2(x-1)+8x(x-1)+3(x-1)=(x-1)(x^2+8x+3),$$

$\therefore$ the required factors are $(x-1)^2$ and x^2+8x+3.

Ex. 2 Factorise $2x^3+11x^2-26x-35$.

The sum of the coefficients of odd powers is $2-26=-24$, and the sum of the coefficients of even powers is $11-35=-24$. Thus $x+1$ is a factor of the given expression; therefore

$$\text{given expression}=2x^2(x+1)+9x(x+1)-35(x+1)$$
$$=(x+1)(2x^2+9x-35)=(x+1)(2x-5)(x+7)$$

Ex. 3. Factorise $15x^4-38x^3+9x^2+38x-24$

The sum of the coefficients of even powers is $15+9-24=0$, and the sum of the coefficients of odd powers is $38-38=0$. Thus x^2-1 is a factor of proposed expression, which is therefore

$$=15x^2(x^2-1)-38x(x^2-1)+24(x^2-1)$$
$$=(x^2-1)(15x^2-38x+24)=(x^2-1)(3x-4)(5x-6).$$

Factorise the expressions

4. $x^4+3x^3-7x^2+x+2$
5. $12x^3+13x^2+4x+3$
6. $20x^4-29x^3-5x^2+17x-3$
7. $3x^5+34x^4-32x^3-94x^2+29x+60$
7. $3x^5-10x^3+15x+8$
9. $5x^6-27x^5+54x^3-15x^2-27x+10$
10. $a^3+4a^2b-5ab^2-2b^3$
11. $8x^3+34x^2y+41xy^2+15y^3$
12. $3a^4-2a^3x-11a^2x^2+2ax^3+8x^4$
13. $x^4-6a^2x^2-8a^3x-3a^4$

Ex 14 Factorise $x^3-15x^2+71x-105$

Now 5 is a factor of 105, put $x=5$, thus given expression

$$=(5)^3-15(5)^2+71\times 5-105=0 \text{ identically},$$

therefore $x-5$ is a factor of the given expression. Hence the expression may be depressed to a *quadratic* and its factors may be found.

Ex 15 Factorise $6x^4-41x^3+95x^2-86x+24$

Now 2 is one of the factors of 24, and by trial we see that when 2 is substituted for x in the proposed expression, it vanishes, $\therefore$ $x-2$ is a factor of the expression. Thus

$$\text{given expression}=6x^3(x-2)-29x^2(x-2)+37x(x-2)-12(x-2)$$
$$=(x-2)(6x^3-29x^2+37x-12)$$

Let us resolve the second factor. One of the factors of 12 is 3 and by trial we see that 3 being put for x, causes this factor to vanish, therefore $x-3$ is a factor of this expression. Therefore

$$6x^3-29x^2+37x-12=6x^2(x-3)-11x(x-3)+4(x-3)$$
$$=(x-3)(6x^2-11x+4)=(x-3)(2x-1)(3x-4)$$

Therefore the proposed expression $=(x-2)(x-3)(2x-1)(3x-4)$

Ex. 16 Factorise $x^4-8x^3+17x^2-8x+16$

The proposed expression vanishes when 4 is put for x; therefore $x-4$ is a factor of the expression, thus

$$x^4-8x^3+17x^2-8x+16=x^3(x-4)-4x^2(x-4)+x(x-4)-4(x-4)$$
$$=(x-4)(x^3-4x^2+x-4)$$

Now x^3-4x^2+x-4 vanishes when again 4 is put for x, therefore $x-4$ is a factor of this expression, thus it

$$=x^2(x-4)+(x-4)=(x-4)(x^2+1)$$
$$\therefore \text{proposed expression}=(x-4)^2(x^2+1)$$

Factorise the expressions

17 $2x^4-9x^3-4x^2+51x-36$. 18. $6x^4+x^3y-x^2y^2-9xy^3+3y^4$

19 $a^4-10a^3x+35a^2x^2-50ax^3+24x^4$

279 Consistency of Equations We have seen that two equations are necessary to determine two variables. But suppose there are two such equations as

$$2x+3y=14 \text{ and } 6x+9y=42$$

A glance at once shews that the second equation is derived from the first by multiplying the latter by 3 Hence we cannot solve these equations definitely, as there is in fact only one independent equation [§ 237] Hence these equations are not *sufficient* to determine x and y, for which purpose, we must have another equation *independent* of the first, *i e*, one which must not be deduced from it by multiplying it by a constant.

Again let there be another pair of equations

$$2x+3y=14 \text{ and } 4x+6y=26.$$

By dividing the second equation by 2, we get $2x+3y=13$, therefore from the first equation, we obtain $14=13$, an absurd result. Hence the proposed equations are not *consistent*, and the values of x and y cannot be detemined from them

These simple cases present no difficulty, but the general ones, *viz*, those where the co efficients of the variables are letters, are not so simple. We therefore proceed to investigate them.

Let there be two equations in their general forms

$$a_1x+b_1y+c_1=0 \qquad (1),$$

$$a_2x+b_2y+c_2=0 \qquad (2).$$

Solving these we get

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\ \ y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1} \qquad (3)$$

Now if $a_1b_2-a_2b_1=0$, *i e*, if the denominators vanish, we get

$$\frac{a_2}{a_1}=\frac{b_2}{b_1}=\lambda \text{ suppose, } \therefore a_2=\lambda a_1,\ b_2=\lambda b_1.$$

Substitute the values of a_2 and b_2 in (2), thus

$$\lambda a_1x+\lambda b_1y+c_2=0,$$

or

$$a_1x+b_1y+\frac{c_2}{\lambda}=0 \qquad (4).$$

Now (4) differs from (1) only in the *constant term*; therefore (1) and (2) are *inconsistent* equations. Hence *the condition of inconsistency of the proposed equations* is

$$a_1b_2-a_2b_1=0, \text{ or } \frac{a_1}{a_2}=\frac{b_1}{b_2} \qquad (5).$$

Again if $\frac{c_2}{\lambda}=c_1$ or $\frac{c_2}{c_1}=\lambda$, then (2) is *consistent* with (1), but now it follows from (1) and is therefore not *independent* of (1). Therefore, if this condition hold, the proposed equations are *insufficient* for finding the values of x and y Hence *the conditions of insufficiency and consequently of consistency of the proposed equations are*

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2} \tag{6}$$

The conditions (5) and (6) of inconsistency and insufficiency of the given equations may be expressed in another form, for if (5) is satisfied, the values of x and y assume the forms $x=\frac{P}{0}$, $y=\frac{Q}{0}$, and if also (6) is satisfied, they assume the forms $x=\frac{0}{0}$, $y=\frac{0}{0}$ [§ 266] The latter forms shew that the values of x and y are *indeterminate* as they should be, since now the equations are no longer independent of one another.

Let us now investigate the condition under which

$$a_3x+b_3y+c_3=0 \tag{7}$$

is consistent with (1) and (2) It is clear that if (7) be consistent with (1) and (2), the values of x and y satisfying these, *i e*, the values (3), must also satisfy (7) Thus the required condition, after reduction is

$$a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0 \tag{8}$$

The above investigation shews that unless this condition is satisfied, (7) will not be consistent with (1) and (2), and (1) and (2) being two independent and consistent equations to determine x and y, a third equation is not at all necessary for the purpose Thus *to determine two variables two, and only two, independent and consistent, equations are necessary and sufficient*

Reasoning similarly we arrive at the general conclusion that *to determine* n *variables,* n *and only* n, *independent and consistent equations are necessary and sufficient*

EXAMPLES

Ex 1 For what value of λ will the equations $5x+3y=21$ and $15x+\lambda y=57$ be inconsistent ?

The condition of inconsistency is [see (5)]

$$5\lambda-45=0, \text{ whence } \lambda=9$$

Ex. 2 Find for what values of a and b, the equations $x+6y=27$ and $2x+ay=b$ will be consistent

The conditions of consistency are [see (6)]

$$\frac{1}{2}=\frac{6}{a}=\frac{27}{b}, \text{ whence } a=12 \text{ and } b=54$$

Ex 3. If the equations

$$x-a=y-b, \frac{x}{a+c}+\frac{y}{b+c}=1, \frac{x}{a-c}+\frac{y}{b-c}=1$$

be consistent, shew that $c^2-ab=0$

We may apply the general formula (8), or we may proceed thus :—Subtract the second equation from the third, thus

$$\left(\frac{1}{a-c}-\frac{1}{a+c}\right)x+\left(\frac{1}{b-c}-\frac{1}{b+c}\right)y=0, \text{ or } \frac{x}{a^2-c^2}+\frac{y}{b^2-c^2}=0 ;$$

whence $$x=k(a^2-c^2), \ y=k(c^2-b^2)$$

where k is some constant Therefore from the second equation

$$\frac{k(a^2-c^2)}{a+c}+\frac{k(c^2-b^2)}{b+c}=1 \text{ or } k=\frac{1}{a-b} ;$$

and therefore $$x=\frac{a^2-c^2}{a-b} \text{ and } y=\frac{c^2-b^2}{a-b}$$

Substitute these values in the first equation, therefore

$$\frac{a^2-c^2}{a-b}-a=\frac{c^2-b^2}{a-b}-b, \quad \text{whence } c^2-ab=0$$

280 Elimination It is the method of finding a relation among the coefficients of the variables We shall explain how this relation may be obtained when the variables occur in expressions of the first degree We have seen that n independent equations in n variables are sufficient to determine the n variables [§ 279] Hence if there be *one more* consistent equation involving the same number of variables, the variables can be *eliminated* from the given equations *by simply substituting their values in the* $(n+1)^{th}$ *equation* This is the general method of elimination when the given equations are *non-homogeneous*, as for instance when they are of the form

$$ax+by+c=0, a'x+b'y+c'=0, a''x+b''y+c''=0$$

But if they be *homogeneous*, *only* n *equations are sufficient*, for then by dividing each of the equations by one of the variables, we can reduce their number to *one* less, and thus have a sufficient number of equations to eliminate the variables Thus

$ax+by=0, a'x+b'y=0$ become $a\frac{x}{y}+b=0$ and $a'\frac{x}{y}+b'=0$, where the ratio $\frac{x}{y}$ ($=z$ say) may, for the purposes of elimination, be considered as one variable

Examples

Ex 1 Eliminate x and y from the equations

$$ax+by=c,\ a'x+b'y=c',\ a''x+b''y=c''.$$

From (1) and (2), we get [§ 244]

$$x=\frac{b'c-bc'}{ab'-a'b},\quad y=\frac{c'a-ca'}{ab'-a'b};$$

substitute in (3), thus

$$a''\frac{b'c-bc'}{ab'-a'b}+b''\frac{c'a-ca'}{ab'-a'b}=c'',$$

$$a''(b'c-bc')+b''(c'a-ca')=c''(ab'-a'b)$$

Ex 2 Eliminate x, y and z from the equations

$$\frac{x}{y+z}=a,\quad \frac{y}{z+x}=b,\quad \frac{z}{x+y}=c$$

From the given equations, we have

$$x-ay-az=0,\ -bx+y-bz=0 \qquad (1),$$

$$-cx-cy+z=0 \qquad (2).$$

These are *homogeneous*, hence the three are sufficient to eliminate the variables x, y and z We have from (1) by § 244,

$$\frac{x}{ab+a}=\frac{y}{ab+b}=\frac{z}{1-ab}=k \text{ suppose},$$

$$\therefore\ x=k(ab+a),\ y=k(ab+b),\ z=k(1-ab)$$

Substitute in (2), thus

$$-ck(ab+a)-ck(ab+b)+k(1-ab)=0,$$

whence by dividing by k, and transposing,

$$ab+bc+ac+2abc=1$$

3 Eliminate x and y from the equations

$$ax+by=c,\ mx-ny=d,\ nx+ay=m$$

4 Eliminate x between the equations

$$(x-a)(x-b)=(x-c)(x-d)=(x-e)(x-f).$$

5. Eliminate x, y and z from the equations

$$ax+by+cz=0,\ bx+cy+az=0,\ cx+ay+bz=0$$

6. Eliminate a, b, c between the equations

$$bz+cy=a,\ cx+az=b,\ ay+bx=c$$

7. If $y+z : z+x : x+y=a : b : c$, and $ax+by+cz=0$, shew that

$$2(a^2+b^2+c^2)=(a+b+c)^2$$

8. Given that

$$x=by+cz+du,\ y=ax+cz+du,\ z=ax+by+du,\ u=ax+by+cz,$$

prove that $$1=\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}.$$

We shall now give some examples of elimination where the variables occur in special expressions of higher degree

Ex. 9 Eliminate x, y and z from the equations

$$\frac{y}{z}+\frac{z}{y}=a,\quad \frac{z}{x}+\frac{x}{z}=b,\quad \frac{x}{y}+\frac{y}{x}=c$$

We know [§ 183, Ex. 9] that

$$\left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z}{x}+\frac{x}{z}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)^2=4+\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right)\left(\frac{x}{y}+\frac{y}{x}\right),$$

$$\therefore\ a^2+b^2+c^2=4+abc$$

Ex. 10 Eliminate x between the equations

$$ax^2+bx+c=0,\ cx^2+ax+b=0$$

By § 244, we have

$$\frac{x^2}{b^2-ca}=\frac{x}{c^2-ab}=\frac{1}{a^2-bc},$$

$$\therefore\ x^2=\frac{b^2-ca}{a^2-bc},\quad x=\frac{c^2-ab}{a^2-bc},$$

whence $$\frac{b^2-ca}{a^2-bc}=x^2=\left(\frac{c^2-ab}{a^2-bc}\right)^2,\ \text{or } a^3+b^3+c^3-3abc=0$$

Ex 11. Eliminate x from the equations

$$32\frac{c}{a}=\left(\frac{x}{a}\right)^5+10\frac{x}{a}+5\left(\frac{a}{x}\right)^3,\quad 32\frac{a}{c}=\left(\frac{a}{x}\right)^5+10\frac{a}{x}+5\left(\frac{x}{a}\right)^3.$$

By addition we have

$$\left(\frac{x}{a}\right)^5+5\left(\frac{x}{a}\right)^3+10\frac{x}{a}+10\frac{a}{x}+5\left(\frac{a}{x}\right)^3+\left(\frac{a}{x}\right)^5=32\left(\frac{c}{a}+\frac{a}{c}\right),$$

or $$\left(\frac{x}{a}+\frac{a}{x}\right)^5=32\left(\frac{c}{a}+\frac{a}{c}\right),\quad \therefore\ \frac{x}{a}+\frac{a}{x}=2\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{1}{5}} \tag{1}$$

Similarly by subtraction, we have $$\frac{x}{a}-\frac{a}{x}=2\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{1}{5}} \tag{2}.$$

Squaring (1) and (2), we get

$$\frac{x^2}{a^2}+\frac{a^2}{x^2}+2=4\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{2}{5}},\quad \frac{x^2}{a^2}+\frac{a^2}{x^2}-2=4\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{2}{5}};$$

$$\therefore\ 4=4\left\{\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{2}{5}}-\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{2}{5}}\right\},\ \text{or}\ \left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{2}{5}}-\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{2}{5}}=1.$$

12. Eliminate x from $x+\frac{1}{x}=a$, $x^3+\frac{1}{x^3}=b^3-3b$

13. Eliminate x and y from $x-y=a$, $x^2+y^2=b^2$, $xy=c^2$

14. Eliminate x and y between the equations

$$px-qy+q=py-qx+p,\ x^3+3xy^2=p^3,\ y^3+3x^2y=q^3.$$

15. Eliminate x, y and z from

$$x^2(y+z)=a^3,\ y^2(z+x)=b^3,\ z^2(x+y)=c^3,\ xyz=abc.$$

16. Eliminate x between the equations

$$\frac{x^3}{a^3}+\frac{a^3}{x^3}+3\left(\frac{x}{a}+\frac{a}{x}\right)=m,\ \frac{x^3}{a^3}-\frac{a^3}{x^3}-3\left(\frac{x}{a}-\frac{a}{x}\right)=n$$

17. Eliminate x and y between the equations

$$ax+by=c\sqrt{(x^2+y^2)},\ a'x+b'y=c'\sqrt{(x^2+y^2)}.$$

18. Eliminate x and y from the equations

$$a=x^3y,\ b=xy^3,\ xy+1=c(x+y)$$

19. Shew that if

$$ax^2+by^2+cz^2=ax+by+cz=yz+zx+xy=0,$$

then $$abc=(b+c-a)(c+a-b)(a+b-c)$$

281 Examples

Ex 1 If $ax^2+\lambda x+c$ be a perfect square with respect to x, find the value of λ

It is evident that the proposed expression must be the square of

$$\sqrt{a}x+\sqrt{c},\ \text{which} =ax^2+2\sqrt{(ac)}x+c\ ,$$

whence by § 272, $$\lambda=2\sqrt{ac}$$

Otherwise .—Proceed as in the extraction of the square root.

$$\begin{array}{r|l}
 & ax^2+\lambda x+c\left(\sqrt{a}x+\dfrac{\lambda}{2\sqrt{a}}\right. \\
 & ax^2 \\
2\sqrt{a}x+\dfrac{\lambda}{2\sqrt{a}} & \lambda x+c \\
 & \lambda x+\dfrac{\lambda^2}{4a} \\
\hline
 & c-\dfrac{\lambda^2}{4a}.
\end{array}$$

Now since the proposed expression is an exact square, the remainder must vanish, therefore

$$c-\frac{\lambda^2}{4a}=0, \text{ or } \lambda^2=4ac$$

Ex. 2. The expression $ax^2+2hxy+by^2+2gx+2fy+c$ will be an exact square, if $$abc=af^2=bg^2=ch^2.$$

Let the proposed expression be the square of $\lambda x+\mu y+\sqrt{c}$. By expanding its square, we get
$$\lambda^2x^2+2\lambda\mu xy+\mu^2y^2+2\lambda\sqrt{c}x+2\mu\sqrt{c}y+c\ ;$$
whence equating the coefficients [§ 272], we have
$$\lambda^2=a,\ 2\lambda\mu=2h,\ \mu^2=b,\ 2\lambda\sqrt{c}=2g,\ 2\mu\sqrt{c}=2f$$
Therefore from the first three, $h^2=ab$, from first and fourth, $g^2=ac$, and from third and fifth, $f^2=bc$ Hence multiplying respectively by c, b and h, we have the required relations

Note We might have assumed the proposed expression to be the square of $\sqrt{a}x+\lambda y+\sqrt{c}$.

Ex 3 Shew that $ax^3+3bx^2+3cx+d$ will be a perfect cube, if $ad=bc$

Let the proposed expression be the cube of $\lambda x+\mu$ By developing the cube of $\lambda x+\mu$, and equating the coefficients [§ 272], we get
$$\lambda^3=a,\ \lambda^2\mu=b,\ \lambda\mu^2=c,\ \mu^3=d\ ,$$
whence $$ad=\lambda^3\mu^3=bc$$

Note Instead of assuming $\lambda x+\mu$ as the cube root of the given expression, we might have evidently assumed $\sqrt[3]{a}x+\sqrt[3]{d}$ as the cube root

Ex 4. If $2a-3y=\dfrac{(z-x)^2}{y}$ and $2a-3z=\dfrac{(x-y)^2}{z}$, x, y, z being unequal, then will $2a-3x=\dfrac{(y-z)^2}{x}$, and $x+y+z=a$.

Subtract the first equation from the second, thus
$$3y-3z=\frac{(x-y)^2}{z}-\frac{(z-x)^2}{y},$$
or $$3(y-z)yz=y(x-y)^2-z(z-x)^2\ ,$$
whence by dividing by $y-z$, which is not $=0$ by supposition, we get
$$x^2+y^2+z^2=2yz+2zx+2xy \qquad (\alpha).$$

From the first equation
$$2a=\frac{(z-x)^2}{y}+3y=\frac{z^2+x^2-2zx+3y^2}{y}=\frac{2y^2+2yz+2xy}{y} \text{ using } (\alpha)$$
$$=2(x+y+z)\ ,$$
$$\therefore\ x+y+z=a \qquad (\beta).$$

Again from (α), $y^2+z^2-2yz=2zx+2xy-x^2$

or $$\frac{(y-z)^2}{x}=2z+2y-x=2a-3x, \text{ from } (\beta).$$

Ex 5 If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$, and $(b-c)^2x+(c-a)^2y+(a-b)^2z=0$,

then will $$(b-c)x=(c-a)y=(a-b)z$$

Eliminating z between the given equations, we have

$$(b-c)^2x+(c-a)^2y-(a-b)^2\frac{xy}{x+y}=0 ;$$

whence after reduction $(b-c)^2x^2-2(b-c)(c-a)xy+(c-a)^2y^2=0$;

or $$\{(b-c)x-(c-a)y\}^2=0 , \qquad (b-c)x=(c-a)y$$

And since the given expressions are symmetrical, we have

$$(c-a)y=(a-b)z ,$$

$$(b-c)x=(c-a)y=(a-b)z$$

Ex. 6 If $a=bz+cy$, $b=cx+az$, $c=ay+bx$, shew that

$$\frac{a^2}{1-x^2}=\frac{b^2}{1-y^2}=\frac{c^2}{1-z^2}$$

By transposition, we get

$$-a+bz+cy=0 \qquad (1),$$

$$az-b+cx=0 \qquad (2),$$

$$ay+bx-c=0 \qquad (3)$$

Thus from (1) and (2), and (2) and (3), we get, by Cross Multiplication [§ 244],

$$\frac{a}{zx+y}=\frac{c}{1-z^2}, \quad \frac{a}{1-x^2}=\frac{c}{zx+y} ,$$

whence $$\frac{a^2}{1-x^2}=\frac{c^2}{1-z^2} \qquad (4)$$

Similarly from (2) and (3), and (3) and (1), we have

$$\frac{a}{1-x^2}=\frac{b}{xy+z}, \quad \frac{a}{xy+z}=\frac{b}{1-y^2},$$

whence $$\frac{a^2}{1-x^2}=\frac{b^2}{1-y^2} \qquad (5)$$

Thus the required relations follow from (4) and (5)

7 For what value of k will $3x^2+kx+1$ have two linear factors ?

8 Find the value of h which will make $2x^2+hx+3$ a complete square.

9 For what value of λ, μ and ν will $4x^2+\lambda xy+9y^2+\mu x+\nu y+16$ be an exact square ?

10. If ax^3+bx^2+cx+d be a perfect cube, shew that $b^2=3ac$, $c^2=3bd$ and $c^3=27ad^2$

11. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$

12. Shew that $a(x+1)^2+bx^2+2cx(x+1)$ is a perfect square with respect to x, if $c^2-ab=0$

13. If $x^4+ax^3+bx^2+cx+d$ be an exact square, then the relations between the coefficients are

$$8c=a(4b-a^2),\ (4b-a^2)^2=64d$$

14. If $ax^4+bx^3+cx^2$ be subtracted from $(x^2+2x+4)^2$, the remainder is an exact square; find a, b, c

15. Shew that $4x(x+a)(x+b)(x+c)+a^2b^2$ is a perfect *square*, if $c=a+b$ and a perfect *fourth* power, if also $c^2=2ab$

16. Find the value of y which will make

$$2(y^2+y)x^2+(11y-2)x+4$$

and $$2(y^3+y^2)x^3+(11y^2-2y)x^2+(y^2+5y)x+5y-1$$

have a common factor, and find the factor

17. If $x+\frac{1}{x}=a+\frac{1}{a}$, shew that $x^n+\frac{1}{x^n}=a^n+\frac{1}{a^n}$

18. If $ax^3=by^3=cz^3$ and $x^{-1}+y^{-1}+z^{-1}=1$, shew that

$$ax^2+by^2+cz^2=(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})^3.$$

19. If $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$, shew that

$$(a+b+c)(x+y+z)=\sqrt{a^3+b^3+c^3-3abc}.$$

20. If $(a^2+bc)^2(b^2+ca)^2(c^2+ab)^2=(a^2-bc)^2(b^2-ca)^2(c^2-ab)^2$, then either $a^3+b^3+c^3+abc=0$, or $a^{-3}+b^{-3}+c^{-3}+a^{-1}b^{-1}c^{-1}=0$

21. If $$(a^2-bc)(b^2-ca)(c^2-ab)=0,$$

prove that $$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{a^3+b^3+c^3}{a^2b^2c^2}$$

22. If $a^2+b^2+c^2+\dots\dots$ to n terms $=2x\left(a+b+c+\dots\dots-\frac{nx}{2}\right)$,

then $a=b=c=\dots\dots$, $a, b, c, \dots x$ being all real

23. If $(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2$, shew that

$$x:y:z=a:b:c$$

24. If $a_1+a_2+a_3+\dots\dots+a_n=\frac{1}{2}ns$, prove that

$$(s-a_1)^2+(s-a_2)^2+\dots\dots+(s-a_n)^2=a_1^2+a_2^2+\dots\dots+a_n^2.$$

25. If $\sqrt{ay^2-a^2}=yz$ and $\sqrt{ax^2-a^2}=xy$, then $\sqrt{az^2-a^2}=xz$

26. If $x+\frac{1}{x}=2\sqrt{1+m^2}$, $y+\frac{1}{y}=2\sqrt{1+n^2}$ and $\frac{m}{a}=\frac{n}{b}$,

shew that $$ay+\frac{b}{x}=bx+\frac{a}{y}$$

27. If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$, then $\frac{1}{x^2-yz}+\frac{1}{y^2-zx}+\frac{1}{z^2-xy}=0$

28. If $x+y+z=0$, then $\frac{x^2}{2x^2+yz}+\frac{y^2}{2y^2+zx}+\frac{z^2}{2z^2+xy}=1$ [*App*]

29. If $xy+yz+zx=1$, then will

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

30. If $x=\frac{1}{1-y}$ and $y=\frac{z}{1-z}$, shew that

$$z=1+x+2x^2+4x^3+ \dots$$

31 If $a=b\frac{y}{z}+c\frac{z}{y}$, $b=c\frac{z}{x}+a\frac{x}{z}$, $c=a\frac{x}{y}+b\frac{y}{x}$, prove that

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}+\frac{1}{xyz}=0 \quad [App]$$

32 If $bc+ca+ab=1$, shew that

$$\left\{1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right\}^2=\frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)}$$

33 If $\frac{ad-bc}{a-b-c+d}=\frac{ac-bd}{a-b-d+c}$, then each $=\frac{1}{4}(a+b+c+d)$.

34 If $(a+b)(x+y)=2(ab+xy)$ and $(c+d)(x+y)=2(cd+xy)$, prove that

$$\left(\frac{x-y}{2}\right)^2=\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}$$

35 If $yz+zx+xy=1$, prove that

$$x\left\{\frac{(1+y^2)(1+z^2)}{1+x^2}\right\}^{\frac{1}{2}}+y\left\{\frac{(1+z^2)(1+x^2)}{1+y^2}\right\}^{\frac{1}{2}}+z\left\{\frac{(1+x^2)(1+y^2)}{1+z^2}\right\}^{\frac{1}{2}}=2$$

36. Given $U=\sqrt{1+x^2}-\sqrt{1+y^2}$ and $V=\frac{\sqrt{1+x^2}-1}{\sqrt{1+y^2}-1}\frac{y}{x}$, shew that

$$\frac{1}{2}xy=\frac{U}{V-V^{-1}}$$

37 Prove that two of the quantities a, b, c, must be equal to one another, if

$$\frac{b-c}{1+bc}+\frac{c-a}{1+ca}+\frac{a-b}{1+ab}=0$$

38 If $\frac{1}{x^3}\left(\frac{y}{z}+\frac{z}{y}\right)=\frac{1}{y^3}\left(\frac{z}{x}+\frac{x}{z}\right)=\frac{1}{z^3}\left(\frac{x}{y}+\frac{y}{x}\right)$, shew that each $=\frac{2}{xyz}$.

39. If $x=\frac{a+1}{a-1}$, $y=\frac{b+1}{b-1}$, $z=\frac{c+1}{c-1}$, shew that

$$\frac{(1+x^2)(1+y^2)(1+z^2)}{(1+xy)(1+yz)(1+zx)}=\frac{(1+a^2)(1+b^2)(1+c^2)}{(1+ab)(1+bc)(1+ca)}.$$

40. If x, y, z, be unequal and $\frac{x^2-yz}{x-xyz}=\frac{y^2-zx}{y-xyz}$, shew that each of these ratios is equal

$$\text{to } \frac{z^2-xy}{z-xyz}, \text{ to } x+y+z, \text{ and to } \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

41. If x, y, z, be unequal and $\frac{yz-x^2}{y+z}=\frac{zx-y^2}{z+x}$, prove that each of these $=\frac{xy-z^2}{x+y}$.

42. If $\frac{x^2-y^2}{a-b}=\frac{xy}{z}$ and $\frac{y^2-z^2}{b-c}=\frac{yz}{x}$, then $\frac{z^2-x^2}{c-a}=\frac{zx}{y}$.

43. If $a+b+c=0$, and

$$a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz),$$

then will $\quad x+y+z=0.$ [*App.*]

CHAPTER XXIII.

QUADRATIC EQUATIONS

282 Quadratic Equation A QUADRATIC EQUATION in a certain variable is one in which, when the equation is reduced to a rational and integral form, the *second and no higher, power* of that variable occurs [see §§ 220 and 221] Thus $x^2+px+q=0$ is a quadratic equation in *one* variable, *viz*, x, and $xy+x+y=a$ is a quadratic equation in *two* variables, *viz*, x and y [see § 229] As has been pointed out before, to ascertain the degree of an equation, we must see whether it is in a rational and integral form as far as the *variable* or *variables* are concerned Hence the following are other examples of quadratic equations—

(i) $x+\frac{1}{x}=a$, (ii) $x+\frac{1}{y}=m$, (iii) $x+\sqrt{x}+c=0$, &c

Note. It is well that the student should at the outset distinguish between a *quadratic expression* and a *quadratic equation*. In the quadratic *expression* ax^2+bx+c, the variable x may have *any* value whatever but *when that expression is equated to zero*, it becomes the quadratic *equation* $ax^2+bx+c=0$ and then the variable x must have a *definite number* of values.

283 Different forms of Quadratics The most general form of a quadratic equation in one variable x is

$$ax^2+bx+c=0 \qquad (1),$$

for when the equation is reduced to a rational and integral form, all of its terms may be transposed to one side, and then the terms involving x^2 may be collected together, those involving x may be collected together, and the constant terms may be collected together, and thus the coefficients of x^2 and x, and the constant terms may be bracketed together, thus the a, b and c will be found [see § 222] Hence a quadratic equation in its general form consists of only *three terms* [§ 267]

For particular values of a, b and c, the equation (1) may assume **special forms** Thus by dividing this equation by a, we have

$$x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

which, when p is put for $\frac{b}{a}$ and q for $\frac{c}{a}$, becomes

$$x^2+px+q=0$$

Thus when the coefficient of x^2 is made *unity* by division, (1) assumes the form

$$x^2+px+q=0 \qquad (2).$$

When $b=0$, (1) reduces to the form

$$ax^2+c=0 \qquad (3)$$

When $c=0$, (1) has the form

$$ax^2+bx=0 \qquad (4)$$

Lastly when $b=0$ and $c=0$, the form of (1) is

$$ax^2=0 \qquad (5)$$

Thus it appears that *any equation which after reduction assumes any one of the above forms is a Quadratic Equation*

Sometimes equations of the form (3) are called PURE QUADRATICS and those of the forms (1) or (2) are called ADFECTED QUADRATICS Thus *a* PURE *Quadratic is one in which only the second power of the variable occurs*, and *an* ADFECTED *Quadratic is one in which both the second and first powers occur*

284 Theorem *The quadratic expression ax^2+bx+c has two linear factors* and no more*

Let $x-a$ be a factor of the given expression ; thus it is divisible by $x-a$, therefore

$$ax^2+bx+c=Q(x-a),$$

* By *factor* is here meant a *factor in x*

where Q is of $(2-1)^{th}$ degree in x, i e, *linear*. Now if $x-\beta$ be another factor of the given expression, then Q is divisible by $x-\beta$; therefore

$$Q \equiv q(x-\beta);$$

thus
$$ax^2+bx+c \equiv q(x-a)(x-\beta),$$

where q is of $(2-2)^{th}$ degree or *zeroth* degree in x, that is, q is a *constant* Hence if the proposed expression has any other factor, that factor must be independent of x, and by comparing the coefficients of x^2 [§ 272], we find $q=a$ Thus

$$ax^2+bx+c \equiv a(x-a)(x-\beta).$$

285 Theorem. *A quadratic equation has only* TWO ROOTS, *and no more*

Let the quadratic equation be $ax^2+bx+c=0$.

Then since $ax^2+bx+c \equiv a(x-a)(x-\beta)$ [§ 284], we have

$$a(x-a)(x-\beta)=0$$

Now $a \neq 0$, for then the quadratic will reduce to $bx+c=0$, which is a simple equation, therefore $a(x-a)(x-\beta)$, cannot $=0$, unless $x-a=0$, or $x-\beta=0$ [§ 265], that is, unless $x=a$ or $x=\beta$. Thus the quadratic has only *two roots*, a and β.

Note 1 From this and the last article, it is evident that the quadratic *expression* $ax^2+bx+c=a(x-a)(x-\beta)$ always, where a and β are the roots of the quadratic *equation* $ax^2+bx+c=0$

This Theorem is usually proved otherwise thus —

If possible let the proposed quadratic have three different roots, a, β and γ, then each of these roots will satisfy it.

Hence
$$a a^2+ba+c=0 \qquad (1),$$
$$a\beta^2+b\beta+c=0 \qquad (2),$$
$$a\gamma^2+b\gamma+c=0 \qquad (3).$$

Subtract (2) from (1), thus

$$a(a^2-\beta^2)+b(a-\beta)=0,$$

or dividing by $a-\beta$, which is by supposition not 0,

$$a(a+\beta)+b=0 \qquad (4).$$

Similarly from (2) and (3), we get

$$a(\beta+\gamma)+b=0 \qquad (5).$$

Subtract (5) from (4), thus

$$a(a-\gamma)=0 \qquad (6).$$

Now either $a=0$, or $a-\gamma=0$; but a cannot be equal to 0, for the reason given above; therefore

$$a-\gamma=0, \text{ i e, } a=\gamma.$$

Thus α and γ are not *two different* quantities but *one and the same* quantity Hence the quadratic equation has only two roots α and β

[This proof fails, when the roots of a proposed quadratic are *equal*]

Note 2. If, however, α be different from γ, then from (6), $a=0$ [§ 265] and therefore from (4), $b=0$ and from (1), $c=0$, and then the equation is clearly satisfied by *all values* of x Thus when a quadratic is satisfied by *more than two values* of the variable, it is not an *equation* but an *identity*

286 Special forms of Quadratics Before we proceed to solve the general equation $ax^2+bx+c=0$, we shall notice some of its particular cases as being simple

If $c=0$, the equation takes the form

$$ax^2+bx=0 \qquad (1),$$

whence $$x(ax+b)=0$$

thus either $x=0$, or $ax+b=0$, which latter gives $x=-\dfrac{b}{a}$

Hence the roots of the equations of the form (1) are 0 and $-\dfrac{b}{a}$

If $b=0$ and $c=0$, the equation reduces to

$$ax^2=0 \qquad (2),$$

whence, since a is not supposed to be 0, $x^2=0$ Hence the roots of equations of the from (3) are 0 and 0

If $b=0$, the equation takes the form of a *pure quadratic*

$$ax^2+c=0 \qquad (3)$$

To solve this, transpose c and divide by a, thus $x^2=-\dfrac{c}{a}$, or $x=\pm\sqrt{-\dfrac{c}{a}}$, that is, the roots are $+\sqrt{-\dfrac{c}{a}}$ and $-\sqrt{-\dfrac{c}{a}}$ Hence the roots of a pure quadratic are *equal in magnitude but of opposite signs.*

REMARK. In extracting the square root, it is sufficient to affect with the double sign *only the right side* of an equation and not *both* of its sides Thus from $x^2=a^2$, we put $x=\pm a$, and not $\pm x=\pm a$, as the first virtually includes all the cases of the second For from $\pm x=\pm a$, we have

$+x=+a$	(i),	$-x=+a$	(iii),
$+x=-a$	(ii),	$-x=-a$	(iv) ;

and when the signs of (iii) and (iv) are changed, they are respectively the same as (ii) and (i)

Examples.

Solve the equations

1. $3x^2-7=41$

2. $6x^2+5=68-x^2$.

3. $5x^2-121=4(26-x^2)$

4. $3(2x-3)^2=4x(2x-9)+43$.

5. $2x+\dfrac{17}{x}=\dfrac{7x-10}{2\frac{1}{3}}+4$

6. $(x-7)(x+7)=31-4x^2$

7. $\dfrac{x-1}{x+2}=\dfrac{2x+1}{5x-2}$

8. $\dfrac{2x-1}{x-2}=\dfrac{x-5}{3x-2}$.

9. $\dfrac{x}{2}+\dfrac{2}{x}=\dfrac{x}{3}+\dfrac{3}{x}$

10. $(x-a)^2+(x-b)^2=a^2+b^2$.

11. $(x+4)(2x+9)=(2x+\frac{17}{5})2x-13$.

12. $\dfrac{a^2}{(x-a)^2}=\dfrac{b^2}{(x-b)^2}$

13. $(2x-5)(3x-2)-(x-2)(2x-3)=4$

14. $\dfrac{2x}{x-2}+\dfrac{x-2}{x}=2$.

15. $\dfrac{x-7}{x-4}+\dfrac{5}{2+x}=\dfrac{2}{x^2-2x-8}$.

16. $\dfrac{x+7}{x(x-7)}-\dfrac{x-7}{x(x+7)}=\dfrac{7}{x^2-7^2}$.

17. $x+\dfrac{2}{1+\dfrac{1}{x}}=3$

18. $\sqrt{3x+1}-\sqrt{x+1}=\sqrt{3x}$

19. $\sqrt{2x+6}-\sqrt{x-1}=2$

20. $x\sqrt{x^2+12}+x\sqrt{x^2+6}=3$

21. $\sqrt{1+\dfrac{bx}{a^2}}+\sqrt{1-\dfrac{bx}{a^2}}=1\frac{3}{5}$.

22. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x$

23. $\dfrac{\sqrt{1-x}}{2-\sqrt{1+x}}=\dfrac{\sqrt{1+x}}{2+\sqrt{1-x}}$

24. $\dfrac{\sqrt{1+x}-1}{\sqrt{1-x}+1}+\dfrac{\sqrt{1-x}+1}{\sqrt{1+x}-1}=2a$.

25. $\dfrac{1+x^3}{(1+x)^2}+\dfrac{1-x^3}{(1-x)^2}=a$.

26. $\sqrt{x^2+9}+\sqrt{x^2-9}=4+\sqrt{34}$

[*For other examples, see* §§ 225 *and* 227].

287. Solutions of Quadratics by Fractorisation. We have seen [§ 282, *Note*] that a *complete* quadratic expression, when equated to 0, gives a quadratic equation whose terms are all transposed to one side. Hence a quadratic equation may be solved by factorising the corresponding quadratic expression, as exemplified below.

Ex 1 Solve the equation $x^2-12x+35=0$

Now $x^2-12x+35=(x-5)(x-7)$,

$\therefore \quad (x-5)(x-7)=0$,

whence [§ 265], $x-5=0$, or $x-7=0$,

that is, $x=5$, or $x=7$

Ex 2 Solve the equation $x^2+3x=54$

Transpose, thus $x^2+3x-54=0$,

$(x-6)(x+9)=0$,

whence $x-6=0$ or $x+9=0$,

$x=6$, or $x=-9$

Ex. 3. Solve the equation $8x^2+3=14x$

Transpose, thus $8x^2-14x+3=0$,

whence $(2x-3)(4x-1)=0$,

thus $2x-3=0$, or $4x-1=0$,

$\therefore \quad x=\frac{3}{2}$ or $x=\frac{1}{4}$

Note. From these examples it is easy to see that if an equation be given *as the product of factors equated to* 0, we can at once put down its roots

Example Solve the equation $(x-1)(x-2)(x-3)=0$

Here either $x-1=0$, or $x-2=0$, or $x-3=0$ [§ 265],

thus $x=1$, or 2, or 3

288 Solution of Quadratics by Completing the Square There are two methods of completing the square—(1) the "*Common Method*" and (2) "*Sridhara's Method*" which is commonly known as the "*Hindu Method*" [see § 127]

Common Method

RULE.—*Reduce the quadratic to the general form, transpose the constant term, divide by the coefficient of* x^2, *and add to both sides the square of half the coefficient of* x

Example, Solve $ax^2+bx+c=0$

Transpose c, thus $ax^2+bx=-c$,

divide by a, thus $x^2+\frac{b}{a}x=-\frac{c}{a}$,

add $\left(\frac{1}{2}\text{ of }\frac{b}{a}\right)^2$, that is, $\left(\frac{b}{2a}\right)^2$ to both sides, thus

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}, \text{ or } \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2},$$

extract the square root, thus

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}\ [286,\ \textit{Remark}],$$

$$\therefore\ x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Hence the roots are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-a-\sqrt{b^2-4ac}}{2a}$.

Hindu Method

RULE —*Reduce the quadratic to the general form, transpose the constant term, multiply by four times the coefficient of* x², *and add to both sides the square of the coefficient of* x

Explanation. x^2 has always a coefficient, whether it be *unity* or some other number

Example Solve $ax^2+bx+c=0$.

Transpose c, thus $\qquad ax^2+bx=-c,$

multiply by $4a$, thus $\qquad 4a^2x^2+4abx=-4ac,$

add b^2 to both sides, thus

$$4a^2x^2+4abx+b^2=b^2-4ac, \text{ or } (2ax+b)^2=b^2-4ac,$$

extract the square root, thus

$$2ax+b=\pm\sqrt{b^2-4ac},$$

transpose b, and divide by $2a$,

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

289. Omitted

290 Examples. The methods of solution given above will of course apply to each particular case as will be presently seen.

Ex. 1. Solve $x^2+7x+12=0$.

Common Method.

Transpose, thus $\qquad x^2+7x=-12,$

add ($\frac{1}{2}$ of 7)², thus $\qquad x^2+7x+(\frac{7}{2})^2=(\frac{7}{2})^2-12,$

or $\qquad (x+\frac{7}{2})^2=\frac{49}{4}-12=\frac{1}{4},$

extract the square root, thus

$$x+\tfrac{7}{2}=\pm\sqrt{\tfrac{1}{4}}=\pm\tfrac{1}{2},$$

transpose, thus $\qquad x=-\frac{7}{2}\pm\frac{1}{2}=-3$ or -4

Hindu Method

Transpose, thus $x^2+7x=-12$,
multiply by 4×1 [see *Explanation*, § 288] or 4, thus

$$4x^2+28x=-48,$$

add 7^2, thus $4x^2+28x+7^2=7^2-48$,
or $(2x+7)^2=49-48=1$,
extract the square root, thus

$$2x+7=\pm\sqrt{1}=\pm 1,$$

transpose, thus $2x=-7\pm 1=-6$ or -8,
divide by 2, thus $x=-3$ or -4

Exercise (i)

Solve the equations

1 $x^2+5x+4=0$ 2 $x^2+3x+2=0$ 3 $x^2+15x+36=0$
4 $x^2+11x+24=0$ 5 $x^2+10x+21=0$. 6. $x^2+12x+35=0$
7. $x^2+24x+80=0$ 8 $x^2+105x+2000=0$

Ex 2 Solve $x^2-48x+527=0$

Transpose, thus $x^2-48x=-527$,
add ($\frac{1}{2}$ of $48)^2$ or $(24)^2$, thus

$$x^2-48x+(24)^2=(24)^2-527,$$

or $(x-24)^2=576-527=49$,
extract the square root, thus

$$x-24=\pm\sqrt{49}=\pm 7,$$

transpose, thus $x=24\pm 7=31$ or 17

[Solve this equation by the Hindu Method]

Exercise (ii)

Solve the equations

1 $x^2-9x+20=0$ 2 $x^2-7x+12=0$ 3 $x^2-27x+170=0$.
4 $x^2-12x+35=0$ 5 $x^2-24x+80=0$ 6 $x^2-50x+429=0$.

Ex 3 Solve $x^2-\frac{8}{3}x-1=0$

Transpose, thus $x^2-\frac{8}{3}x=1$,
add ($\frac{1}{2}$ of $\frac{8}{3})^2$ or $(\frac{4}{3})^2$, thus $x^2-\frac{8}{3}x+(\frac{4}{3})^2=(\frac{4}{3})^2+1$,
that is, $(x-\frac{4}{3})^2=\frac{16}{9}+1=\frac{25}{9}$,
extracting the square root of both sides, we get

$$x-\frac{4}{3}=\pm\sqrt{\tfrac{25}{9}}=\pm\tfrac{5}{3},$$

$$x=\tfrac{4}{3}\pm\tfrac{5}{3}=3 \text{ or } -\tfrac{1}{3}$$

Exercise (iii)

Solve the equations

1. $x^2-\frac{1}{3}x-8=0$ 2 $x^2+\frac{15}{4}x+\frac{7}{2}=0$ 3. $x^2+\frac{13}{4}x=3$
4. $x^2+\frac{15}{2}x=81.$ 5 $x^2-\frac{35}{6}x+6=0.$ 6. $x^2-\frac{3}{4}x=\frac{1}{4}.$

Ex 4. Solve $2x^2-5x+3=0$

Divide by 2, thus $x^2-\frac{5}{2}x+\frac{3}{2}=0,$

transpose $\frac{3}{2}$, thus $x^2-\frac{5}{2}x=-\frac{3}{2},$

add $(\frac{1}{2}\text{ of }\frac{5}{2})^2$, thus $x^2-\frac{5}{2}x+(\frac{5}{4})^2=(\frac{5}{4})^2-\frac{3}{2},$

or $(x-\frac{5}{4})^2=\frac{25}{16}-\frac{3}{2}=\frac{1}{16},$

whence $x-\frac{5}{4}=\pm\sqrt{\frac{1}{16}}=\pm\frac{1}{4},$

$$x=\frac{5}{4}\pm\frac{1}{4}=\frac{3}{2}\text{ or }1$$

Ex. 5 Solve $3x^2+10x=88.$

Divide by 3, thus $x^2+\frac{10}{3}x=\frac{88}{3},$

add $(\frac{5}{3})^2$, thus $x^2+\frac{10}{3}x+(\frac{5}{3})^2=(\frac{5}{3})^2+\frac{88}{3},$

whence $(x+\frac{5}{3})^2=\frac{25}{9}+\frac{88}{3}=\frac{289}{9},$

extract the square root, thus

$$x+\frac{5}{3}=\pm\sqrt{\frac{289}{9}}=\pm\frac{17}{3}.$$

$$\therefore\ x=-\frac{5}{3}\pm\frac{17}{3}=4\text{ or }-\frac{22}{3}$$

Exercise (iv).

Solve the equations

1 $4x^2+11x+6=0.$ 2. $5x^2+19x=4.$ 3. $9x^2+52x=12.$
4. $6x^2-11x=10$ 5 $42x^2-41x=20.$ 6. $10x^2-31x=-15.$
7 $12x^2-13x=4.$ 8. $24x^2-55x=-14$ 9 $3x^2-26x=169$
10 $2x^2-3x-54=0$ 11. $\frac{2}{3}x^2+\frac{3}{4}x-105=0.$ 12 $\frac{6}{5}x^2-\frac{31}{10}x=-2.$

Ex. 6. Solve $x^2-6x+7=0.$

Transpose, thus $x^2-6x=-7,$

add 3^2, thus $x^2-6x+3^2=3^2-7$, or $(x-3)^2=2,$

whence $x-3=\pm\sqrt{2},$

$$\therefore\ x=3\pm\sqrt{2}$$

REMARK Here the two roots are *irrational* but *real* as $\sqrt{2}$ can be found though approximately.

Ex. 7. Solve $9x^2-12x+8=0$

Transpose, thus $\qquad 9x^2-12x=-8,$

divide by 9, thus $\qquad x^2-\frac{4}{3}x=-\frac{8}{9},$

add $(\frac{1}{2}\times\frac{4}{3})^2$, thus $\qquad x^2-\frac{4}{3}x+(\frac{2}{3})^2=(\frac{2}{3})^2-\frac{8}{9},$

or $\qquad (x-\frac{2}{3})^2=\frac{4}{9}-\frac{8}{9}=\frac{-4}{9},$

extract the square root, thus

$$x-\frac{2}{3}=\pm\sqrt{\frac{-4}{9}}=\pm\frac{\sqrt{-4}}{3}$$

$$x=\frac{2}{3}\pm\frac{\sqrt{-4}}{3}=\frac{2\pm\sqrt{-4}}{3}.$$

REMARK. Here the two roots are not *real* but *imaginary*, as -4 has no square root

Ex 8. Solve $x^2+px+q=0$

Transpose q, thus $\qquad x^2+px=-q,$

add $(\frac{1}{2}p)^2$, thus $\qquad x^2+px+(\frac{1}{2}p)^2=(\frac{1}{2}p)^2-q,$

that is, $\qquad \left(x+\frac{p}{2}\right)^2=\frac{p^2}{4}-q=\frac{p^2-4q}{4},$

whence $\qquad x+\frac{p}{2}=\pm\sqrt{\frac{p^2-4q}{4}}=\pm\frac{\sqrt{p^2-4q}}{2},$

$$x=-\frac{p}{2}\pm\frac{\sqrt{p^2-4q}}{2}=\frac{-p\pm\sqrt{p^2-4q}}{2}$$

REMARK From the last two examples, it is clear that the roots of this equation will be *real* if $p^2>4q$, and *imaginary* if $p^2<4q$, also in the *first* case, they will be *rational* or *irrational* according as p^2-4q is a *perfect* square or not Thus when $p=-9$ and $q=20$, the roots are *real* and *rational*, when $p=4$ and $q=1$, they are *real* but *irrational*, and when $p=2$ and $q=3$, they are *imaginary*

Exercise (v)

Solve the equations

1 $2x^2+3x+5=3x^2+4x-1$ $\qquad$ 2. $5x(x+1)-2=x(x-2)$

3 $16x^2+3p^2=16px.$ $\qquad$ 4. $12x^2+5ax=3a^2$

5 $5x^2+7ax=20a^2-x^2$ $\qquad$ 6. $12(x^2+ax-2a^2)=11a(a+x)$

7 $\frac{x^2}{3}+\frac{5x}{2}=27$ $\qquad$ 8. $\frac{1}{x}-7x=6$ $\qquad$ 9 $\frac{3}{x}+2x=7$

10 $15x+\frac{96}{x}=76$ $\qquad$ 11 $x+\frac{28}{x}+4=2x-7.$ $\qquad$ 12 $1-\frac{3}{x}=\frac{2}{x}+\frac{6}{x^2}$

13 $\frac{x}{4}+\frac{3}{x}=2$ $\qquad$ 14 $\frac{x}{5}+\frac{5}{x}=\frac{3}{5}+\frac{5}{3}$ $\qquad$ 15. $x-\frac{1}{x}=\frac{3}{2}$ $\qquad$ 16 $x+\frac{1}{x}=4.$

Solve the equations

17. $x^2-2ax+a^2-b^2=0$

18 $ax^2+2bx+c=0$

19 $(a^2-b^2)(x^2+1)=2(a^2+b^2)x.$

20. $(a+b+x)^2=2bx+a^2+2b^2.$

21 $(x-7)(x-16)=0$

22 $(2x+3)(3x-4)=0.$

23 $(a-x)(2m-x)=(a-x)(x-2n).$

24 $\frac{x}{a}\left(1+\frac{x}{b}\right)+1+\frac{x}{b}=0$

25 $(x-1)(x-2)=12$

26 $(x-8)(x-10)=5\times 3$

27. $(x-a)(x-2a)=12a^2.$

28 $(x-1)(x-2)=2(x-3)(x-4).$

29 $(x-1)(x+3)=(2x-5)(3x-5).$

30 $x^2-(a-b)x=(c-a)(c-b)$

31 $(a+x)(x-b)=2(x-b)^2+ab.$

32. $(a-b)x^2-(a+b)x+2b=0.$

33 $\frac{x+11}{x}=7-\frac{9+4x}{x^2}.$

34 $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}.$

35. $\frac{x}{6}+\frac{6}{x}=\frac{5(x-1)}{4}.$

36 $\frac{2x-1}{x-3}=\frac{5x+2}{x+1}.$

37. $\frac{x-7}{2(x+3)}=\frac{x-6}{x+24}$

38 $\frac{x+1}{x-1}=\frac{4x-3}{x+9}.$

39. $2x-5+\frac{1}{2x-5}=3\frac{1}{3}$

40 $\frac{x}{x+1}+\frac{x+1}{x}=2\frac{1}{30}.$

41. $\frac{x-2}{x}+\frac{x}{x-2}=\frac{5}{2}$

42. $\frac{1}{x-2}+\frac{2}{x-1}=\frac{6}{x}.$

43 $\frac{x+2}{x-1}-\frac{4-x}{2x}=2\frac{1}{3}$

44. $\frac{1}{x}+\frac{3}{x+1}=\frac{5}{x+2}.$

45 $\frac{2}{x+1}+\frac{3}{x+2}=\frac{8}{x+3}.$

46 $\frac{6}{x+2}+\frac{7}{x+3}=\frac{16}{x+4}.$

47. $\frac{2}{x-3}+\frac{1}{x-4}=\frac{7}{6},$

48 $\frac{4}{x-1}=\frac{1}{18}+\frac{3}{x+7}.$

49. $\frac{x+1}{x-1}+\frac{x+2}{x-2}=7$

50 $\frac{x+1}{x+2}+\frac{x+2}{x+3}=1\frac{5}{12}.$

51. $\frac{2x+3}{3x+2}-\frac{2x-3}{3x-2}=1.$

52. $\frac{x+2}{x-2}+\frac{2x-3}{2(x-1)}=\frac{23}{6}.$

53. $\frac{2x-7}{2x-8}+\frac{5}{2}=\frac{3x+1}{3(x-5)}.$

54 $\frac{2x+7}{3(x-3)}-3=\frac{x+6}{2x-3}.$

55. $\frac{4x^2(2x-3)+3}{2(x^2-1)-x}=4x(x-1)+15.$

56. $ax\left(\frac{ax}{b^2}-\frac{1}{c}\right)+\frac{1}{c}\left(\frac{b^2}{c}-ax\right)=0$

Solve the equations

57. $\dfrac{5}{2x-a}+\dfrac{1}{2x-5a}=\dfrac{2}{a}$

58. $\dfrac{a^2(x-b)}{a-b}+\dfrac{b^2(x-a)}{b-a}=x^2.$

59. $\dfrac{b}{x-a}+\dfrac{a}{x-b}=2$

60. $x+\dfrac{1}{x}=a+\dfrac{1}{a}$

61. $(b+c)x^2+ax-a-b-c=0$

62. $(b-c)x^2+(c-a)x+(a-b)=0$

63. $a(b-c)x^2+b(c-a)x+c(a-b)=0$

64. $(a+b)x^2+(a-b)x=\dfrac{ab}{a+b}$

65. $\dfrac{1+a}{1-ax}+\dfrac{1-a}{1+ax}=1$

66. $\dfrac{1}{a}+\dfrac{b}{x+ab}+\dfrac{b}{2x+ab}=0$

67. $\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{1}{x-1}-\dfrac{1}{x+1}\right)=\dfrac{2}{x}$

68. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}=\dfrac{1}{a+b+x}$

69. $\dfrac{x+2}{x-2}+\dfrac{x-2}{x+2}=\dfrac{6x+16}{3x}$

70. $\dfrac{2x-1}{x+1}-\dfrac{x-7}{x-1}=4-\dfrac{3x-1}{x+2}$

71. $\dfrac{2x}{x-1}+\dfrac{3x-1}{x+2}-\dfrac{5x-11}{x-2}=0$

72. $\dfrac{7x-11}{4x-7}+\dfrac{3x-2}{12x-1}=\dfrac{2x+5}{x+2}.$

73. $x^2-2(a^2+b^2+c^2)x+(a+b+c)(b+c-a)(c+a-b)(a+b-c)=0$

74. $(c+a-2b)x^2+(a+b-2c)x+(b+c-2a)=0$

75. $(x+a)(x+b)(x+c)=abc$

76. $x^3=(x-a)(x-b)(x-c)$

77. $\dfrac{1}{x-a}+\dfrac{1}{x-b}+\dfrac{1}{x-c}=0$

78. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x+c}{x-c}=3$

79. $\left(\dfrac{2a}{x}-\dfrac{x}{a}-1\right)\left(1-\dfrac{a}{x}+\dfrac{2x}{a}\right)=0$

80. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{a}{b}+\dfrac{b}{a}$

81. $\dfrac{(x+a)(x+a+b)}{(x+c)(x+c+b)}=\dfrac{(x-a)(x-a-b)}{(x-c)(x-c-b)}$

82. $\dfrac{a+b}{x+b}+\dfrac{a+c}{x+c}=\dfrac{2(a+b+c)}{x+b+c}.$

83. $\dfrac{ax^2-b}{ax+b}+\dfrac{a+bx^2}{a-bx}=\dfrac{2(a^2+b^2)}{a^2-b^2}$

84. $x=\cfrac{3}{4-\cfrac{3}{4-\cfrac{3}{4-x}}}$

85. $\sqrt{16-x}+\sqrt{2x-5}=6$

86. $\sqrt{x+3}+\sqrt{3x-3}=10$

87. $\sqrt{3x+1}-\sqrt{4x+5}+\sqrt{x-4}=0$

88. $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}$

89. $\sqrt{3x-3}+\sqrt{5x-19}=\sqrt{3x+4}$

90. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}$

91. $\sqrt{a+x}-\sqrt{2a-x}=\sqrt{4a+x}$

291 Formula for Solving Quadratics The solution of the general quadratic $ax^2+bx+c=0$, viz.,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

may be used as a FORMULA to find the roots of any particular quadratic.

Ex. 1. Solve $x^2+7x+12=0$ [§ 290, Ex. 1]

Here $a=1$, $b=7$ and $c=12$, therefore

$$x=\frac{-7\pm\sqrt{49-4.1.12}}{2.1}=\frac{-7\pm\sqrt{1}}{2}=\frac{-7\pm1}{2}=-3 \text{ or } -4.$$

Ex. 2. Solve $3x^2+10x=88$ [§ 290, Ex. 5]

By transposition, we have $3x^2+10x-88=0$, which is of the form $ax^2+bx+c=0$.

Hence here $a=3$, $b=10$, $c=-88$, therefore

$$x=\frac{-10\pm\sqrt{(10)^2-4.3(-88)}}{2.3}=\frac{-10\pm\sqrt{100+1056}}{6}$$

$$=\frac{-10\pm\sqrt{1156}}{6}=\frac{-10\pm34}{6}=\frac{24}{6}\text{ or }-\frac{44}{6}=4\text{ or }-\frac{22}{3}.$$

Ex. 3. Solve $\dfrac{7-x}{2x+1}+\dfrac{3x+2}{2x-1}-3=\dfrac{7x-1}{4x^2-1}+2$

We must first reduce this equation to the general form.

Multiply by $4x^2-1$, thus

$$(7-x)(2x-1)+(3x+2)(2x+1)-3(4x^2-1)=7x-1+2(4x^2-1),$$

clear and transpose all the terms to one side, thus $16x^2-15x-1=0$. Now proceed as before.

292 Theorem. *In a quadratic of the form* $x^2+px+q=0$,* *the sum of the roots is equal to the coefficient of the second term with its sign changed, and the product of the roots is equal to the constant term.*

Let α and β denote the roots of $ax^2+bx+c=0$, that is, of

$$x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

then we have to prove that $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

* That is, one in which the coefficient of x^2 is made unity by division

Put for a and β respectively the roots found in § 288; thus

$$a=\frac{-b+\sqrt{b^2-4ac}}{2a}, \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a},$$

$$\therefore \quad a+\beta=\frac{-2b}{2a}=-\frac{b}{a} \qquad \text{(i)},$$

and
$$a\beta=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a} \qquad \text{(ii)}$$

If the quadratic be of the form $x^2+px+q=0$, then of course

$$a+\beta=-p \text{ and } a\beta=q.$$

This Theorem, which gives the above *relations between the roots and coefficients of a quadratic*, is very important One of its uses may be seen from the following examples.

Ex 1. Solve $(a-b)x^2-(a+b)x+2b=0$. [§ 290, Ex (v), Ex. 32]

Evidently $x=1$ satisfies the given equation, therefore 1 is a root of the equation The product of its roots is $\frac{2b}{a-b}$, therefore the other root is $\frac{2b}{a-b}$

Ex 2 Solve $(a+b)x^2+(a-b)x=\frac{ab}{a+b}$. [§ 290, Ex (v), Ex 64]

The equation may be written $x^2+\frac{a-b}{a+b}x-\frac{ab}{(a+b)^2}=0$

Hence the sum of the roots is $-\frac{a-b}{a+b}$, which are therefore $\frac{-a}{a+b}$ and $\frac{b}{a+b}$, for their product is $-\frac{ab}{(a+b)^2}$, as it should be

Ex 3. Solve $\frac{ax^2-b}{ax+b}+\frac{a+bx^2}{a-bx}=\frac{2(a^2+b^2)}{a^2-b^2}$ [§ 290, Ex (v), Ex 83]

Here $x=1$ obviously satisfies the given equation, therefore 1 is a root Again after reduction to the integral form the equation becomes

$$(a^2+2ab-b^2)x^2-(a^2-b^2)x-2ab=0$$

Hence the product of the roots is $\frac{-2ab}{a^2+2ab-b^2}$, and as one of them is 1, the other must evidently be $\frac{-2ab}{a^2+2ab-b^2}$

[*Solve equations Nos.* 61, 62, 63, 74, 80 *and* 82 of § 290, *Ex* (v)]

APPENDIX.

MISCELLANEOUS EXAMPLES WORKED OUT.

Ex. 1. Prove that $(ax+by+cz)^2+(ay-bx)^2+(bz-cy)^2+(cx-az)^2$
$$=(a^2+b^2+c^2)(x^2+y^2+z^2).$$

Expn. $=(ax+by)^2+2cz(ax+by)+c^2z^2+(ay-bx)^2+(bz-cy)^2+(cx-az)^2$
$$=(ax+by)^2+(ay-bx)^2+2acxz+2bcyz+c^2z^2+b^2z^2+c^2y^2-2bcyz$$
$$+c^2x^2+a^2z^2-2acxz$$
$$=(a^2+b^2)(x^2+y^2)+c^2(x^2+y^2)+(a^2+b^2+c^2)z^2$$
$$=(a^2+b^2+c^2)(x^2+y^2)+(a^2+b^2+c^2)z^2$$
$$=(a^2+b^2+c^2)(x^2+y^2+z^2)$$

Ex. 2. If $x=(b-c)(a-d)$, $y=(c-a)(b-d)$, $z=(a-b)(c-d)$, find the value of $x^3+y^3+z^3-3xyz$

We have

$x+y+z=(b-c)(a-d)+(c-a)(b-d)+(a-b)(c-d)$

$=a(b-c)+b(c-a)+c(a-b)-d\{(b-c)+(c-a)+(a-b)\}=0$ [§ 116],

thus $x+y+z$ which is a factor of the proposed expression, is 0, therefore the proposed expression is 0

Ex. 3. Shew that $(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3$
$$-3(x^2-yz)(y^2-zx)(z^2-xy)=(x^3+y^3+z^3-3xyz)^2.$$

Left side

$=\frac{1}{2}(x^2-yz+y^2-zx+z^2-xy)\times$
$\quad\{[y^2-z^2+x(y-z)]^2+[z^2-x^2+y(z-x)]^2+[x^2-y^2+z(x-y)]^2\}$

$=\frac{1}{2}(x^2+y^2+z^2-yz-zx-xy)\times$
$\quad\{(x+y+z)^2(y-z)^2+(x+y+z)^2(z-x)^2+(x+y+z)^2(x-y)^2\}$

$=\frac{1}{2}(x^2+y^2+z^2-yz-zx-xy)\times$
$\quad\{(x+y+z)^2\times2(x^2+y^2+z^2-yz-zx-xy)\}$

$=(x+y+z)^2(x^2+y^2+z^2-yz-zx-xy)^2$

$=\{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)\}^2=(x^3+y^3+z^3-3xyz)^2.$

Ex. 4. If $2s=a+b+c$, prove that
$$a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)$$
$$=a(s-a)^2+b(s-b)^2+c(s-c)^2.$$

From the given relation, we have
$$a=(s-b)+(s-c),$$
$$b=(s-c)+(s-a),$$
$$c=(s-a)+(s-b);$$
$$\therefore\ a(s-b)(s-c)=(s-b)^2(s-c)+(s-b)(s-c)^2,$$
$$b(s-c)(s-a)=(s-c)^2(s-a)+(s-c)(s-a)^2,$$
$$c(s-a)(s-b)=(s-a)^2(s-b)+(s-a)(s-b)^2;$$

whence by addition

$$a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)$$
$$=(s-a)^2(2s-b-c)+(s-b)^2(2s-c-a)+(s-c)^2(2s-a-b)$$
$$=a(s-a)^2+b(s-b)^2+c(s-c)^2$$

Ex. 5 Prove that

(i) $\{(x-y)^2+(y-z)^2+(z-x)^2\}^2=2\{(x-y)^4+(y-z)^4+(z-x)^4\}$,

(ii) $(x-y)^4+(y-z)^4+(z-x)^4$
$$=2\{(x-y)^2(y-z)^2+(y-z)^2(z-x)^2+(z-x)^2(x-y)^2\}.$$

Put $a=y-z$, $b=z-x$, $c=x-y$, thus $a+b+c=0$,

whence $\qquad a^2+b^2+c^2+2bc+2ca+2ab=0$ [§ 115] ;

transpose and square, thus

$$(a^2+b^2+c^2)^2=4(bc+ca+ab)^2$$
$$=4(b^2c^2+c^2a^2+a^2b^2) \text{ [115, Cor 2]}$$

expand, thus $\qquad a^4+b^4+c^4=2(b^2c^2+c^2a^2+a^2b^2) \qquad (\alpha)$,

add $a^4+b^4+c^4$ to both sides, thus

$$2(a^4+b^4+c^4)=a^4+b^4+c^4+2(b^2c^2+c^2a^2+a^2b^2)$$
$$=(a^2+b^2+c^2)^2 \text{ [§ 115]} \qquad (\beta)$$

Thus (α) proves the *second* identity and (β) proves the *first* identity.

Ex. 6 If $\qquad (by-cx)^2=(b^2-ac)(y^2-cz)$,

prove that $\qquad (bx-ay)^2=(b^2-ac)(x^2-az)$

From the proposed relation, we get

$$b^2y^2-2bcxy+c^2x^2=b^2y^2-acy^2-b^2cz+ac^2z ,$$

whence cancelling b^2y^2, and dividing by c, we have

$$-2bxy+cx^2=-ay^2-b^2z+acz ,$$

multiply by a, and add b^2x^2 to both sides , thus

$$b^2x^2-2abxy+acx^2=b^2x^2-a^2y^2-ab^2z+a^2cz ,$$

transpose, thus $b^2x^2-2abxy+a^2y^2=b^2x^2-ab^2z-acx^2+a^2cz$,

$$(bx-ay)^2=b^2(x^2-az)-ac(x^2-az)$$
$$=(b^2-ac)(x^2-az)$$

Ex 7 Find the H C D of

$$(ax+by)^2-(a-b)(x+z)(ax+by)+(a-b)^2xz$$

and
$$(ax-by)^2-(a+b)(x+z)(ax-by)+(a+b)^2xz$$

First expression

$$=(ax+by)^2-\{(a-b)x+(a-b)z\}(ax+by)+(a-b)x\times(a-b)z$$
$$=\{(ax+by)-(a-b)x\}\{(ax+by)-(a-b)z\} \text{ [§ 109]}$$
$$=b(x+y)\{(ax+by)-(a-b)z\}$$

Similarly second expression

$$=\{(ax-by)-(a+b)x\}\{(ax-by)-(a+b)z\}$$
$$=-b(x+y)\{(ax-by)-(a+b)z\}.$$

The H. C. D. required evidently $=b(x+y)$

Note The second expression differs from the first only in the sign of b; therefore its factors might have been obtained by writing $-b$ for b in the factors of the first expression

Ex 8 Assuming that $\dfrac{a+b-c}{a+b}=\dfrac{b+c-a}{b+c}=\dfrac{c+a-b}{c+a}$, and that $a+b+c$ is not$=0$, shew that $a=b=c$ [*Cal*, 1873]

From the given relations, we have

$$2-\frac{a+b-c}{a+b}=2-\frac{b+c-a}{b+c}=2-\frac{c+a-b}{c+a},$$

or $$\frac{a+b+c}{a+b}=\frac{a+b+c}{b+c}=\frac{a+b+c}{c+a},$$

divide by $a+b+c$, which, by supposition, is not 0, therefore

$$\frac{1}{a+b}=\frac{1}{b+c}=\frac{1}{c+a},$$

whence $$a+b=b+c=c+a,$$

that is, $$a=b=c$$

Ex. 9 Shew that if $\dfrac{a-b}{c}+\dfrac{b-c}{a}+\dfrac{c+a}{b}=1$ and $a-b+c$ is not$=0$,

then $$\frac{1}{a}=\frac{1}{b}+\frac{1}{c}\quad [\textit{Cal}, 1875]$$

We have from the given relation,

$$\frac{a-b}{c}+1+\frac{b-c}{a}-1+\frac{c+a}{b}-1=0,$$

or $$\frac{a-b+c}{c}+\frac{b-c-a}{a}+\frac{c+a-b}{b}=0,$$

whence $$(a-b+c)\left\{\frac{1}{c}-\frac{1}{a}+\frac{1}{b}\right\}=0,$$

thus either $a-b+c=0$, or $\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}=0$ [§ 265], but by supposition $a-b+c$ is not $=0$, therefore

$$\frac{1}{b}+\frac{1}{c}-\frac{1}{a}=0, \text{ or } \frac{1}{a}=\frac{1}{b}+\frac{1}{c}.$$

Ex. 10 Shew that $(1+x)(1+x^2)(1+x^4)(1+x^8)=\dfrac{1-x^{16}}{1-x}$.

Let
$$y=(1+x)(1+x^2)(1+x^4)(1+x^8),$$
$$\begin{aligned}(1-x)y&=(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\\&=(1-x^2)(1+x^2)(1+x^4)(1+x^8)\\&=(1-x^4)(1+x^4)(1+x^8)=(1-x^8)(1+x^8)=1-x^{16},\end{aligned}$$
$$\therefore\ y=\frac{1-x^{16}}{1-x}$$

Ex. 11 Express $\alpha\beta+\sqrt{(\alpha^2-1)(\beta^2-1)}$ in terms of x and y, when $2\alpha=x+x^{-1}$ and $2\beta=y+y^{-1}$.

Proposed expression
$$\begin{aligned}&=\frac{1}{2}\left(x+\frac{1}{x}\right)\times\frac{1}{2}\left(y+\frac{1}{y}\right)+\sqrt{\left\{\frac{1}{4}\left(x+\frac{1}{x}\right)^2-1\right\}\left\{\frac{1}{4}\left(y+\frac{1}{y}\right)^2-1\right\}}\\&=\frac{1}{4}\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+\sqrt{\frac{1}{4}\left(x-\frac{1}{x}\right)^2\times\frac{1}{4}\left(y-\frac{1}{y}\right)^2}\\&=\frac{1}{4}\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+\frac{1}{4}\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)\\&=\frac{1}{4}\left\{\left(xy+\frac{y}{x}+\frac{x}{y}+\frac{1}{xy}\right)+\left(xy-\frac{y}{x}-\frac{x}{y}+\frac{1}{xy}\right)\right\}\\&=\frac{1}{4}\left(2xy+\frac{2}{xy}\right)=\frac{1}{2}\left(xy+\frac{1}{xy}\right)\end{aligned}$$

Ex. 12 Find the first 4 terms of the square root of a^2+x^2, and from the result deduce the square root of 101 correct to 6 places of decimals. [*Cal* 1877].

$$\begin{array}{r|l}
 & a^2+x^2\,\left(\,a+\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}+\dfrac{x^6}{16a^5}\right.\\
 & a^2\\
2a+\dfrac{x^2}{2a} & x^2\\
 & x^2+\dfrac{x^4}{4a^2}\\
2a+\dfrac{x^2}{a}-\dfrac{x^4}{8a^3} & -\dfrac{x^4}{4a^2}\\
 & -\dfrac{x^4}{4a^2}-\dfrac{x^6}{8a^4}+\dfrac{x^8}{64a^6}\\
2a+\dfrac{x^2}{a}-\dfrac{x^4}{4a^3}+\dfrac{x^6}{16a^5} & \dfrac{x^6}{8a^4}-\dfrac{x^8}{64a^6}\\
 & \dfrac{x^6}{8a^4}+\dfrac{x^8}{16a^6}-\dfrac{x^{10}}{64a^8}+\dfrac{x^{12}}{256a^{10}}
\end{array}$$

$101=100+1=(10)^2+1^2$; thus here $a=10$, $x=1$, therefore required

square root $=10+\dfrac{1}{2\times10}-\dfrac{1}{8\times(10)^3}+\dfrac{1}{16\times(10)^5}$

$=10+\dfrac{\cdot5}{10}-\dfrac{\cdot125}{(10)^3}+\dfrac{\cdot0625}{(10)^5}$

$=10+\cdot05-\cdot000125+\cdot000000625$

$=10+\cdot05-\cdot000125$ (rejecting the last term as it cannot affect the required result, for there are 6 zeros after the decimal point)

$=10\cdot049875$.

Ex 13. Divide $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}+y^{2^{n-1}}$. [*Cal*, 1879]

Dividend $=(x^{2^{n-1}})^2-(y^{2^{n-1}})^2$ [for by § 185, $(x^{2^{n-1}})^2$ is x raised to the power expressed by the number $2^{n-1}\times2$ or 2^n]

$=(x^{2^{n-1}}+y^{2^{n-1}})(x^{2^{n-1}}-y^{2^{n-1}})$;

$\therefore$ the second factor = quotient required.

Ex 14. A man receives $\dfrac{x}{y}$ths of 10 rupees and afterwards $\dfrac{y}{x}$ths of 10 rupees. He then gives away 20 rupees. Shew that he cannot lose by the transaction. [*Cal*, 1881]

He receives $\left(10\dfrac{x}{y}+10\dfrac{y}{x}\right)$ rupees and gives away 20 rupees, therefore he has $\left(10\dfrac{x}{y}+10\dfrac{y}{x}\right)$ rupees $-$ 20 rupees

$=10\left(\dfrac{x}{y}+\dfrac{y}{x}-2\right)$ rupees $=10\dfrac{x^2+y^2-2xy}{xy}$ rupees $=\dfrac{10(x-y)^2}{xy}$ rupees.

Now x and y being supposed positive, $(x-y)^2$ is always positive whatever values x and y may have; hence $\dfrac{10(x-y)^2}{xy}$ is a positive quantity. Therefore $\left(10\dfrac{x}{y}+10\dfrac{y}{x}\right)$ rupees is *greater than* 20 rupees [§ 45]; and consequently the man cannot lose.

Ex. 15. Solve $\dfrac{x}{\sqrt{1+x}+\sqrt{1-x}}=\dfrac{2a^2}{1-\sqrt{1-x^2}}$.

Rationalise the denominator of left side, and multiply numerator and denominator of right side by 2, thus

$$\frac{x(\sqrt{1+x}-\sqrt{1-x})}{2x}=\frac{4a^3}{2-2\sqrt{1-x^2}}=\frac{4a^3}{(\sqrt{1+x}-\sqrt{1-x})^2},$$

$$\therefore (\sqrt{1+x}-\sqrt{1-x})^3=8a^3,$$

$$\sqrt{1+x}-\sqrt{1-x}=2a,$$

$$2-2\sqrt{1-x^2}=4a^2,$$

$$\sqrt{1-x^2}=1-2a^2,$$

$$1-x^2=1-4a^2+4a^4,$$

$$x^2=4a^2(1-a^2),$$

$$\therefore x=2a\sqrt{1-a^2}$$

Ex. 16 Solve $\sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2}$

We have the identical relation

$$(a^2-x^2)-x^2(a^2-1)=a^2(1-x^2) \qquad (1)$$

divide (1) by the proposed equation, thus

$$\therefore \sqrt{a^2-x^2}-x\sqrt{a^2-1}=\sqrt{1-x^2} \qquad (2)$$

subtract (2) from the given equation, thus

$$2x\sqrt{a^2-1}=(a^2-1)\sqrt{1-x^2},$$

$$2x=\sqrt{a^2-1}\sqrt{1-x^2},$$

$$4x^2=(a^2-1)(1-x^2),$$

$$x^2(a^2+3)=a^2-1,$$

$$\therefore x=\sqrt{\frac{a^2-1}{a^2+3}}$$

Ex 17. Solve $(a+x)\sqrt{1+a}+(a-x)\sqrt{1-a}=2\sqrt{a^2+x^2}$

Since $a^2+x^2=\frac{1}{2}(a+x)^2+\frac{1}{2}(a-x)^2$ [§ 101, (iv)], we have by squaring

$$(a+x)^2(1+a)+(a-x)^2(1-a)+2(a^2-x^2)\sqrt{1-a^2}=2(a+x)^2+2(a-x)^2,$$

whence by transposition,

$$(a+x)^2(1-a)+(a-x)^2(1+a)-2(a^2-x^2)\sqrt{1-a^2}=0,$$

or $$\{(a+x)\sqrt{1-a}-(a-x)\sqrt{1+a}\}^2=0,$$

thus $$(a+x)\sqrt{1-a}-(a-x)\sqrt{1+a}=0,$$

whence $$\frac{a+x}{a-x}=\frac{\sqrt{1+a}}{\sqrt{1-a}},$$

$$\therefore \frac{x}{a}=\frac{\sqrt{1+a}-\sqrt{1-a}}{\sqrt{1+a}+\sqrt{1-a}} [\S\ 227]=\frac{(\sqrt{1+a}-\sqrt{1-a})^2}{2a} ; \&c.$$

Ex. 18 Solve $\dfrac{1+x-\sqrt{2x+x^2}}{1+x+\sqrt{2x+x^2}}=a^3\dfrac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}$.

Multiply numerator and denominator of first side by 2, thus

$$\frac{2+2x-2\sqrt{2x+x^2}}{2+2x+2\sqrt{2x+x^2}}=a^3\frac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}},$$

$$\frac{(\sqrt{2+x}-\sqrt{x})^2}{(\sqrt{2+x}+\sqrt{x})^2}=a^3\frac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}},$$

$$\frac{(\sqrt{2+x}-\sqrt{x})^3}{(\sqrt{2+x}+\sqrt{x})^3}=a^3,$$

$$\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}+\sqrt{x}}=a,$$

$$\frac{\sqrt{2+x}}{\sqrt{x}}=\frac{1+a}{1-a},$$

$$\frac{2+x}{x}=\left(\frac{1+a}{1-a}\right)^2,$$

$$\frac{2}{x}=\left(\frac{1+a}{1-a}\right)^2-1=\frac{4a}{(1-a)^2} ; \therefore \&c.$$

Ex 19 Solve

$$cx+(a+b)y+(a-b)z=a \quad (1),$$

$$cx+(b+c)y+(b-c)z=b \quad (2),$$

$$bx+(c+a)y+(c-a)z=c \quad (3).$$

Adding the equations together, we get

$$(a+b+c)x+2(a+b+c)y=a+b+c,$$

or

$$x+2y=1 \quad (4).$$

Again multiply (1) by $a-b$, (2) by $b-c$, (3) by $c-a$, and add, thus

$$\{(a-b)^2+(b-c)^2+(c-a)^2\}z=a^2+b^2+c^2-bc-ca-ab,$$

whence

$$z=\tfrac{1}{2}.$$

Substitute the value of z in (1), thus

$$cx+(a+b)y=a-\tfrac{1}{2}(a-b)=\tfrac{1}{2}(a+b) \quad (5).$$

Multiply (4) by c, and subtract from (5), thus

$$(a+b-2c)y=\tfrac{1}{2}(a+b)-c=\tfrac{1}{2}(a+b-2c),$$

$$\therefore y=\tfrac{1}{2}$$

Hence from (4),

$$x=1-2y=0$$

Ex 20 Solve

$$(b+c)x+by+cz=(c+a)y+cz+ax=(a+b)z+ax+by \qquad (1),$$

$$xyz-(c+a-b)yz-(a+b-c)zx-(b+c-a)xy=2xyz\frac{a^2+b^2+c^2}{(a+b+c)^2} \qquad (2).$$

From equations (1) by transposition, we get

$$(b+c-a)x=(c+a-b)y=(a+b-c)z=\lambda \text{ suppose} \qquad (3)$$

From (2) by division by xyz, we have

$$1-(c+a-b)\frac{1}{x}-(a+b-c)\frac{1}{y}-(b+c-a)\frac{1}{z}=\frac{2(a^2+b^2+c^2)}{(a+b+c)^2},$$

or $$(c+a-b)\frac{1}{x}+(a+b-c)\frac{1}{y}+(b+c-a)\frac{1}{z}=1-\frac{2(a^2+b^2+c^2)}{(a+b+c)^2}$$

$$=\frac{2bc+2ca+2ab-a^2-b^2-c^2}{(a+b+c)^2},$$

substitute the values of x, y, z in terms of λ from (3), thus

$$(c+a-b)\frac{b+c-a}{\lambda}+(a+b-c)\frac{c+a-b}{\lambda}+(b+c-a)\frac{a+b-c}{\lambda}$$

$$=\frac{2bc+2ca+2ab-a^2-b^2-c^2}{(a+b+c)^2},$$

$$\frac{1}{\lambda}\{c^2-(a-b)^2+a^2-(b-c)^2+b^2-(c-a)^2\}$$

$$=\frac{2bc+2ca+2ab-a^2-b^2-c^2}{(a+b+c)^2},$$

whence $$\lambda=(a+b+c)^2 \qquad (4)$$

Thus from (3) and (4), the values of x, y, z are known.

Ex. 21 Solve

$$(b+c)(y+z)=a(x+1) \qquad (1),$$
$$(c+a)(z+x)=b(y+1) \qquad (2),$$
$$(a+b)(x+y)=c(z+1) \qquad (3)$$

By transposition

$$-ax+(b+c)y+(b+c)z=a,$$
$$(c+a)x-by+(c+a)z=b,$$
$$(a+b)x+(a+b)y-cz=c,$$

whence by addition,

$$(a+b+c)x+(a+b+c)y+(a+b+c)z=a+b+c,$$

or $$x+y+z=1 \qquad (4).$$

From (1) and (4), $$(b+c)(1-x)=a(x+1),$$

or $$x=\frac{-a+b+c}{a+b+c}$$

Similarly from (2) and (3), the values of y and z may be found

Note. Observe that (1), (2) and (3) are collaterally symmetrical

Ex 22. The expression $ax-3b$ is equal to 30 when x is 3, and to 42 when x is 7; what is its value when $x=4\frac{1}{3}$, and for what value of x is it zero? [*Cal.*, 1874]

By the first condition, $\quad 3a-3b=30 \quad$ (1).

By the second condition, $\quad 7a-3b=42 \quad$ (2).

Subtract (1) from (2), thus $4a=12$ or $a=3$. Hence from (2), $b=-7$.

Therefore when $x=4\frac{1}{3}=4\frac{1}{3}$, the required value

$$=3\times 4\tfrac{1}{3}-3\times -7=34.$$

Again, $\quad 3x-3\times -7=0$ gives, when solved, $x=-7$.

Ex. 23. If $x+y+z=0$, then $\dfrac{x^2}{2x^2+yz}+\dfrac{y^2}{2y^2+zx}+\dfrac{z^2}{2z^2+xy}=1$.

From the given relation, $x=-(y+z)$; $\therefore\ x^2=-x(y+z)$; hence

$$2x^2+yz=x^2+x^2+yz=x^2-x(y+z)+yz=(x-y)(x-z).$$

Similarly $\quad 2y^2+zx=(y-z)(y-x)$, $2z^2+xy=(z-x)(z-y)$.

$\therefore$ Left side $=\dfrac{x^2}{(x-y)(x-z)}+\dfrac{y^2}{(y-z)(y-x)}+\dfrac{z^2}{(z-x)(z-y)}=1$. [See § 180, Ex. 1.]

Ex 24. If $\dfrac{bx-ay}{cy-az}=\dfrac{cx-az}{by-ax}=\dfrac{z+y}{z+x}$, then each of these ratios $=\dfrac{x}{y}$, unless $b+c=0$.

Let each ratio $=l$, thus

$$l=\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{a(z+y)}{a(z+x)}\ [\S\ 168]$$

$$=\frac{(bx-ay)+(cx-az)+(az+ay)}{(cy-az)+(by-ax)+(az+ax)}\ [\S\ 261,\ \text{Cor. } 1]$$

$$=\frac{(b+c)x}{(b+c)y}=\frac{x}{y},\ \because\ b+c \text{ is not } =0.$$

Note. If $b+c=0$, then $l=\dfrac{0\times x}{0\times y}=\dfrac{0}{0}$; thus the value of l would be *indeterminate* [see § 266].

Ex 25. If $\dfrac{x}{l(mb+nc-la)}=\dfrac{y}{m(nc+la-mb)}=\dfrac{z}{n(la+mb-nc)}$,

then $\quad \dfrac{l}{x(by+cz-ax)}=\dfrac{m}{y(cz+ax-by)}=\dfrac{n}{z(ax+by-cz)}$.

Divide the terms of the first fraction by l, of the second fraction by m, and of the third fraction by n, thus

$$\frac{\frac{x}{l}}{mb+nc-la}=\frac{\frac{y}{m}}{nc+la-mb}=\frac{\frac{z}{n}}{la+mb-nc}=\lambda \text{ say };$$

$$k=\frac{\frac{y}{m}+\frac{z}{n}}{2la}=\frac{\frac{z}{n}+\frac{x}{l}}{2mb}=\frac{\frac{x}{l}+\frac{y}{m}}{2nc}$$

$$=\frac{ny+mz}{2lmna}=\frac{lz+nx}{2lmnb}=\frac{mx+ly}{2lmnc},$$

hence $\quad 2\lambda lmn=\dfrac{ny+mz}{a}=\dfrac{lz+nx}{b}=\dfrac{mx+ly}{c}$.

now multiply the terms of the first fraction by x, of the second fraction by y, and of the third fraction by z, thus

$$2\lambda lmn=\frac{nxy+mxz}{ax}=\frac{lyz+nxy}{by}=\frac{mxz+lyz}{cz}$$

$$=\frac{2lyz}{by+cz-ax}=\frac{2mxz}{cz+ax-by}=\frac{2nxy}{ax+by-cz}$$

Next multiply the terms of the first, second and third members of the last equality by x, y and z respectively, thus

$$\frac{2lxyz}{x(by+cz-ax)}=\frac{2mxyz}{y(cz+ax-by)}=\frac{2nxyz}{z(ax+by-cz)};$$

divide by $2xyz$, thus the required relations follow

Ex 26 If $a=b\frac{y}{z}+c\frac{z}{y}$, $b=c\frac{z}{x}+a\frac{x}{z}$, $c=a\frac{x}{y}+b\frac{y}{x}$,

prove that $\qquad \dfrac{1}{x^3}+\dfrac{1}{y^3}+\dfrac{1}{z^3}+\dfrac{1}{xyz}=0$

From the first two equations by transposition, we get

$$-a+b\frac{y}{z}+c\frac{z}{y}=0,\ a\frac{x}{z}-b+c\frac{z}{x}=0,$$

whence by Cross Multiplication [§ 244]

$$\frac{a}{\frac{y}{z}\times\frac{z}{x}+\frac{z}{y}}=\frac{b}{\frac{z}{y}\times\frac{x}{z}+\frac{z}{x}}=\frac{c}{1-\frac{xy}{z^2}}=\lambda \text{ suppose},$$

$$\therefore\ a=k\left(\frac{y}{x}+\frac{z}{y}\right),\qquad b=\lambda\left(\frac{x}{y}+\frac{z}{x}\right),\qquad c=\lambda\left(1-\frac{xy}{z^2}\right),$$

substitute a, b, c in the third equation, thus

$$\lambda\left(1-\frac{xy}{z^2}\right)=\lambda\left(\frac{y}{x}+\frac{z}{y}\right)\frac{x}{y}+\lambda\left(\frac{x}{y}+\frac{z}{x}\right)\frac{y}{x};$$

divide by l and transpose, thus

$$\frac{yz}{x^3}+\frac{zx}{y^2}+\frac{xy}{z^2}+1=0\ ;$$

divide now by xyz, thus the required relation follows

Ex. 27 If $a+b+c=0$ and

$$a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz),$$

then will $$x+y+z=0.$$

From the first and second members, by transposition,

$$a(a+b)x-b(a+b)y+c(b-a)z=0$$

From the second and third members, by transposition,

$$a(b-c)x-b(b+c)y+c(b+c)z=0$$

Hence by Cross Multiplication [§ 244]

$$\frac{x}{-bc(a+b)(b+c)+bc(b+c)(b-a)}=\frac{y}{ca(b-c)(b-a)-ca(a+b)(b+c)}$$
$$=\frac{z}{-ab(a+b)(b+c)+ab(a+b)(b-c)},$$

or after reduction,

$$\frac{x}{-2abc(b+c)}=\frac{y}{-2abc(c+a)}=\frac{z}{-2abc(a+b)},$$

whence $$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}=k \text{ suppose,}$$

thus $$x=k(b+c),\ y=k(c+a),\ z=k(a+b)\ ;$$

therefore $$x+y+z=k(b+c)+k(c+a)+k(a+b)$$
$$=2k(a+b+c)=0,\ \because\ a+b+c=0$$

Miscellaneous Examples for Exercise.

1. If $a=1$, $b=\frac{2}{3}$, $x=7$, $y=3$, find the value of
$$5(a-b)\sqrt[3]{(a+x)y^2}-b\sqrt{(a+x)y}$$

2. Multiply $(x^2+x-6)^2$ by x^2+4x+4

3 Shew that $(3x^2-4x+2)^2-(2x^2+9x+3)^2$ is divisible by x^2+x+1 without performing the operation of division; and find the quotient

4. Find the H C D of $a(a-1)x^2+(2a^2-1)x+a(a+1)$ and $(a^2-3a+2)x^2+(2a^2-4a+1)x+a(a-1)$.

5. If $2s=a+b+c$, prove that
$$(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)=s^2-\tfrac{1}{2}(a^2+b^2+c^2).$$

6. Shew that
$$a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3=3abc(b-c)(c-a)(a-b).$$

7 Shew that the sum of

$$\frac{x}{x+a}+\frac{x}{x+b}+\frac{x}{x+c} \text{ and } \frac{a}{x+a}+\frac{b}{x+b}+\frac{c}{x+c}$$

is independent of x

8 If $x\pm\frac{1}{x}=y$, shew that $x^3\pm\frac{1}{x^3}=y^3\mp 3y$

9 If $\frac{1-2ax+a^2}{1-a^2}=\frac{1-a^2}{1+2ay+a^2}$, then $\frac{x-y}{1-xy}=\frac{2a}{1+a^2}$

10 Find the square root of $x^2+8x-64x^{-1}+64x^{-2}$

11 Find the expression whose square is $3x-1+2\sqrt{2x^2+x-6}$.

12 Solve $\frac{6x-7}{9x+6}-\frac{5(x-1)}{12x+8}=\frac{1}{12}$

13 Solve $\frac{5x+6}{10}-\frac{11y-5}{21}=11$, $\frac{1}{25}(55y-12)=\frac{7x}{5}-37$

14 If $a : b=c : d$, prove that

$$a^2d-b^2c+b^2d : a^2c+b^2d=d : c+d$$

15 An article is bought and sold so as to gain 5 per cent If it had been bought at 5 per cent less and sold for 1s less, 10 per cent would have been gained Find the cost price

16 Find the value of

$$\frac{4y}{5}(y-x)=35\left[\frac{3x-4y}{15}-\frac{1}{10}\{3x-\tfrac{5}{7}(7x-4y)\}\right]$$

when $x=-\frac{1}{2}$, $y=2$

17. Multiply

$$\sqrt{(2x)}+\sqrt{2(2x-1)}-\frac{1}{\sqrt{(2x)}} \text{ by } \frac{1}{\sqrt{(2x)}}+\sqrt{2(2x-1)}-\sqrt{(2x)}.$$

18 Divide $\dfrac{\left(\frac{3x+x^3}{1+3x^2}\right)^2-1}{\frac{3x^2-1}{x^3-3x}+1}$ by $\dfrac{\frac{9}{x^3}-\frac{33-x^2}{3x^2+1}}{\frac{3}{x^2}-\frac{2(x^3+3)}{(x^3-x)^2}}$

19. If

$$a^2=(x+y-2z)(y+z-2x),$$
$$b^2=(y+z-2x)(z+x-2y),$$
$$c^2=(z+x-2y)(x+y-2z),$$

then

$$(a^2+b^2+c^2)(x+y+z)=3(3xyz-x^3-y^3-z^3)$$

20 If $2s=a+b+c$, then $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.

21. If

$$\frac{a}{x}(b-c)+\frac{b}{y}(c-a)+\frac{c}{z}(a-b)=0,$$

then

$$\frac{x}{a}(y-z)+\frac{y}{b}(z-x)+\frac{z}{c}(x-y)=0$$

22. If $\left(a+\frac{1}{a}\right)^2=3$, shew that $a^3+\frac{1}{a^3}=0$

23. Find the value of $27x+48x^3-8x^4$ when $x=\frac{1}{4}(\sqrt{21}-3)$

24. Find the square root of $57-12\sqrt{15}$

25. Solve $\frac{x}{2}+\frac{5x^2-15x-8}{10(x-3)}=\frac{5x-9}{5}+1$

26 Solve $\frac{ax}{a+x}+\frac{by}{b+y}=\frac{(a+b)c}{a+b+c}$, $x+y=c$.

27. If $x : y=a : b$, shew that $(x^2-a^2)(y^2-b^2)=(xy-ab)^2$

28. A and B shoot by turns at a target, A puts in 3 arrows out of 7 and B 2 arrows out of 5 how many must each shoot, that they may put in 29 arrows between them?

29. Find the value of x^2-6x+7 when $x=3-\sqrt{3}$

30. Shew that $a(a-x)(a-2x)$
$$=(a-b)(a-b-x)(a+2b-2x)+b(b-x)(3a-2b-2x).$$

31. Find the coefficients of x^2, x^3 and x^4 in $(x+a)^3(x-a)^5$.

32 Find the value of
$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)-\left(a-\frac{1}{a}\right)\left(b-\frac{1}{b}\right)\left(c-\frac{1}{c}\right)$$

33. Find the square root of $(a^2+b^2)(x^2+y^2)-(ax+by)^2$.

34 Given $2s=a+b+c+d$, shew that
$$4(bc+ad)^2-(b^2+c^2-a^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d)$$

35 Prove that
$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(xy+\frac{1}{xy}\right)^2=4+\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(xy+\frac{1}{xy}\right).$$

36 If $\frac{a-b}{1+ab}=\frac{x-y}{1+xy}$, shew that
$$\frac{a-x}{1+ax}=\frac{b-y}{1+by} \text{ and } \frac{a+y}{1-ay}=\frac{b+x}{1-bx}.$$

37. Simplify $\left\{\frac{\sqrt{x+a}}{\sqrt{x-a}}-\frac{\sqrt{x-a}}{\sqrt{x+a}}\right\}\frac{\sqrt{x^3-a^3}}{\sqrt{(x+a)^2-ax}}$.

38. Prove that $\frac{3}{2}(\sqrt{3}+1)^2-2(\sqrt{2}+1)^2=\sqrt{59-24\sqrt{6}}$.

39. Solve $\frac{1}{3x-1}+\frac{2(x+1)}{x-1}=1+\frac{3x^2+1}{3x^2-4x+1}$.

40 Solve $x+y+z=0$, $ax+by+cz=0$, $\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=3$

41. If $a^2+c^2 : ab+cd=ab+cd : b^2+d^2$, prove that $a : b=c : d$

42 The crew of a ship consisted of her complement of sailors and a number of soldiers, there were 22 soldiers to every 3 guns and 10 over, also the whole number of hands was 5 times the number of sailors and guns together. After an engagement in which the slain were $\frac{1}{4}$ of the survivors, there wanted 5, to be 13 men to every 2 guns Required the number of guns, soldiers and sailors.

43 Find the value of $\frac{a}{b}-\sqrt{\frac{1+a}{1-b}}+\sqrt[6]{\frac{5a^3}{2b^4}}$, when $a=\frac{1}{4}$ and $b=\frac{1}{5}$

44 Simplify $4\left\{a-\frac{3}{2}\left(b-\frac{4c}{3}\right)\right\}\ \left\{\frac{1}{2}(2a-b)+2(b-c)\right\}$.

45 Find the four factors of $(1+y)^2-2(1+y^2)x^2+(1-y)^2x^4$

46 If $s=a+b+c$, prove that
$$(s-3a)^2+(s-3b)^2+(s-3c)^2=3\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

47 Shew that $\frac{a(x^2+ab)}{b(a^2+b^2)}-\frac{x^2}{ab}+\frac{b(x^2-ab)}{a(a^2+b^2)}$ is independent of x

48. If $x=\frac{a+b}{c-d}$, shew that $(a-cx)^2+(x^2-1)(b^2-d^2)$ is a complete square

49. If $\frac{1}{s}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots\ldots$to n terms, shew that
$$\frac{s-a}{a}+\frac{s-b}{b}+\frac{s-c}{c}+\ldots\ldots\text{to } n \text{ terms }=1-n$$

50. If $t=\frac{2}{2-w}$, $w=\frac{2}{2-z}$, $z=\frac{2}{2-y}$, $y=\frac{2}{2-x}$, then $t=x$

51 Solve $\dfrac{1}{1-\frac{1}{x}}-\dfrac{\frac{1}{x}}{x-1}=6$.

52 Solve
$$x+y+z=1,$$
$$ax+by+cz=d,$$
$$a^2x+b^2y+c^2z=d^2.$$

53. If $a : b=c : d$, prove that
$$(la^3+mb^3)\left(\frac{l}{a^3}+\frac{m}{b^3}\right)=(ld^3+mc^3)\left(\frac{l}{d^3}+\frac{m}{c^3}\right)$$

54 Two passengers have together 5 cwt of luggage, and are charged for the excess above the weight allowed 5s 2d, and 9s. 10d, respectively, had the luggage all belonged to one of them, he would have been charged 19s 2d How much luggage is allowed free of charge and what amount of weight had each passenger?

55 Simplify $3a-[b+\{2a-(b-c)\}]+\frac{1}{2}+\frac{2c^2-\frac{1}{2}}{2c+1}$.

56 Resolve $\{x^2-(b+c)x+bc\}^2-(x-c)^2(x-a)^2$

57 If $s=a+b+c$, prove that
$$(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2.$$

58 Prove that

$$\left(x+\frac{a}{b}y\right)\left(x+\frac{b}{c}y\right)\left(x+\frac{c}{a}y\right)-\left(x+\frac{b}{a}y\right)\left(x+\frac{c}{b}y\right)\left(x+\frac{a}{c}y\right)$$
$$=\frac{xy(x-y)(a-b)(b-c)(c-a)}{abc}.$$

59. Find the value of $\dfrac{3^{\frac{2}{3}}+3^{\frac{1}{3}}+1}{3^{\frac{1}{3}}+1}+\dfrac{3^{\frac{2}{3}}-3^{\frac{1}{3}}+1}{3^{\frac{1}{3}}-1}$.

60 Simplify $\dfrac{\frac{x}{y}+1+\frac{y}{x}}{\frac{x}{y}-1+\frac{y}{x}}\times\dfrac{1+\frac{y^3}{x^3}}{1-\frac{y^3}{x^3}}\div\dfrac{(x+y)^2}{x^2-y^2}$.

61. Find the value of $(x^a)^{b-c}(x^b)^{c-a}(x^c)^{a-b}$.

62 The coefficient of x in the expansion of

$(x-a)(x-b)(x-c)(x-d)(x-f)$ is $\dfrac{1}{A}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}+\dfrac{1}{f}\right)$,

find the value of A.

63. If $a+b+c=0$, prove that $\dfrac{a^2+b^2+c^2}{a^3+b^3+c^3}+\dfrac{2}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=0$

64. Solve $\dfrac{2x}{3}-\dfrac{1-\frac{x}{2}}{4x}=\dfrac{8x+1}{12}$.

65 Solve $x+y+z=a+b+c$,

$$bx+cy+az=cx+ay+bz=bc+ca+ab$$

66 If a, b, c are in continued proportion, prove that

$$\left(\frac{b}{c}+\frac{c}{a}\right):\left(\frac{c}{a}+\frac{a}{b}\right)$$

is a ratio of equality.

67. At an election, where each elector may give 2 votes to different candidates, but only one to the same, it is found on casting up the poll that of 3 candidates A, B and C, A has 158 votes, B 132, and C 118 Now 26 electors voted for A alone, 30 for B alone, and 28 for C alone How many voted for A and B jointly, how many for A and C, and how many for B and C?

68 Find the value of $\sqrt{\dfrac{2xz(x-z)}{y^3}}$, when $x=1$, $y=\cdot 2$, $z=\cdot 02$.

69 Resolve $ab(x^2+y^2)+(a^2+b^2)xy+(a-b)(x-y)-1$

70. Simplify

$$(a+x)^4+4(a+x)^3(a-x)+6(a^2-x^2)^2+4(a+x)(a-x)^3+(a-x)^4.$$

71 If $3s=a+b+c$, prove that $(s-a)^4+(s-b)^4+(s-c)^4$
$$=2\{(s-a)^2(s-b)^2+(s-b)^2(s-c)^2+(s-c)^2(s-a)^2\}$$

72 Simplify $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}$ 73 Reduce $\dfrac{a^5+2a^4x+a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$

74 Find the value of c which will make $x^4+5x^3+7x^2+cx-2c$ divisible without remainder by x^2+3x+2

75 Establish the identity
$$\frac{x^2-xy+y^2}{x^2y^2}\left\{\frac{1}{x}+\frac{1}{y}\right\}=\frac{y^2-yz+z^2}{y^2z^2}\left\{\frac{1}{y}+\frac{1}{z}\right\}+\frac{x^2+xz+z^2}{x^2z^2}\left\{\frac{1}{x}-\frac{1}{z}\right\}$$

76 Shew that $(2x+y^{-1})(2y+x^{-1})=\left(2x^{\frac{1}{2}}y^{\frac{1}{2}}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)^2$

77 Simplify $\left\{\dfrac{a^{\frac{x}{x-y}}\sqrt[y]{b^{x-y}}}{\sqrt[y]{a^{x+y}b^{x}}\times a^{\frac{y}{x-y}}}\right\}^{-y}$.

78 Solve $3-\dfrac{2}{4+\dfrac{3}{1-x}}=\dfrac{11}{3}$

79 Solve $\dfrac{5\sqrt{x+y}}{x}+\dfrac{5\sqrt{x+y}}{y}=10\frac{2}{3}$, $\dfrac{3\sqrt{x-y}}{y}-\dfrac{3\sqrt{x-y}}{x}=\dfrac{4}{5}$

80 If $a : b=c : d$, shew that
$$la+mb : lc+md=\sqrt{(pa^2+qb^2)} \quad \sqrt{(pc^2+qd^2)}$$

81 A letter carrier has to go daily from P to Q in a prescribed time If he goes a mile an hour faster than his ordinary rate, he arrives at Q half an hour before the time But if he goes a mile an hour slower, he arrives three-quarters of an hour too late Find his ordinary rate and the distance from P to Q

82 Find the value, when $a=2$, $b=3$, of
$$a^{a-1}b^{b+1}-(a+1)^b(b-1)^a+a^{b-1}b^{a+1}$$

83 Simplify $(a+b+c)(x+y+z)+(b+c-a)(y+z-x)$
$$+(c+a-b)(z+x-y)+(a+b-c)(x+y-z)$$

84 Resolve into factors
$$(x-y)(x-2y)(x-3y)+9y(x-y)(x-2y)+18y^2(x-y)+6y^3$$

85 Expand $\{(x^2+x+1)(x^2-x+1)(x^2-1)\}^2$ in powers of x

86 Find the square root of $(x^2-3x+2)(x^2-4x+3)(x^2-5x+6)$

87. If $2s=a+b+c$, $2\sigma^2=a^2+b^2+c^2$, shew that

$$(\sigma^2-a^2)(\sigma^2-b^2)+(\sigma^2-b^2)(\sigma^2-c^2)+(\sigma^2-c^2)(\sigma^2-a^2)=4s(s-a)(s-b)(s-c)$$

88. Simplify $\left\{\frac{\sqrt{x}}{x^{\frac{3}{2}}-1}-\frac{1}{x^{\frac{3}{2}}+x+\sqrt{x}}\right\} \div \left\{\frac{x^{\frac{3}{2}}+1}{x}\div\frac{x^{\frac{3}{2}}-1}{\sqrt{(x+1)}}\right\}$.

89 If $x+y=a$ and $xy=b^2$, express x^3+y^3 and x^4+y^4 in terms of a and b.

90 Prove the identity

$$\frac{1}{a+x}+\frac{2x}{a^2+x^2}+\frac{4x^3}{a^4+x^4}+\frac{8x^7}{a^8+x^8}=\frac{1}{a-x}-\frac{16x^{15}}{a^{16}-x^{16}}$$

91. If $x=\frac{1}{\sqrt{a^2+b^2}}$, then $\frac{a^3}{1\mp bx}+\frac{b^3}{1\pm ax}=(a+b)(a^2+b^2)$.

92. Solve $\frac{x}{a^2+b^2}+\frac{x}{c^2+d^2}=\frac{a^2+b^2+c^2+d^2}{(ac+bd)^2+(bc-ad)^2}$

93. Solve $5^{x-2}+3^{y-2}=14$, $5^{x+1}-3^{y+2}+104=0$

94. If $a : b=c : d=e : f$, then

$$(a^2+b^2)(ce+df)^2=(c^2+d^2)(ae+bf)^2=(e^2+f^2)(ac+bd)^2$$

95 A criminal having escaped from prison, travelled 10 hours before his escape was known He was pursued so as to be gained upon 3 miles an hour After his pursuers had travelled 3 hours, they met an Express going at the same rate as themselves, who met the criminal 2 hours 24 minutes before In what time after the commencement of the pursuit will they overtake him?

96 Find the value of $\sqrt[5]{\tfrac{3}{5}(3b+c)(b^2-2a-c)(2b-a)^2}$, when $a=5$, $b=4$, $c=3$

97. Multiply $x^2+(a-1)x+(a+1)$ by $(a-1)x-(a^2+a+1)$.

98 Shew that $(a+b+c)(ax^2+by^2+cz^2)-(ax+by+cz)^2$

$$=ab(x-y)^2+bc(y-z)^2+ca(z-x)^2.$$

99 Resolve into factors $(a^2x^2-b^2y^2)^2+4abxy(bx+ay)^2-(bx+ay)^4$

100. Shew that

$$\frac{a^2}{(a-b)(a-c)(1+ax)}+\frac{b^2}{(b-c)(b-a)(1+bx)}+\frac{c^2}{(c-a)(c-b)(1+cx)}$$

$$=\frac{1}{(1+ax)(1+bx)(1+cx)}.$$

101. If $2s=a+b+c$, $2\sigma^3=a^3+b^3+c^3$, prove that

$$(\sigma^3-a^3)(s-a)+(\sigma^3-b^3)(s-b)+(\sigma^3-c^3)(s-c)=a^4+b^4+c^4-s\sigma^3.$$

102. Extract the square root of $\frac{a^2}{b^2c^2}+\frac{b^2}{c^2a^2}+\frac{c^2}{a^2b^2}+2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$

103. Simplify $\dfrac{a^2}{(x-a)^n}+\dfrac{2a}{(x-a)^{n-1}}+\dfrac{1}{(x-a)^{n-2}}$.

104. Find the value of $\dfrac{(x^{a+b})^2(x^{b+c})^2(x^{c+a})^2}{(x^a x^b x^c)^4}$.

105. If $x=\dfrac{2ac}{a+c}$, shew that the value of $\dfrac{(x-a)^2+(x-c)^2}{a^2+c^2}+\dfrac{4ac}{(a+c)^2}$ is the same for all values of a and c.

106. Simplify

$$\frac{\sqrt{a+x}-\sqrt{a-x}}{a+x+\sqrt{a^2-x^2}}\times\frac{\sqrt{a^2-x^2}}{\sqrt{a}+\sqrt{x}}\times\frac{x}{a-\sqrt{a^2-x^2}}$$

107. If $x+y=xy=1$, prove that $x^3=y^3=-1$.

108. Solve $\dfrac{1-\dfrac{1-\dfrac{a}{a-x}}{3-x}}{x}=\dfrac{1}{x-a}$.

109. Solve $(a+b)x-(a-b)y=3ab$, $(a+b)y-(a-b)x=ab$.

110. If $a:b=x:y=p:q$, shew that $\left(\dfrac{a^3+x^3}{b^3+y^3}\right)^2=\dfrac{p^6}{q^6}$.

111. A, B, C start at the same time from P to run to Q, their rates being such that B is always as much behind A as he is in advance of C. After A has reached Q he returns at once to P at the same rate, and meets B at a point whose distance from Q is equal to one-fourth of PQ. Shew that A meets C at a distance from P equal to one-third of PQ.

112. Find the value of $(\sqrt{x^2+y^2}-z)(\sqrt{x^2+y^2}+z)$, when $x=4$, $y=5$, $z=6$.

113. Simplify $\frac{1}{4}\{a-5(b-a)\}-\dfrac{3}{2}\left\{\dfrac{1}{3}\left(b-\dfrac{a}{3}\right)-\dfrac{2}{9}\left[a-\dfrac{3}{4}\left(b-\dfrac{4a}{5}\right)\right]\right\}$.

114. Resolve $(1+xz)^2(1+yz)^2+\{(1-xz)(1-yz)+2xyz\}^2$.

115. Find the square root of

$$x^4+2x^3(y+z)+x^2(y^2+z^2+4yz)+2xyz(y+z)+y^2z^2.$$

116. Find the value of $\sqrt{12}+\frac{1}{10}\sqrt{75}+6\sqrt{\frac{1}{12}}$.

117. Simplify $\dfrac{\left\{\dfrac{x}{1+x}+\dfrac{1-x}{x}\right\}-\left\{\dfrac{x}{1+x}-\dfrac{1-x}{x}\right\}}{\left(1+\dfrac{x}{1-x}\right)\left(1-\dfrac{x}{1+x}\right)}$.

√ 118. If $x+y+z=0$, prove that $\dfrac{x^2}{yz}+\dfrac{y^2}{zx}+\dfrac{z^2}{xy}=3$.

119. Shew that

$$\frac{(ab-cd)(a^2-b^2+c^2-d^2)+(ac-bd)(a^2+b^2-c^2-d^2)}{(a^2-b^2+c^2-d^2)(a^2+b^2-c^2-d^2)+4(ab-cd)(ac-bd)}$$

$$=\frac{(a+d)(b+c)}{(a+d)^2+(b+c)^2}$$

120. If $c=a\sqrt{1-b^2}+b\sqrt{1-a^2}$, prove that

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c)=4a^2b^2c^2.$$

121. Solve $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$

122. Solve $ax+by+cz=a+b$, $bx+cy+az=b+c$, $cx+ay+bz=c+a$.

123. If $2a-b : a+2b=3c-d : c+2d$, then

$$3a+b : a-3b=4c+d : 2c-3d$$

124. A was following B, who after a time turned and without changing his pace walked in the opposite direction; A now approached B six times as fast as before. Compare their rates of walking.

125. Find the value of $x^4-2a(a-b)x^2+(a^2+b^2)(a-b)x-a^2b^2$, when $a=1$, $b=-2$, $x=3$.

126. Divide $9a^2b^3-12a^4b+3b^5+2a^3b^2+4a^5-11ab^4$

by $3b^3+4a^3-2ab^2$.

127. Resolve into linear factors

$$(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$$

128. Simplify $\dfrac{(yz-a)^2-(ca-y^2)(ab-z^2)}{(bc-1)(yz-a)-(z-by)(y-cz)}$.

129. Reduce $\left(2-\dfrac{3x}{y}+\dfrac{9x^2-2y^2}{y^2+3xy}\right)\div\left\{\dfrac{1}{y}-\dfrac{1}{y-2x-\dfrac{4x^2}{x+y}}\right\}$.

130. Find the fourth root of

$$x^8-\frac{3a^2x^7}{b}+\frac{27a^4x^6}{8b^2}-\frac{27a^6x^5}{16b^3}+\frac{81a^8x^4}{256b^4}$$

131. Simplify $1+\sqrt{8}+\sqrt{2}-\sqrt{27}-\sqrt{12}+\sqrt{75}-\sqrt{19+6\sqrt{2}}$.

132. If $x^y=y^x$, prove that $\left(\dfrac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$; and that $y=2$, if also $x=2y$.

133. If $x=\frac{1}{2}\left\{a+\sqrt{\left(\dfrac{4b^3-a^3}{3a}\right)}\right\}$ and $y=\frac{1}{2}\left\{a-\sqrt{\left(\dfrac{4b^3-a^3}{3a}\right)}\right\}$, find the value of x^3+y^3.

134. Solve $\sqrt{\dfrac{x}{a}}+\sqrt{\dfrac{(b-c)(ac-bx)}{abc}}=1$

135. Solve $(a+b)x-(a-b)y=m(x^2-y^2)$,
$$(a-b)x+(a+b)y=m(x^2-y^2)$$

136. If $\dfrac{x}{b-c+a}=\dfrac{y}{c-a+b}=\dfrac{z}{a-b+c}$, then each
$$=\frac{a(y+z)+b(z+x)+c(x+y)}{2(bc+ca+ab)}$$

137 A fruiterer sold for 19*s* 6*d* a certain number of oranges and apples, of which the latter exceeded the former by 180 He sells the apples at the rate of 5 for 3*d*, and 15 oranges bring him in $1\frac{1}{2}d$ more than 35 apples How many are there of each sort, and what are the oranges worth a piece ?

138 From $\{m(2m-3p)-2n(4n-3p)\}x+\{m(p-m)-p(2n+p)\}y$ take $3\{p\left(2n-\frac{3p}{2}\right)-\frac{p}{2}(2m-3p)\}x-\{p(p-m)+2n(2n+p)\}y$, and find the value of this difference when $x=n=\frac{1}{2}$, $y=m=-2$

139. Reduce $\dfrac{12x^4-4ax^3-23a^2x^2+9a^3x-9a^4}{8x^4-14a^2x^2-9a^4}$

140 Multiply $\dfrac{x^3}{2}-5\dfrac{x^2}{3}+\dfrac{x}{12}+3$ by $\dfrac{x^2}{2}-x+3$

141 Resolve into linear factors $2x^2-7xy-22y^2-5x+35y-3$

142 Simplify $\dfrac{\left(\dfrac{x}{y}+1\right)^2}{\dfrac{x}{y}-\dfrac{y}{x}}\times\dfrac{\dfrac{x^3}{y^3}-1}{\dfrac{x^2}{y^2}+\dfrac{x}{y}+1}-\dfrac{\dfrac{x^3}{y^3}+1}{\dfrac{x}{y}+\dfrac{y}{x}-1}$

143 If x be the L C M and y the H C D of a and b, and if $x+y=ma+\dfrac{b}{m}$, prove that $x^3+y^3=m^3a^3+\dfrac{b^3}{m^3}$

144 Find the value of $\left\{\dfrac{c^{-n}}{d^{-3q}}+\dfrac{a^{-\frac{m}{2}}}{b^{-2p}}\right\}\left\{\dfrac{a^{-\frac{m}{2}}}{b^{-2p}}-\dfrac{c^{-n}}{a^{-3q}}\right\}$.

145. If $x(y-z)^2-z(y+z)^2=0$, then $\dfrac{\sqrt{x}+\sqrt{z}}{\sqrt{x}-\sqrt{z}}=\dfrac{y}{z}$

146 Solve $\dfrac{3-2x}{1-2x}-\dfrac{2x-5}{2x-7}=1-\dfrac{4(x^2-1)}{7-16x+4x^2}$

147. Solve $x+y+z=(a-b)(a-c)(b-c)$,
$$ax+by+cz=a^2x+b^2y+c^2z=0$$

148. If a, b, c, d be in continued proportion, shew that
$$\left(\frac{a-b}{c}+\frac{a-c}{b}\right)^2-\left(\frac{d-b}{c}+\frac{d-c}{b}\right)^2=(a-d)^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)$$

149 The sides of a right-angled triangle are as 5 to 12, and the hypotenuse is 130, find the sides

150 Find the value of $a^4+3a^3b+4a^2b^2+3ab^3+b^4$,

when $$a=\frac{5+\sqrt{13}}{2} \text{ and } b=\frac{5-\sqrt{13}}{2}$$

151 Multiply $x^5+a^5-ax(x^3+a^3)$ by $x^3+a^3-ax(x+a)$.

152 Find $(x^2-2xy+4y^2)^{\frac{1}{2}}$ in terms of a and b, where $x=9a^2+12ab$ and $y=2b^2+6ab$

153 Multiply together $\sqrt{x}-3\sqrt{y}$, $\sqrt{x}-2\sqrt{y}$, $\sqrt{x}-\sqrt{y}$, $\sqrt{x}+\sqrt{y}$, $\sqrt{x}+2\sqrt{y}$, and $\sqrt{x}+3\sqrt{y}$

154 Extract the square root of

$$9\frac{x}{y}-24\sqrt{\frac{x}{y}}+34-24\sqrt{\frac{y}{x}}+9\frac{y}{x}.$$

155 Shew that $\dfrac{x^2y^{-2}-xy^{-1}-x^{-1}y+x^{-2}y^2}{x^2y^{-2}-2+x^{-2}y^2}=1-\dfrac{xy}{(x+y)^2}$.

156 If $\sqrt{a^2+b^2}+a=bx$, find $x-x^{-1}$ and $x+x^{-1}$ in terms of a and b

157. Shew that $\sqrt[3]{(\sqrt{5}+2)}-\sqrt[3]{(\sqrt{5}-2)}=1$.

158. Solve $\dfrac{x+\frac{1}{x}}{x^2+1}-\dfrac{1}{x+1}=\dfrac{3}{x^2+2x+1}$

159. Solve $x^3+y^3+z^3-3xyz=0$, $3a-x+z=3b-y+x=3c-z+y$.

160. If $\dfrac{x}{2a-b+2c}=\dfrac{y}{2b-c+2a}=\dfrac{z}{2c-a+2b}$,

then $$\frac{a}{2x+2y-z}=\frac{b}{2y+2z-x}=\frac{c}{2z+2x-y}.$$

161 A has twice as many pennies as shillings B, who has 1s 7d. more than A, has twice as many shillings as pennies together they have one more penny than they have shillings How much has each ?

162. Prove that $(ax-by+cz-du)^2+(ay+bx-cz-du)^2$

$$+(az-bu-cx+dy)^2+(au+bz+cy+dx)^2$$
$$=(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+u^2)$$

163 Divide $x^8+\dfrac{x^6}{y^2}+\dfrac{x^4}{y^4}+\dfrac{x^2}{y^6}+\dfrac{1}{y^8}$ by $x^4-\dfrac{x^3}{y}+\dfrac{x^2}{y^2}-\dfrac{x}{y^3}+\dfrac{1}{y^4}$.

164 Find a value of x which will make $x^4+6x^3+11x^2+3x+31$ a perfect square.

165. Shew that the difference of (e^x+e^{-x}) and (e^x-e^{-x}) is independent of x, and that the sum of $\left(\frac{1}{x}+x\right)\left(\frac{1}{x}-x\right)$ and $\left(\frac{1}{x}+x\right)^2$ diminishes as x increases.

166. Find the square root of $(3x+1)(3x+4)(3x+7)(3x+10)+81$

167. Simplify $\dfrac{a^2\left(\frac{1}{b}-\frac{1}{c}\right)+b^2\left(\frac{1}{c}-\frac{1}{a}\right)+c^2\left(\frac{1}{a}-\frac{1}{b}\right)}{a\left(\frac{1}{b}-\frac{1}{c}\right)+b\left(\frac{1}{c}-\frac{1}{a}\right)+c\left(\frac{1}{a}-\frac{1}{b}\right)}$.

168. Find the value of $\dfrac{1+x}{1+\sqrt{1+x}}+\dfrac{1-x}{1+\sqrt{1-x}}$, when $x=\dfrac{\sqrt{3}}{2}$

169. If $m=a^x$, $n=a^y$, $a^2=(m^y n^x)^z$, shew that $xyz=1$.

170. Prove that $\sqrt{y+\sqrt{2xy-x^2}}+\sqrt{y-\sqrt{2xy-x^2}}=\sqrt{2x}$

171. Solve $\dfrac{a}{x+b-c}+\dfrac{b}{x+a-c}=\dfrac{b-c}{x+a}+\dfrac{a-c}{x+b}$.

172. Solve $\dfrac{x+y-z}{b+c}=\dfrac{y+z-x}{c+a}=\dfrac{z+x-y}{a+b}=a+b+c$

173 If $$\frac{x}{a+2b+c}=\frac{y}{2a+b-c}=\frac{z}{4a-4b+c},$$
then $$\frac{a}{x+2y+z}=\frac{b}{2x+y-z}=\frac{c}{4x-4y+z}$$

174. A waterman finds that he can row with the tide from A to B, a distance of 18 miles, in an hour and a half, and that to return from B to A, against the same tide, though he rows back along the shore where the stream is only three-fifths as strong as in the middle, takes him just 2 hours and a quarter Find the rate at which the tide runs in the middle where it is strongest

175. Multiply $$(2a-3c)y^2-(a-c)y+(2a+c)$$
by $$(2a+3c)y^2+(a+c)y-(2a-c)$$

176 Find the coefficient of x^4 in $\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}\right)^2$

177 Resolve
$$(b-c)(b+c-2a)^2+(c-a)(c+a-2b)^2+(a-b)(a+b-2c)^2$$

178. Simplify
$$\frac{a^4}{(a-b)(a-c)}+\frac{b^4}{(b-a)(b-c)}+\frac{c^4}{(c-a)(c-b)}.$$

179. Reduce $\dfrac{e^{x-y}+xy^{-1}+yx^{-1}+e^{y-x}}{xy^{-1}e^{x-y}+2+yx^{-1}e^{y-x}}$.

180 If $y=x+\frac{1}{x}$, express $x^5+\frac{1}{x^5}$ in terms of y.

181 If $x=\left(\frac{a+b}{a-b}\right)^{\frac{2mn}{m-n}}$, then $\frac{1}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)(\sqrt[m]{x}+\sqrt[n]{x})=\left(\frac{a+b}{a-b}\right)^{\frac{m+n}{m-n}}$.

182 If $x=\left\{-\frac{a}{2}+\sqrt{\left(\frac{a^2}{4}-\frac{b^3}{27}\right)}\right\}^{\frac{1}{3}}+\left\{-\frac{a}{2}-\sqrt{\left(\frac{a^2}{4}-\frac{b^3}{27}\right)}\right\}^{\frac{1}{3}}$,
prove that
$$x^3-bx+a=0.$$

183. Solve
$$(x-9)(x-7)(x-5)(x-1)=(x-2)(x-4)(x-6)(x-10).$$

184 Solve $\sqrt{y}-\sqrt{a-x}=\sqrt{y-x}$,
$$\sqrt{y-x}+\sqrt{a-x} : \sqrt{a-x} :: 5 : 2.$$

185 Two persons A and B run round a field, starting from the same point in opposite directions; A reaches the starting-point 4 minutes and B 9 minutes after they meet; if they continue to run at the same rate, in what time will they meet at the starting-point?

186. Find the product of $\frac{1}{1-ax}$, $\frac{1}{1-bx}$ and $\frac{1}{1-cx}$, and keep it in an *integral form*

187. Reduce $\frac{px^3-(p-1)qx^2+(p-q^2)x-pq}{pqx^3+(p^2+q^2)x^2+2pqx+p^2}$.

188. Shew that $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)-1=\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)\left(\frac{x}{y}+\frac{y}{z}\right)$.

189 If
$$\frac{x}{a}=\frac{y}{b} \text{ and } \frac{x(1-a)}{a(1-x)}+\frac{y(1-b)}{b(1-y)}=1,$$
then
$$\frac{x}{a}=\frac{y}{b}=\frac{1}{1-\sqrt{(1-a)(1-b)}}.$$

190 If $x=\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}$, then shall $\frac{1-x}{1+x}=\frac{e}{1+\sqrt{1-e^2}}$

191. Find the value of
$$\left\{\left(\frac{m^{-1}-n^{-1}}{a^{-2}-a^{-1}b^{-1}+b^{-2}}\right)^{-3}\right\}^{-4}\left\{\left(\frac{m^{-2}-n^{-2}}{a^{-3}+b^{-3}}\right)^{2}\right\}^{-6}.$$

192. Prove that $\dfrac{\frac{x}{\sqrt{1+x^2}}-\frac{x}{\sqrt{1-x^2}}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\frac{1}{x}\left\{1-\frac{1}{\sqrt{1-x^4}}\right\}$.

193. Solve $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$.

194 Solve

$$x-\frac{y+12}{5} : y-\frac{24-5x}{8} \quad 1 . 5, \quad \frac{x-2}{2}-\frac{19-y}{3}=\frac{\frac{x}{2}+8}{6}-\frac{y-9}{4}$$

195. Fifteen guineas should weigh 4 oz , but a parcel of light gold having been weighed and counted, was found to contain 9 more guineas than was supposed from the weight and a part of the whole, exceeding the half by four guineas and a half, was found to be $1\frac{1}{2}$ oz deficient in weight. What was the number of guineas ?

196 If
$$A=c^2(ax+b)^3\{3a(cx+d)-c(ax+b)\},$$
$$B=a^2(cx+d)^2\{3c(ax+b)-a(cx+d)\},$$
find the simplest value of $(A-B)^{\frac{1}{3}}$

197 Multiply
$$(a+b)x^2-abxy+(a-b)y^2,$$
by
$$(a-b)x^2+abxy+(a+b)y^2.$$

198 Find the coefficient of x^6 in the product of
$$1+x+x^2+\dots \text{ and } 1-x+x^2- \dots .$$

199 Resolve
$$(y-z)^3(y+z-2x)+(z-x)^3(z+x-2y)+(x-y)^3(x+y-2z)$$

200 Simplify $\dfrac{x^4(y-z)+y^4(z-x)+z^4(x-y)}{x^2(y-z)+y^2(z-x)+z^2(x-y)}$

201 Reduce to its simplest form
$$\frac{1-x}{1+x}+\frac{x-y}{x+y}+\frac{y-1}{y+1}+\frac{(1-x)(x-y)(y-1)}{(1+x)(x+y)(y+1)}.$$

202 Find the value of
$$\frac{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}, \text{ when } x=\sqrt{\frac{a^4+1}{2}}.$$

203 If
$$u=x\sqrt{1+y^2}+y\sqrt{1+x^2},$$
prove that
$$\sqrt{1+u^2}=xy+\sqrt{(1+x^2)(1+y^2)}$$

204. Solve
(1) $2ax^n-(b-1)x^{n+1}=(a-1)x^n+bx^{n+1}$

(2) $\sqrt{2}+1-\dfrac{1}{\dfrac{1}{2^x}-1}=0.$

205 Solve
$$x^my^n=a^{2m+n}b^n, \quad x^ny^m=a^nb^{2m+n}$$

206 If
$$a(y+z)=b(z+x)=c(x+y),$$
then
$$\frac{y-z}{a(b-c)}=\frac{z-x}{b(c-a)}=\frac{x-y}{c(a-b)}.$$

207. Two loaded waggons were weighed, and their weights were found to be in the ratio of 4 to 5; parts of their loads, which were in the ratio of 6 to 7, being taken out, their weights were then found to be as the numbers 2 and 3, and the sum of their weights was then 10 tons. What were the weights at first?

208 Shew that $(x^2+y^2-1)^2+(x'^2+y'^2-1)^2+2(xx'+yy')^2$
$$=(x^2+x'^2-1)^2+(y^2+y'^2-1)^2+2(xy+x'y')^2.$$

209. Resolve into four factors $x^2+y^2+z^2-2xy-2yz-2xz$.

210 Reduce $\dfrac{(1-xy)(1+xy)-(x-y)(x+y)}{(1+xy)^2+(x-y)^2}$ to its lowest terms

211 Simplify $\dfrac{a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)}{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}$.

212 Find the square root of $\dfrac{9x^{-\frac{4}{m}}}{4}-a^{-\frac{1}{n}}x^{\frac{2a+m}{am}}+\dfrac{x^{-\frac{2}{a}}}{9a^{\frac{2}{n}}}$.

213 If $xyz=1$, shew that
$(1+x+y^{-1})^{-1}+(1+y+z^{-1})^{-1}+(1+z+x^{-1})^{-1}=1$.

214 Simplify $\left\{\sqrt[4]{\dfrac{(x+y)^{\frac{1}{2}}+(x-y)^{\frac{1}{2}}}{(z+y)^{\frac{1}{2}}+(z-y)^{\frac{1}{2}}}}\times\dfrac{\sqrt{x+y}-\sqrt{x-y}}{\sqrt{z+y}-\sqrt{z-y}}\right\}^2$

215 Solve $\sqrt[4]{\dfrac{a+x}{a-x}}+\sqrt[4]{\dfrac{a-x}{a+x}}=\sqrt[4]{b}$.

216. Solve $\left(\dfrac{x}{a}\right)^m\left(\dfrac{y}{b}\right)^n=c,\ \left(\dfrac{x}{b}\right)^n\left(\dfrac{y}{a}\right)^m=d.$

217 If $x-z : y-z=x^2 : y^2$, shew that $x+z : y+z=\dfrac{x}{y}+2 : \dfrac{y}{x}+2$

218 A pedestrian finding that he could walk forwards 4 times as fast as he could backwards, undertook to walk a certain distance ($\frac{1}{4}$ of it backwards) in a given time But the ground being bad, he found that his rate per hour backwards was $\frac{1}{8}$th of a mile less than he had supposed, and that to have won his wager, he must have walked forwards 2 miles an hour faster than he did What was his rate per hour backwards?

219. If $x+y+a=0$, $xy-b=0$, then $(1+x^2)(1+y^2)=a^2+(1-b)^2$.

220 Resolve $(a+b+c)^3abc-(bc+ca+ab)^3$

221 Reduce $\dfrac{(1-x^2)(1-y^2)(1-z^2)-(z+xy)(y+zx)(x+yz)}{1-x^2-y^2-z^2-2xyz}$.

222 Simplify

$$\frac{bc}{a(a^2-b^2)(a^2-c^2)}+\frac{ac}{b(b^2-a^2)(b^2-c^2)}+\frac{ab}{c(c^2-b^2)(c^2-a^2)}.$$

223 Find the square root of $\left(1-\frac{1}{1-x^2}\right)\left\{\frac{1}{1-\frac{1}{x^2}}-1\right\}$.

224 Prove that

$$\frac{x^3-3x+(x^2-1)\sqrt{x^2-4}-2}{x^3-3x+(x^2-1)\sqrt{x^2-4}+2}=\frac{x+1}{x-1}\sqrt{\frac{x-2}{x+2}}.$$

225 If $x\sqrt{a^2-y^2}+y\sqrt{a^2-x^2}=a^2$, then $x^2+y^2=a^2$.

226 Solve $\sqrt{1+a}\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{1-a}\left(\frac{1-x}{1+x}\right)^{\frac{1}{4}}=2\sqrt[4]{1-a^2}$

227. The trinomial ax^2+bx+c becomes 8, 22 and 42 respectively when x becomes 2, 3, 4, what does it become when $x=-\frac{1}{3}$?

228 If $\frac{x}{y}-\frac{y}{x}=a$, and $\frac{x^2}{y^2}-\frac{y^2}{x^2}=b$, then $a^4+4a^2=b^2$

229 Reduce to its lowest terms $\frac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}$

230 If $y=\frac{ax-b}{a-bx}$, shew that $\frac{x-y}{1-xy}=\frac{b}{a}$

231 Solve $(\sqrt[4]{x+a}+\sqrt[4]{x-a})^3(\sqrt[4]{x+a}-\sqrt[4]{x-a})=2c$

232 Solve $(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3=0$.

233 If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that

$$\frac{ax-by}{(a-b)(x+y)}+\frac{by-cz}{(b-c)(y+z)}+\frac{cz-ax}{(c-a)(z+x)}=3$$

234 If $x+y : 3y=6x-y : 8x$, find the ratio $x : y$

235 A person, sculling in a thick fog, meets one barge and overtakes another which is going at the same rate as the former; if a be the greatest distance to which he can see, and b, b' the distances that he sculls between the times of his first seeing and passing the barges, shew that

$$\frac{2}{a}=\frac{1}{b}+\frac{1}{b'}.$$

ADDENDUM.

SOLUTIONS OF UNIVERSITY PAPERS.

1 Calcutta University, 1892, Q 1.(ii).

By § 228, we get

$$\frac{x+a}{x+b}=\frac{(x+3a)-(x+a)}{(x+a+b)-(x+b)}=\frac{2a}{a}=2,$$

$$\therefore\ x+a=2(x+b),$$

whence $$x=a-2b$$

2. Punjab University, 1887, Q 4.

By Cross Multiplication [§ 244], we have from the first two equations

$$\frac{x}{bc(mr-qn)}=\frac{y}{ca(mp-rl)}=\frac{z}{ab(lq-pm)}=k \text{ suppose.}$$

Thus $$x=kbc(mr-qn),$$
$$y=kca(mp-rl),$$
$$z=kab(lq-pm).$$

Multiply successively by ax, by and cz, and add ; thus

$$ax^2+by^2+cz^2=kabc\,x(mr-qn)+kabc\,y(mp-rl)+kabc\,z(lq-pm)$$

But the left side=0 (hypothesis) ; therefore

$$kabc\,x(mr-qn)+kabc.y(mp-rl)+kabc\,z(lq-pm)=0\ ;$$

divide by $kabc$, thus the required relation follows

3. Punjab University, 1888. Q 7. [*See* § 281, *Ex* 42]

From the given relations, we have [§ 252]

$$x(y^2-z^2)=yz(b-c) \qquad (1),$$
$$y(z^2-x^2)=zx(c-a) \qquad (2).$$

Multiply (1) by x and (2) by y, and add ; thus

$$z^2(y^2-x^2)=xyz(b-a)$$

Divide by z and change signs , thus

$$z(x^2-y^2)=xy(a-b),$$

or $$\frac{x^2-y^2}{xy}=\frac{a-b}{z}\ ;$$

i e , $$x^2-y^2 : xy :: a-b : z.$$

4 Madras University, 1889, Q 3

$$a^2+x^2=a^2+ab+bc+ca=a^2+a(b+c)+bc=(a+b)(a+c),$$

similarly $b^2+x^2=(b+c)(b+a)$, and $c^2+x^2=(c+a)(c+b)$;

$$\text{proposed expression } =(a+b)(c+a)\times(b+c)(a+b)\times(c+a)(b+c)$$
$$=(a+b)^2(b+c)^2(c+a)^2$$
$$=\{(b+c)(c+a)(a+b)\}^2$$

5 Bombay University, 1882, Q 5 (i).

Transpose, thus

$$\frac{a}{x+a}-\frac{a-c}{x+a-c}=\frac{b+c}{x+b+c}-\frac{b}{x+b},$$

$$\frac{ax+a(a-c)-(a-c)x-a(a-c)}{(x+a)(x+a-c)}=\frac{(b+c)x+b(b+c)-bx-b(b+c)}{(x+b)(x+b+c)}$$

$$\frac{cx}{(x+a)(x+a-c)}=\frac{cx}{(x+b)(x+b+c)},$$

hence either $cx=0$, whence $x=0$ [one solution],

or
$$\frac{1}{(x+a)(x+a-c)}=\frac{1}{(x+b)(x+b+c)},$$

whence
$$(x+a)(x+a-c)=(x+b)(x+b+c),$$

or
$$x^2+x(2a-c)+a(a-c)=x^2+x(2b+c)+b(b+c),$$
$$2x(a-b-c)=b^2-a^2+c(a+b)$$
$$=(b-a+c)(a+b),$$
$$\therefore x=-\tfrac{1}{2}(a+b) \text{ [another solution]}$$

6 Calcutta University, 1896, Q 4 (i)

Since $3=1+1+1$, we have by transposition

$$\left(\frac{x-a}{3b+5c}-1\right)+\left(\frac{x-3b}{5c+a}-1\right)+\left(\frac{x-5c}{a+3b}-1\right)=1,$$

$$\frac{x-a-3b-5c}{3b+5c}+\frac{x-a-3b-5c}{5c+a}+\frac{x-a-3b-5c}{a+3b}=0,$$

whence $x-a-3b-5c=0$ [see § 265, Ex 2], &c

7 Calcutta University, 1902, Q 4. (i)

By transp, we have

$$\frac{bc(ax-1)}{b+c}-a+\frac{ca(bx-1)}{c+a}-b+\frac{ab(cx-1)}{a+b}-c=0,$$

$$\therefore \frac{abcx-bc-ca-ab}{b+c}+\frac{abcx-bc-ca-ab}{c+a}+\frac{abcx-bc-ca-ab}{a+b}=0,$$

or $(abcx-bc-ca-ab)\left(\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}\right)=0$,

whence
$$abcx-bc-ca-ab=0,$$
$$x=\frac{bc+ca+ab}{abc}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

ANSWERS.

18. Ex. (i) [*pp* 4—5]

8 15. 9 23. 10. 12 11 22 12. 4. 13. 2. 14. 19
15 2. 16. 0. 17. 43. 18 15. 19. 55. 20. 530
21 282. 22 210 23. $2\frac{1}{3}$ 24 $\frac{1}{6}$ 25. $\frac{19}{20}$ 26 3
27. $1\frac{3}{5}$ 28. $2\frac{1}{3}$ 29 $\frac{4}{5}$.

18. Ex. (ii) [*p*. 6]

5 168 6 108. 7. 0, 8. 0 9 84. 10. $\frac{1}{3}$. 11 $\frac{1}{2}$.
12 1. 13 56. 14 1120 15 0 16. $4\frac{1}{2}$. 17. 165
18. 0. 19 10. 20 64 21 6. 22. 60 23. $\frac{8}{25}$
24 0 25 $10\frac{10}{49}$ 26 6 27. 88. 28. 39 29. 98
30 42. 31. 0 ; 0 , 0 32. $\frac{17}{18}$ 33. $\frac{4}{11}$ 34 $3\frac{1}{9}$.

18 Ex. (iii) [*p* 7].

1 15 2 1. 3. 49. 4 16 5. 23. 6 6. 7. 2.
8 8 9. 8 10 16. 11 1 12 14. 13 6 14. 8
15. 2. 16. 2 17 26. 18 11. 19. 1. 20 48. 21. 24.
22 33. 23. $4\frac{3}{4}$

18 Ex (iv). [*p* 7]

1. $3\frac{1}{4}$ 2 $21\frac{3}{4}$. 3 $2\frac{77}{82}$. 4. $3\frac{37}{205}$. 5 122 6. $6\frac{5}{24}$
7. $2\frac{31}{240}$ 8. 10 55. 9. 035 10. 3 4599.

21 [*p* 9]

6. 125 7 64 8. 784. 9. 1728 10. 6250000.
11. 729 12. 32. 13. 1536 14 224 15 5832 16 0.
17. 82944 18. $115\frac{1}{5}$. 19. 396 20. 0. 21. 20480
22. 196608. 23. 0 24 25. 25 64 26 648. 27. 5.
28. 4096. 29. 1000000 30. 129600000000 31 11025.
32 15552. 33. $112\frac{1}{2}$ 34. 180 35 2688. 36 291600000000.
37 18144. 38. 24192 39 $7\frac{19}{32}$ 40 $\frac{15}{32}$ 41 $\frac{8}{27}$. 42. $\frac{9}{100}$
43. $\frac{4}{27}$

22 [*pp* 10—11].

5 12 6 432 7. 300. 8 108 9. 4860 10. 6.
11. 120 12 810. 13 27 14. 24. 15 4 16. $\frac{1}{2}$

22. [*pp.* 10—11]

17. 6 18 $-\frac{1}{12}$. 19. $\frac{1}{81}$. 20. $\frac{1}{96}$. 21. 0 22. $2\frac{2}{3}$.
23 $\frac{4}{9}$. 24. 4 25 $\frac{1}{10}$. 26 $\frac{1}{8}$ 28 125. 29 109
30 23. 31 13 32 17 33 37. 34. 73. 35 3.
36 27 37. 28 38. 64 39 80.

27 [*p* 12].

7. $24\frac{2}{3}$. 9. $10x+y$, for $45=10\times4+5$ 11 2, 3, a, $b+c$ 12 18
13. 6 14 No, simple expression. 15. 0.

28 [*pp* 12—13]

1. $247\frac{1}{2}$ 2 1025 3. 1250. 4 $15\frac{2}{3}$. 5 136 or 64
6. 255 or 225 7 442 8 308 9 161 10 157 11 0
12 142 13 106. 14 95 15. 462. 16 450. 17. $2\frac{18}{25}$
18. 25 19 250 20. 159 21 722. 22. 204 23 24575
24. 29 25 230 26 20 27 39 28 161. 29. 723
30 65 31 15 32. 4069 33 62 34 840 35 290
36. 243. 37. 232. 38. $4\frac{11}{12}$ 39 $6\frac{5}{12}$. 40. $36\frac{7}{8}$ 41 $\frac{3}{16}$.
42. $1\frac{7}{36}$. 43. $3\frac{3}{10}$. 44 $\frac{23}{24}$ 45 Each = 233. 46 Each $=24\frac{7}{12}$
49 5 50 86 51 $3\frac{4}{15}$ 52. $2\frac{4}{15}$

29 [*p* 15]

8. $4\frac{1}{2}$ 9 150 ; 25 10. 5. 11 £3 12 8*s*. 2*d*
13 20. 14. 17 15 $1\frac{1}{2}$ 16 32 17 1 mile
18. 72 sq ft 19 20 sq yds 20. 16.

37 [*pp.* 18—19].

7. He gains -5 rupees, he loses $+5$ rupees. 8. -2 miles
9. -1 min *i e*, 1 min. to 12 noon. 10. Assets of 35 rupees.
11. -45.

39. [*p* 21].

4 $+13$, $+3$, -3, -13 5 $+21$, -21; -4, 0 6 -13
7 -4, $+5$. 8 -3 9 -14. 10 -23 11. $+3$
12. -42 13. $+33$ 14 -128. 15 $+22$ 16. -240.
17. -49

42 [p 23]

4 $+15$; $+7$; -4 5. $+6$; -25; $+18$; -6. 6. -6.
7. -2. 8 $+4$ 9. $+7$

43. [p 23]

4 $+5$; 0 5 -2; -2 6 0 7 0 8 $+2$ 9. -16.
10. $+16$ 11 -24 12 $+14$. 13. $+26$

44. [p 24]

1. $+13$; -13; -1; $+1$. 2 $+5m$; $+11m$; $-11m$; $-5m$.

45 [p 25].

1. -2, -1; $+1$. 2 -4; -1, -100 3 The latter by $+2$.
4 The former 5. Yes; by $+4$ 6 The former; by $+3$

48. [p 28]

3 -3 4 11 5. 23 6 13 7. -11

50 [p 29].

5 -3, $a+b-c-3$; $+13$; $-11a$, $-8xy$

51. Ex. (i) [p. 30]

3 $56x$ 4 $26a$. 5 $24m$ 6 $37n$ 7 $33r$ 8 $46y$
9 $-42c$ 10 $-25n$ 11 $-21y$. 12. $-25ab$ 13. $-16xy^2$
14. $-73max$ 15 $\frac{77}{2}ab$ 16 $\frac{24}{5}xyz$ 17 $-\frac{617}{28}mn$ 18 $\frac{83}{35}pq$.
19. $(a+b+c+d)x$. 20. $(a+p+9)x$

51. Ex. (ii). [p 31]

2 a. 3 $2x$ 4 $28m$ 5. $15xy$. 6 abc 7. 0
8. $\frac{17}{10}axy$. 9. 0 10 $\frac{17}{40}mn$ 11 $-8a$ 12 $-17z$
13 $-56by$. 14. $-\frac{11}{4}ab$. 15. $(a-b+c-d+f)x^2$
16. $(4a+4p-q)x$ 17 $(7+4a)x^2y$.

54. [p 32]

3. $43ab+5ax+20xy+18pq$ 4. $12abc-axy+20pqr+15mr+29df$.
5. $3a^2+10b^2-18c^2+25ab-bc+17ca$ 6. $5ab-3xy+8x^2-2a^2c+4abc$. 7. $3a+2b-c+7d$ 8. $-xy+3yz+8xz$ 9 $ax-2by-3cz$
10. $6mn-11np+10nq$. 11. $2abc+2abx+7xyz$.

55 [pp 33—35]

5 $2x$ 6 58 7 $5a$ 8 $5ax+by$ 9 $ab-\frac{11}{8}xy$ 10. $38a+2b+4c$ 11 $-ab-2cd$ 12 $-a+15b-8c$. 13 $3y$ 14 $4x^2 1$ 15 $-9ax+6by-7cz$ 16 $15a+3b-6c+5d$ 17 $7a^2+25ax+2y^2$ 18 $\frac{73}{6}x^2-\frac{53}{3}ab+\frac{37}{4}a^2$ 19 $12x^2+7y^2$ 20 $6+8ab-4bc+15ac$ 21 $-9x^3+2ax^2-31a^2x+16a^3$ 22 $a-e$ 23 $3x-7z$ 24 $2b+d-f$ 25 $11b^2$ 26 $10x^3-6mn+11r^2$ 27 $-x^2+2xy+4x-y$ 28 $-2bx^2+6d-1$ 29 $7x^3-2xy-2y^2$ 30 $4r^2-2q^2$ 31 $5x^3+4y^3+2z^3-24xyz$

56 [p 35]

2 3, $-4a$, $2ab^3$, $+1$, $-3a^2bc$, m^3, -1; $a+2$ 3. 2 4 -1 0 5 3, $3a+1$ 7 Each $=9a-2b+7c$, when 3 is written for x 8 x^3-2x^2-5x+1. 9 $-3x^2+6x-3y$ 10 $(2x+3)$ rupees 11. $(3a-2)$ rupees 12 $(m+n+2p)$ rupees

57 [pp 36—37]

8. $a+x$ 9 $2y$ 10 $15a+12$ 11 $2a+1$ 12 $\frac{1}{2}x+\frac{3}{2}y-2$ 13 $-mn-mx+my+2n$ 14 $x+5y-7z$ 15 $-x-3y+4z$ 16 $5a-5b+c+1$ 17. $a+b+2c-5d+9$ 18 $pq+1$ 19 $\frac{1}{4}ax-xy+1$ 20. $-x^2+2y^2+3z^2$. 21 $3abc-3ab-2ac-1$ 22 a^2-2ax 23 $2ar-6r^2-2a^2$ 24 $(a-p)x^3-(b-q)x^2+(1-r)x$ 25. x^3+2x^2+4x-2 26 $3x^2+2x+3$ 27 $-\frac{1}{3}xy+\frac{1}{2}y^2-\frac{1}{5}$ 28 $-\frac{5}{2}y-\frac{21}{4}a-\frac{1}{12}x+\frac{1}{4}b$ 29 $7a^4+2a^3b-2ab^3-7b^4$ 30 $2x^4-5x^3-2x$ 31 $(a-1)x^5+2x^2y^3+(b+5)y^5$

59 [p 38]

4 y 5 $a-x$ 6 $ax+1$ 7 $2a+2c$ 8 $3a-2b-e$ 9 $-x+y+3z$ 10 $7a-13b$ 11 $4a-2b+2c+3d$ 12 6. 13 $2ax+2cy$ 14 y^2

60 [p 39]

2 $x-y$ 3. $3a-7b$ 4. $1-2x$ 5 $5x-3a$ 6 $x-a$ 7 $2x$ 8 $2a-2b+y$ 9 y^2 10 $12a-b$ 11 $10a-4b$ 12 $12a-3x$ 13 $2bce+2bef$. 14 $4c$ 15 $21+3x$ 16 $4a-5b+c$ 17 $-25x+2y$ 18 $a-2b$ 19. $a+8b-c$. 20 $m-11n$

61. [p. 40]

3. (1) $a-b+(c-d-e+f)$; (2) $a-b-(d+e-c-f)$.

4. (1) $(a-b)+(c-d)-(e-f)-(g-h)$.
(2) $a-(b-c)-(d+e)+(f-g)+h$. (3) $(a-b+c)-(d+e-f)-g+h$.
(4) $a-(b-c+d)-(e-f+g)+h$. (5) $a-b+c-(d+e-f+g-h)$
(6) $a-(b-c+d+e)-g+h$ (7) $a-b+c-d-(e-f+g-h)$.
(8) $a-(b-c+d+e-f+g-h)$

62 [p 40]

1 $-3a-2b+x-4$. 2. $1-3x^2+3xy-3y^2$. 3. $4x-3y+1$.
4. $-x^2+2x-1$. 5 $-3x^2+4x-1$; -1. 6. $3x-5$; 0
7. $a-(x+y)=(a-x-y)$Rs 8 $(a-x)$ years. 9. $(1500-x)$ rupees.
10 $2a+2b$

Miscellaneous Examples I. [pp. 41—43].

1. 1. 2 $6ax^2+2a^2x-ax$; $5a+2a^2$. 3 a^2+5a-b ; 8
4 $2x+3z$. 5 $-2y$ 6. 5 7. $b-a+c$. 8 $2x-3x^2$; -1.
9. $2p-q$ 10 $9x^2-2xy-y^2$; x^2-2y^2 11. 0. 12. $4x-y+9$
13 $3-8a+6b$ 14. $3a-8b-2c$. 15 x^2+ax-1. 16. 50
17. $b+6c$ 18 $x-2y+9z-4$. 19. $x-2y+3z+1$. 20 $\frac{5}{3}a^2-ab-\frac{3}{4}b^2$
21 -1. 22 $x+y+5z$ 23. $-8a+3b+c$. 24. x^2+6x-6.
25. $5xy-x^2-2y^2$. 26. 0. 27 $-5a-3b+3c+2d$ 28 a^4-a^2
$-ab-b^2+2$. 29. $m-3n$. 30 $3a-ab+2b$ 31. 6 32. $8x^2$
$-3ax+a^2$ 33. $-\frac{x}{4}-\frac{y}{3}+\frac{11z}{4}+1$. 34 $6a+3b-4c$ 35 $\frac{1}{2}x^2-y^2$
36. 3. 37 $2a^2-b^2+\frac{8}{3}c^2$. 38. $-b-2c$; $1-2c$. 39. $15x-9y-2z$
41 3 42 $\frac{5a}{2}-\frac{b}{2}+\frac{17c}{4}$; $-\frac{3a}{2}+\frac{b}{2}-\frac{17c}{4}$ 43 x^3-x^2-3.
44. $2x-5y+10z$ 45 (1) 0, (2) $-a^2+4ab+5b^2$.

64. [p 45]

3 16. 4 90 5. 25. 6 -64. 7. -120 8. -216
9. 16. 10. -648. 11. 750. 12. -200 13. -45.
14. -540 15 64. 16 -1440, 17. -192. 18 -108
19. 0 20 152 21. -5 22. 595. 23 0 24 27
25. -125 26. 247. 27. -16. 28 164 29. 0. 30 0

82 [pp 56—57]

6 $3ax^3$. 7. $-10xy^2$ 8 $-104apx^2$ 9 $-2a^3bcx$ 10 $56ab^2c^2$.
11. pq^2r^2x 12 $-775abcxyz$ 13 $17472mnpq$ 14 $648abcdyz$.
15 $-36ax^3yz$. 16 $80a^2b^2pqxy$ 17 $-340abr^2suz$ 18. $6a^6$
19. $60x^{10}$. 20. $-60y^6$ 21. $-10a^5$ 22 $72m^{17}$ 23 $77z^{12}$.
24 $-3a^3x^3$. 25 $-6x^4y^4$. 26 $2a^2b^3c^4$. 27 $4m^4n^2x^4$
28 $-6x^4yz^3$. 29 $24a^3b^5$. 30 $-2a^4b^3c^4d^5$ 31 $12a^4x^4b^4y^6$
32 $-21a^6b^7m^8$. 33 $160a^3xy^6z^4$ 34 $1, -1, 1, -1$
35. $16a^2, -8a^3b^3; -a^5b^5c^5; 81x^8$ 36 a^2bcx^3 37. $30abcdef$.
38. $-120abcd$ 39 $960a^3b^3c^3$ 40 $360abcpx$ 41. $-640abcxyz$
42 $-2160abcrsxyz$ 43 $19200a^2b^3c^3x^3y^2$ 44 $6x^8$ 45 $-224mc^7$
46 $-60a^{14}$. 47 $3a^3bx^3y^2z$ 48 $-216a^7b^3c^3x^5y^6$ 49 $5a^4b^2c^nmx^4y^6z^2$

83, Ex (i) [pp 57—58]

4 $3a^2x-3ax^2$ 5 $-ax+ay$ 6 $ab+b^2$ 7 $2a^2xy-aby^2$.
8. $12a^3b^2c^3-3abc^4$ 9 $5ab^2c+5bc^2d$. 10 $-2ax+2ay-4a$
11 $-bx^2+3b^2x+ab^4x$ 12 $-ab^3c^3-a^2bc^3+a^3b^2c$ 13. $-6x^2yz$
$+3xy^2z+12xyz^2$ 14 $2a^4b^2c^2+2a^2b^4c^2+2a^2b^2c^4$ 15 $3a^2d+2abd$
$-4acd+ad^2$ 16 $-2a^3b^2c^2+3a^2b^3c^2+4a^2b^2c^3$ 17 $\frac{3}{4}x^2-\frac{3}{10}xy+\frac{3}{5}xz$
18. $-\frac{5}{3}a^3bc+\frac{5}{4}a^2b^2c+\frac{5}{6}a^2bc$ 19 $\frac{2}{7}a^2b+\frac{2}{7}ab^3+\frac{6}{7}ab$ 20 $\frac{1}{3}a^3x^5-\frac{5}{8}a^4x^4$.
21. $-\frac{2}{5}a^6b^4+\frac{9}{10}a^5b^5+\frac{4}{5}a^3b^6$ 22 $\frac{14}{5}a^2b^2cx^4+\frac{4}{3}b^3cx^3y^2-\frac{14}{15}b^2c^3x^3z^3$.
23 $10a^5b^3c+15a^2b^6c+20a^2b^3c^4$ 24 $6a^3b^4c^3-24ab^6c^4-15a^2b^3c^5$.
25. $-4a^4x^8+8a^3x^{10}+12a^6x^6$ 26 $16x^4y^4z^7-14x^3y^2z^6+6x^2y^5z^4$.
27 $8x^4y-24x^3y^2+24x^2y^3-8xy^4$ 28 $-9a^4b^3c^2d+3ab^4c^3d^2+6a^3bc^4d^3$
$-12a^3b^2cd^4$ 29. $-20a^3b^4cxy+12a^2bx^4y^2+4a^2b^3x^2y-4abcx^5y^5$

83, Ex (ii) [pp 58—59]

3. $-4a^4$ 4 x^3+x^2-2 5 $4a^2x-4ax^2$ 6 $21ax^3+21bx^3$.
7. $18a^2-24b^2$ 8 $32x^2+12y^3$ 9 $2x^2y^2$. 10 $4a^3-a^2b+2a^2c$
11; 12; 13 0.

88, Ex (i) [p 62]

10 $x^2+2x-15$ 11 x^2+x-56 12 $x^2-20x+99$.
13. $-a^2+8a-12$ 14 $x^2+(a-b)x-ab$ 15. $x^2-(a-b)x-ab$
16 $x^2-(a+b)x+ab$. 17 $x^2+16x+63$ 18 $ac+bc-ad-bd$
19 $ac-bc-ad+bd$ 20 $ac-bc+ad-bd$ 21 $2m^2+17m-117$

22. $6x^2-13xy+6y^2$. 23. $a^2x^2-b^2y^2$. 24. $\frac{3}{10}x^2-\frac{13}{20}ax+\frac{1}{2}a^2$.
25 $\frac{1}{4}abx^2+(\frac{5}{6}a-\frac{3}{10}b)x-1$. 26. $2x-4x^2$. 27. $-11qr$
28 x^2-5x+4. 29 $7a-a^2$.

88, Ex. (ii) [p 63].

5. $a^3+7a^2+17a+35$ 6 x^3+8 7. x^3-27. 8. x^4-1. 9. $p^3+6p^2q+12pq^2+8q^3$. 10. $6x^3-19x^2y+29xy^2-21y^3$. 11 $125x^3-150x^2y+60xy^2-8y^3$. 12 a^5+3a^2-4a+6. 13 $x^7-4x^3y^2-x^2y^3+2y^4$ 14 $35a^5-49a^4+10a^3-24a+64$ 15. $125a^3-8b^3$. 16. $125x^3+512y^3$. 17. a^6+b^6. 18 $ax^3+(a^2-b)x^2-2abx+b^2$.

88, Ex. (iii) [p. 65].

3 $x^4+a^2x^2+a^4$ 4 $a^4-2a^2b^2+b^4$ 5 $10a^4+a^3b+18a^2b^2-72ab^3-27b^4$. 6. $x^4-7x^2y^2+9y^4$. 7. $x^4+2abx^2+a^2b^2-b^4$ 8. $x^3+y^3+3xy-1$. 9 $a^3+b^3+c^3-3abc$. 10. $apx^3+2(aq-bp)x^2+(cp-4bq)x+2cq$. 11. $x^6-5x^4+3x^3+6x^2-7x+2$ 12. $3x^4-5x^3y+6x^2y^2+5xy^3-3y^4$. 13. $4x^6-10x^5y+10x^4y^2-21x^3y^3-5x^2y^4+5xy^5+y^6$. 14 $6a^5b-7a^4b^2-11a^3b^3+9a^2b^4-5ab^5$ 15. $a^4-2a^2b^2+4abc^2+b^4-c^4$. 16 $35a^2+11a^3-15a^4+18a^5-68a^6+28a^7$. 17 $abx^4-(bp-ac)x^3+(bq-cp-ar)x^2+(cq+rp)x-rq$. 18. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$. 19. $x^5-px^4+qx^3-qx^2+px-1$. 20. $4+5x+8x^2+10x^3-8x^4+5x^5-4x^6$. 21. $1-\frac{1}{6}x-\frac{1}{6}x^2+\frac{5}{18}x^3-\frac{1}{9}x^4$. 22. $1+\frac{1}{6}x-\frac{1}{6}x^2+\frac{25}{36}x^3-\frac{1}{3}x^4$. 23 $x^6-\frac{5}{2}x^5+\frac{21}{4}x^4-\frac{65}{8}x^3+\frac{21}{4}x^2-\frac{5}{2}x+1$. 24. $c^2x^8+(2ac-b^2)x^6+(2af-2bd+c^2)x^4+(2cf-d^2)x^2+f^2$. 25. $lx^3+my^3+mx^2y+lxy^2+(2lg-1)x^2+(2fm-1)y^2+2(lf+mg)xy+(l-2g)x+(m-2f)y-1$.

89. [p. 67]

7. $abc+bcx+acy+abz+cxy+bxz+ayz+xyz$. 8. $1-(a+b+c)x+(ab+ac+bc)x^2-abcx^3$. 9. $a^2b+ab^2+b^2c+bc^2+a^2c+ac^2+2abc$.
10 $ab^2-a^2b+bc^2-b^2c+ca^2-c^2a$. 11. $x^4-4x^3-7x^2+22x+24$
12. $x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$
$+(abc+acd+bcd+abd)x+abcd$.
13 $x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$
$-(abc+acd+bcd+abd)x+abcd$
14. $1-(ax+by+cz)+abxy+acxz+bcyz-abcxyz$.
15 $a^3x^3-a^2(b-c+d)x^2y-(abc-abd+acd)xy^2+bcdy^3$.

16 $a^2(b+c)+b^2(c+a)+c^2(a+b)-a^3-b^3-c^3-2abc$.

17 $9x^4-52x^2y^2+64y^4$. 18 a^6-x^6 19 $x^{16}-a^{16}$ 20 $a^8+a^4x^4+x^8$.

21. $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$

89a [pp 69—70].

11 $x, -2x, -3a^{-\frac{3}{2}}x^{\frac{5}{2}}$ 12 $a^{-3}, -3a^{1-x}b^{\frac{1}{3}}, 5a^{2n+1}$ 13. $6a^{m-1}b^{-1}$; $-4ab^{\frac{5}{2}}c^{-\frac{7}{4}}$. 14 $-24a^{m+1}$; $54x^{m+4}$; $32y^{2n+1}$. 15 $-6a^4x^{m+1}$; $-20a^{m+2}b^{2n+1}$, $42ax^my^n$ 16 $-a^0=-1$ 17 x^2 18. $-16ab^{-\frac{1}{2}}$

19 x^{6a+5} 20 $10x^{m+1}y^{n+1}z^4$ 21 $12a^{m+1}x^{3n}y^{2n+1}$

22. $a^{m+4}b^{2m+2}c^{m+5}$ 23 a^2+acb^{n-m}. 24 $b^n-a^{1-m}b$

25. $y^{c+1}z^{1-a}+x^{b-a}$ 26 $24a^{x+1}m^{y+1}-8a^3m^{x+1}+16a^xm^y$.

27. $-5a^3x^{n+3}+10a^4x^n-15a^5x^{n+1}$. 28. $x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ 29. $4x-9y$

30. a^3-b^3 31 $x^{\frac{3}{2}}-y^{\frac{3}{2}}$ 32 $x-y$ 33. $a^{\frac{3}{2}}-b^{-1}$. 34 $a^{-1}+1$.

35. $2x^{m+2}-3x^{m+1}-2x^m+3x^{m-1}+x^2-1$. 36 $x^3-x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3$

37 $a^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$ 38 $x+x^{-1}+1$ 39 $a^2b^{-2}+1+b^2a^{-2}$

40 $4a-13a^{\frac{1}{2}}x^{\frac{1}{2}}+9x$ 41 $a+b+c-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$. 42 $8x^2-y^2+27z^2+18x^{\frac{2}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}}$. 43 $x^{4a}+x^{2a}y^{2a}+y^{4a}$ 44 $x^{4m}-x^{2m}a^{2n}+2x^ma^{3n}-a^{4n}$

92 [p. 71]

1. 7, 6, 7, 3 2 3, 1, 5, 2, 3, 5 3 (1) Homogeneous. second; (3) not homogeneous, fourth, (3) homogeneous, third.
4 $-1+4x-2x^2+x^3+3x^4$, $3x^4+x^3-2x^2+4x-1$ 5 No, see § 90.
6 $5(x+10)=5x+50$ 7 $(3x-24)$ years 8. $8(a+b)$, $a+9b$.
9 $12(x+1)$ rupees

93 Ex (i) [p 72]

6 byz 7 $-8ad$ 8 $-17mx$ 9 $-\frac{5cv}{2y}$ 10. $6y$

11 $-\frac{2bq}{3px}$ 12 $-\frac{c}{a}$ 13 $\frac{2nx}{3az}$

93 Ex (ii). [p 73]

5 $-a^2x^2$ 6. $3ab$ 7 $-4ax^2y$ 8 $-4a^5bcd$ 9 $43xz^2$.

10 $-\frac{3xz}{2a}$. 11 $\frac{3}{5}ax^3y$ 12. $-4xyz^0$ 13 $\frac{2}{3}ab^2m^3$ 14. $\frac{4}{5}a^2x^2y$

94 [p. 73].

4. $a+c$. 5. $-2a+b$. 6. $3a-2b$. 7. $3x^2-x$. 8. $5b-c+d$.
9. x^2-ax+a^2. 10. $-2x^3+x^2-3x$. 11. $\frac{2}{3}x^2+2xy-3y^2$.
12. $8a^2x+a-2xy$. 13. $12a^2+9ab-8b^2$. 14. $-5abx^2+3x-2a^2yz^2$.
15. $3a^2-4ab+5b^2$. 16. $4ay-\frac{3}{2}z+x^2$. 17. $5b^2-4c^2+2cx-1$.
18. $-p^3+3p^2q-pq^2+\frac{2}{3}q^3$. 19. $4x^3y^3-\frac{4}{5}x^2y^2+2xy+1$.

95. [pp. 76—79].

5. $x+3$. 6. $x+3$. 7. $2x+1$. 8. $x+5$. 9. $2x+1$.
10. $x-3y$. 11. $2x+3$. 12. $9x^2+3x+1$. 13. $4a^2+4a+1$.
14. $2x^2-2x+1$. 15. $2x^2-3x+4$. 16. m^2-4m+3. 17. $2x^2-x+1$.
18. x^2-ax+a^2. 19. $2ab-3b^2$. 20. $27-18x+12x^2-8x^3$.
21. x^2-ax+1. 22. x^2+ax-a. 23. $ax^2+bx-ab$. 27. $x+6$.
28. $-2x^2+8x+1$. 29. $3x^2-4x-2$. 30. $-8x^2+5ax+a^2$.
31. $1-a+b$. 32. $x^2+y^2+a^2$. 33. x^2-xy+y^2. 34. $a^2+2ab+2b^2$.
35. $2x^2y^2+2xy+1$. 36. $16x^2-4xy+y^2$. 37. $1-2x+4x^2$.
38. $a^2+3ab+6b^2$. 39. $2x^3-3x^2+2x$. 40. $2x^3+2x^2-5x$.
41. $a^3+2a^2b+3ab^2+4b^3$. 42. $2a^2+3ab+2b^2$. 43. x^2-xy+y^2.
44. $y-2x-a$. 45. $x^4-2x^3y+3x^2y^2-2xy^3+y^4$. 46. $5x^4-4x^3+3x^2-2x+1$. 47. $m^4+2m^3n+3m^2n^2+4mn^3+5n^4$. 48. $x^2-2ax-bx+a^2+ab+b^2$. 49. $a^2-ab+b^2-a-b+1$. 50. $1-x+2y+x^2+2xy+4y^2$. 51. $1-4x^2-y^2-z^2-yz+2zx-2xy$. 52. $-a^2-b^2-9c^2+3bc-3ca-ab$. 53. $7x^2+4x-9$. 54. $3a^3-4a^2x+6ax^2+5x^3$.
55. $x^4+3ax^3+8a^2x^2-8a^4$. 56. $4x^2-5xy+2y^2$. 57. $x^2+y^2+z^2+1$.
58. $a^2+b^2+c^2+d^2$. 59. $a^2-2ab+b^2-c^2$. 60. $x^3+x^2y+xy^2+y^3$.
61. $x^4-x^3y+x^2y^2-xy^3+y^4$. 62. $a+b-c$. 63. $4x^2-2xy^2+y^4$.
64. $x^2+y^2+z^2+xy-xz-2yz$. 65. $9a^4+6a^2x^3+4x^6$.
66. $x^2+(m+1)x+1$. 67. $a^4-a^3b+a^2b^2-ab^3+b^4$.
68. $a^3-3a^2x+x^3$. 69. $y^4-(m-1)y^3-(m-n-1)y^2-(m-1)y+1$.
70. $x^2+a(1-p)x+a^2$. 71. $\frac{1}{3}+2a+3a^2$. 72. $1-\frac{1}{2}x+\frac{2}{3}x^2$.
73. $x^2+(a-2b)x+a^2+3b^2$. 74. $x^2-a^2+(n+1)ax$. 75. $a-b$.
76. $ac-bc+a^2-b^2$. 77. y^2-yz+z^2. 78. y^2-xy+1.
79. $(a+b)(x+y)+1$.

95a [p 80]

4 $a, x^3, 2a^{\frac{1}{4}}, 3a^{\frac{3}{2}}y^{-4}$. 5 $-2a^{\frac{1}{3}}x^{\frac{1}{4}}, \frac{1}{3}a^4b^{-1}, -\frac{3}{4}x^{-\frac{7}{4}}y^{a-1}z^{\frac{5}{2}}$.
6 $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ 7 $x^{\frac{1}{3}}-y^{\frac{1}{3}}$. 8. $x^{-2}-2$. 9. $x^{\frac{3}{2}}-xy^{\frac{1}{2}}$
$+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$ 10 $a^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$ 11. $x^{\frac{3}{2}}-x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}$ 12. $a^{\frac{2}{3}}+b^{\frac{2}{3}}$
$+c^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}-c^{\frac{1}{3}}a^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}$ 13 a^m-c^n 14 $2x^n-y^{-n}$.
15 $x^{-1}+y^{\frac{1}{3}}$. 16. $x^{-3}+x^{-2}y^{\frac{2}{3}}+y$ 17 $x^n+2xy^{-\frac{1}{2}}-y^{\frac{3}{4}}$.

96 [pp 83—84]

8. 3 9. $-y^3$ 10 $5x-3$ 11 $2y^3$ 12 $a^5+a^4+2a^3+3$.
13 $m^2+2mn+1$. 14 $x-pq+1$ 15 $1-\frac{3x}{a}+\frac{6x^2}{a^2}-\frac{12x^3}{a^3}+$.
16 $1+ax+a^2x^2+a^3x^3+\ldots$. 17. $1+x^2+x^3+x^4+\ldots$; n^{th} term
x^n, last 3 terms, $-\frac{1}{x^2}-\frac{1}{x}-x$ 18 $1-(b+c)\frac{x}{a}+c(b+c)\frac{x^2}{a^2}-c^2(b+c)\frac{x^3}{a^3}$
$+\ldots$ 19. -216 20. $x+3$

97a [pp 84—85]

2 $x^2+\frac{4}{5}x-2$ 3 $a-2b+\frac{c}{x}$. 4 $(x-y)-xy$; $\frac{1}{y}-\frac{1}{x}$ 5 No, see § 97 6 $D=dQ$ 7 $x^4+2ax^3+(a^2-1)x^2+2ax-a^2$.
8 $2a^2-3ab+4b^2$ 9 ax^2+3x-1 10 $\frac{1}{2}x$ 11 $\frac{1}{3}(x+y-1)$.

Miscellaneous Examples II [pp 85—87]

1 7 2 $3a^2-2xy+3c^2$ 3 $bx-ay$ 4 $x^3-a(a+1)x+a^2$.
5 p^2-pq+1 7 1 8 0 9 $2bx+2(c-d)y$ 10. $-3y^4$
$-4xy^3+12x^3y^2-16x^3y+16x^4$ 11 $1-3x+2x^2-x^3$ 13 4.
14 $9x^2-y^2$ 15. $3(a+x)-20+6(x+y)$ 16 $x^8-2x^6+2x^5+4x^4$
$-2x+4$ 17 ax^2-bx-a 19 50 20 $27x-15y+34z$
21 $x-3$ 22 $36x^4-24x^3y+4x^2y^2-16y^4$ 23 $3a^2-ax+2x^2$.
25 22 26 $2x^2-2y^2+2z^2-4xz$ 27 $3x-1$ x^2+1 28 $2x-3y$.
29 $a^2-2a-ab+b^2+b+1$ 30 x 31 11 32 $-a^3+6a^2x-4ax^2$
33 $-x(23x-14y)=-23x^2+14xy$ 34 $4m^2+2mn+3n^2$ 36. 9
37 $a^2+6ac+3b^2$, 79 38 x^2+2x+1 39 $x^2+y^2+z^2-yz-zx-xy$
40 0, 0 42 $1\frac{7}{8}$ 43 $-2x^4+3x^3y-3xy^3+2y^4$ 44 a^4+a^3x
$-ax^3-x^4$, $a^3+2a^2x+2ax^2+x^3$ 45 $2x^2+x+3$; $2x^2-7x+9$

48 $x^2+2x-24$ 48. 1. 49 -72 50 $12x^3+12$ 51 $2a-b+3c$ 52 x^2-xy+y^2; $(a+b)^2-(a+b)c+c^2=a^2+2ab+b^2-ac-bc+c^2$.

99 [p. 89]

9. $4x^2+28x+49$ 10 $a^2+4ab+4b^2$. 11 $4x^2+12ax+9a^2$ 12 $a^2x^2+2ax+1$. 13. $9+12xy+4x^2y^2$. 14. $l^4+2l^2mn+m^2n^2$ 15. $4p^2q^2+4pqr^2+r^4$ 16 $a^2b^2+4abc+4c^2$ 17. $4a^2x^2+12abxy+9b^2y^2$ 18 $a^4+2a^2x^2+x^4$. 19. $a^4x^2+2a^3x^3+a^2x^4$ 20. $4a^6+4a^3b^3+b^6$. 21. $x^6+6x^3yz+9y^2z^2$. 22. $x^8+2x^4y^4+y^8$ 23. $a^4+2a^3b+3a^2b^2+2ab^3+b^4$. 24. $9a^2+25b^2+16c^2+30ab+24ac+40bc$ 25 $4x^2+36y^2+z^2+24xy+4xz+12yz$. 26 $a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2$ 27 $9m^2x^2+12mnxy+6mxz^2+4n^2y^2+4nyz^2+z^4$. 28. $x^2+4y^2+9z^2+16u^2+4xy+6xz+8xu+12yz+16yu+24zu$ 29. $16a^2+25b^2+c^2+4d^2+40ab+8ac+16ad+10bc+20bd+4cd$. 30 $6xy+8xz+9y^2+24yz+16z^2$. 31. $16a^2+56ab+49b^2+4ac+7bc$ 32. $x^2y+2x^2y^2+2xy^3+y^4$ 33 $a^2+b^2+6a+4b+13$. 34 $p^2+q^2+2p+2q+2$. 35 16 36 0. 37 0 38. 49. 39. $25a^2$. 40. $4x^2$ 41. $16x^2$. 42 $4x^2+8xz+4z^2$.

100 [p. 91]

9 $9x^2-24x+16$. 10 $x^2-10xy+25y^2$ 11 $64-48ax+9a^2x^2$ 12 $9a^2-24ab+16b^2$. 13. $a^2b^2-2abxy+x^2y^2$. 14 $1-2abc+a^2b^2c^2$ 15. $c^4-4abc^2+4a^2b^2$. 16. $9p^2q^2-14pqr^3+16r^6$. 17. $x^6-2a^3x^3+a^6$ 18 $a^8-2a^4b^4+b^8$. 19 $a^6-6a^5b+9a^4b^2$ 20 $m^4n^4-2m^2n^2l^3+l^6$ 21 $4m^2x^4-12m^3x^3+9m^4x^2$. 22 $a^2x^2+b^2y^2+c^2z^2-2abxy+2acxz-2bcyz$ 23 $a^2b^2+x^2y^2+c^4-2abxy-2abc^2+2c^2xy$ 24. $9x^2+4y^2+9z^2-12xy-18xz+12yz$ 26 $1-4x+10x^2-12x^3+9x^4$ 26 $a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd$ 27. $x^4-4x^3y+2x^2y^2-2x^2+4xy^3-4xy-2y^2+y^4+1$. 28. $9x^6-12ax^5-6a^3x^3+4a^4x^2+a^6$ 29. $20ax^3-37a^2x^2+30a^3x-9a^4$ 30. $2a^2-4b^2-12a+20b-7$ 31 $4p^2-9q^2-4p+6q$ 32 0 33 529. 34. 0. 35. 411 36 x^2. 37 $16y^2$. 38. $(2by-6z)^2=4b^2y^2-24byz+36z^2$

101. [pp. 93—94]

12. 484 13. 1. 14. 15 15 85; 97; 170 16 26. 17 63001. 18 529282; 235389. 19. $6a^2b^2+2(a^4+b^4)$

20 19 21. 37 22 $3a^2b^2+c^2d^2$; $a^2b^2+3c^2d^2$ 28 (1) $(x+5)^2-5^2$, (2) $(x+9)^2-9^2$, (3) $(x+3a)^2-(2a)^2$, (4) $(x+a)^2-(2a)^2$, (5) $(x^2+5x+7)^2-(2x+5)^2$, or $(x^2+5x+\frac{11}{2})^2-(x+\frac{5}{2})^2$, or $(x^2+5x+5)^2-1$
31 $(a+b)^4+c^4$

102 [pp 95—96]

7 x^2-25 8 $4x^2-49$ 9 $9x^2-16y^2$ 10. a^2x^2-9
11. $4a^2x^2-9b^2y^2$ 12 x^4-4y^2 13 a^4-x^4 14 $4a^2b^2-c^4$
15 $4a^6-9b^6$ 16 $a^2-(b-c)^2=$&c 17 $(a-c)^2-b^2=$&c
18 $(a+c)^2-b^2=$&c 19 $x^2-(2y+3z)^2=$&c 20 $(x-2y)^2-(3z)^2=$&c 21 $x^2-4y^2-9z^2+12yz$ 22 $a^4-2a^2b^2+b^4$ 23 $x^4+2abx^2+a^2b^2-b^4$ 24 a^4-x^4 25 x^8-y^8 26 x^8+x^4+1.
27 $a^2+b^2-c^2-d^2-2ab+2cd$ 28 $x^2-y^2-z^2+2yz+2x+1$
29 (1) $(1+x)(1-x)$ (2) $(m+4)(m-4)$ (3) $(8+q)(8-q)$
(4) $(1+9z)(1-9z)$ (5) $(5y+1)(5y-1)$ (6) $(4a+3b)(4a-3b)$
(7) $(5ax+7b)(5ax-7b)$ (8) $(12x+13a)(12x-13a)$
(9) $(9q+8r)(9q-8r)$ (10) $(25ax^2+11)(25ax^2-11)$
(11) $(9p+10q)(9p-10q)$ (12) $(12x+11y)(12x-11y)$.
(13) $(7ac+9d^2)(7ac-9d^2)$ (14) $(lm+nq)(lm-nq)$
(15) $(3z^2+4xy)(3z^2-4xy)$ (16) $(5az+2cy)(5az-2cy)$
(17) $(a-x)(a+x)(a^2+x^2)$ (18) $(4x^2+5a^3)(4x^2-5a^3)$
31 $4a(6c-2b)=24ac-8ab$ 32. $2x^2(2xy-4y^2)=$&c
33 $8x(4z-3y)=$&c 34 a^2-d^2 35 $4a^2$
36 $4(b^2+2bc+c^2)$ 40 $4(a^4-a^3b+ab^3-b^4)$ 41. $a^2b^2+3c^2d^2$.

103 [pp 97—98]

4. $(2x+3y)^3=8x^3+36x^2y+54xy^2+27y^3$ 5 $(5m+2n)^3=125m^3+150m^2n+60mn^2+8n^3$ 6 $27a^3+54a^2b+36ab^2+8b^3$ 7 $1+6x+12x^2+8x^3$ 8 $1+3x^2+3x^4+x^6$. 9 $(ax+by)^3+3(ax+by)^2+3(ax+by)+1$; [see Ex 3, also Ex 3, § 99] 10 $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$
11. $1+6x+21x^2+44x^3+63x^4+54x^5+27x^6$ 12 $8x^3+y^3+27z^3+12x^2y+6xy^2+36x^2z+54xz^2+9y^2z+27yz^2+36xyz$ 13 $b^3c^3+c^3a^3+a^3b^3+3b^2c^3a+3bc^3a^2+3b^3c^2a+3b^3ca^2+3c^2a^3b+3ca^3b^3+6a^2b^2c^2$ 14 $a^6+3a^5b+6a^4b^2+7a^3b^3+6a^2b^4+3ab^5+b^6$ 16 $(a+1)^3$ 17 $(1+y)^3$
18 $(2x+y)^3$ 19 $(3x+2a)^3$ 21 $(2x)^3=8x^3$ 22 $(3a)^3=27a^3$
23. $64x^3$ 24 $a^3+3a^2b+3ab^2+b^3$ 25 $27(x+1)^3=27(x^3+3x^2+3x+1)$ 26 $8a^3$ 27 $64x^3$ 29 -125 30 0

31. −144 32. 43 36. −1 37. 64 40. 2 ; 40.
41. −296 42. 3

104 [*pp* 99—100].

4 $(x-2)^3=x^3-6x^2+12x-8$ 5 $125x^3-150x^2y+60xy^2-8y^3$.
6. x^3-3x^2+3x-1 7 $8x^3-36x^2+54x-27$ 8 $1-12x+48x^2-64x^3$.
9 $8x^3-36x^2y+54xy^2-27y^3$. 10 $8-12x^2+6x^4-x^6$. 11 $a^3x^3-3a^2x^2y^2+3axy^4-y^6$. 12 $x^3-y^3+z^3-3x^2y+3xy^2+3x^2z+3xz^2+3y^2z-3yz^2-6xyz$ 13 $(2x-3y)^3+3(2x-3y)^2+3(2x-3y)+1=$&c, [see Ex 9] 14 $1-3x+7x^3-3x^5-x^6$ 15 $(1-x)^3$. 16 $(2x-a)^3$
17. $(2x-1-2x-3)^3=(-4)^3=-64$ 18 $(a+b-a+b)^3=(2b)^3=8b^3$
19 $8a^3$ 20 $(b-a)^3=$&c 22 0 23 −740 24 269
28 c^3 31 26 32 100 33 −9.

105 [*p*. 101]

4. x^3+y^3 5 a^3+8x^3. 6 $27x^3+1$. 7 $1+64x^3$ 8. x^6+1
9 $64a^6+125x^3$ 10 $125x^3+512y^3$. 11. x^3+27. 12 $8a^3+1$
13 $27m^3+8$ 14 $8a^3+b^3$ 15 $x+y$ 17 $(a+4)(a^2-4a+16)$.
18 $(3x+a)(9x^2-3ax+a^2)$ 19 $(2a+1)(4a^2-2a+1)$.
20 $(1+3k)(1-3k+9k^2)$ 21 $(5a+3x)(25a^2-15ax+9x^2)$.
22 $(x+4y)(x^2-4xy+16y^2)$. 23 $(4a+5b)(16a^2-20ab+25b^2)$.
24. $(7x+2)(49x^2-14x+4)$ 25. $(x^2+y)(x^4-x^2y+y^2)$

106. [*p* 102]

4 x^3-y^3 5 a^3-8x^3 6 $8a^3-27b^3$ 7 $1-8x^6$ 8 a^3-1
9 x^6-1 10. $8x^3-27$ 11 $27x^6-8y^3z^3$ 12. x^3-8.
13 x^3-64 14 $8m^3-729$ 15 $27a^3-125b^3$ 16. x^3-8y^3.
17 $2a^3+2ax^2-x^3$ 19 $(x-2y)(x^2+2xy+4y^2)$.
20 $(3a-b)(9a^2+3ab+b^2)$ 21. $(4k-1)(16k^2+4k+1)$.
22. $(8x-3)(64x^2+24x+9)$ 23. $2(3m-1)(9m^2+3m+1)$
24 $(ax-4y)(a^2x^2+4axy+16y^2)$ 25 $(2ac^2-b)(4a^2c^4+2abc^2+b^2)$

107. [*p* 103]

4. $x^2+10x+21$ 5. $x^2+19x+88$ 6 $x^2+8ax+15a^2$ 7. $9x^2+18x+8$. 8 $25x^2+55x+10$ 9 $m^2x^2+7mx+12$ 10. $4x^2+2x(a+b)+ab$ 11 $x^4+(c+d)x^2+cd$. 12 $4x^6+24x^3+27$. 13. $(x+a)^2+3(x+a)+2=$&c 14. $(x^2+2x)^2+7(x^2+2x)+12=$&c 15 $(x^2-xy)^2+11(x^2-xy)+28=$&c. 16. $(2a^2-ab)^2+11(2a^2-ab)+24=$&c.

108 [p 104]

4. x^2-x-20 5 $m^2-3m-54$ 6 $y^2-5y-24$ 7 $4m^2+4m-3$ 8 $49a^2+7a-12$ 9. $a^2m^2-6am-55$ 10 $16x^2+4x(a-b)-ab$ 11 $x^4+(a^2-b^2)x^2-a^2b^2$ 12 $9p^6-9p^3-4$. 13 $(x+y)^2-5(x+y)-6=$&c 14 $(ax+b)^2-4(ax+b)-21=$&c 15 $(2x-3y)^2-2(2x-3y)-24=$&c. 16. $(3x^2-y^2)^2-(3x^2-y^2)-72=$&c. 18 $a^2+2a-15$ 19 $q^2+3q-40$ 20 $x^2+3ax-28a^2$. 21 $25x^2+5x-6$ 22 $36x^2-24x-77$ 23 $64l^2-8l-42$. 24 $x^4-2x^2yz-15y^2z^2$ 25 $(3x+2y)^2-(3x+2y)-28=$&c 26 $(4x-5y)^2-3(4x-5y)-54=$&c 27 $(ax+yz)^2+2(dx+yz)-3=$&c

109 [pp 104—105]

4. x^2-5x+6 5 $z^2-15z+50$ 6 $y^2-11ay+28a^2$ 7 $4a^2-20a+21$ 8 $25x^2-85x+66$ 9 $m^2x^2-18mx+80$ 10. $4x^2-2(a+b)x+ab$ 11. $x^4-(c+d)x^2+cd$ 12 $4x^6-24x^3+27$ 13 $(x+2y)^2-4(x+2y)+3=$&c 14 $(2a-b)^2-11(2a-b)+28=$&c 15 $(ax+by)^2-12(ax+by)+32=$&c. 16 $(3x^2-4x)^2-14(3x^2-4x)+48=$&c 17 $(x^2+4z^2)^2-5(x^2+4z^2)3y^2+6y^4=$&c 18 $(a^2+3ab)^2-(a^2+3ab)(c^2+d^2)+c^2d^2=$&c

110. [p 105]

3 $15x^2+22x+8$ 4. $6x^2+13x+6$ 5 $32x^2+28x-15$ 6 $15x^2+2x-24$ 7 $6x^2-25x+24$ 8 $18x^2+43x-5$ 9 $12a^2-25a+12$ 10 $35x^2+33x-54$ 11 $16x^2-134x+105$

111 [p 105]

2 $m^3+6m^2+11m+6$. 3 $x^3+15x^2+66x+80$
4 $x^3+14x^2+55x+42$ 5 $x^3+13x^2+39x+27$
6 $x^3+6ax^2+11a^2x+6a^3$ 7 $x^3+9ax^2+23a^2x+15a^3$
8 $125x^3+225x^2+130x+24$. 9 $a^3x^3+23a^2x^2+159ax+297$

112 [p 106]

2 $x^3-6x^2+11x-6$ 3 $x^3-16x^2+73x-90$ 4 $x^3-29x^2+278x-880$ 5 $x^3-21x^2+126x-216$ 6 $x^3-9ax^2+23a^2x-15a^3$ 7 $x^3-14ax^2+56a^2x-64a^3$ 8 $27x^3-135x^2+162x-40$ 9 $a^3x^3-12a^2x^2+31ax-21$ 12 $x^3+6x^2-9x-14$ 13 x^3+4x^2+x-6 14 $x^3-7x^2-14x+120$ 15 $x^3-6x^2-79x+504$ 16 $l^3+2l^2-11l-12$

17. $x^3+13x^2+34x-48$. 18. $y^3-4y^2-17y+60$. 19. $m^3-16m^2+53m+70$. 20. $p^3+2p^2-91p+88$. 21. $z^3-2z^2-11z+12$. 22. $h^3-h^2-30h+72$. 23. $x^3-7a^2x-6a^3$. 24. $y^3-4my^2+m^2y+6m^3$. 25. $x^3-8ax^2+4a^2x+48a^3$. 26. $h^3-5nh^2-2n^2h+24n^3$.

114. [p. 103]

2. $a^3+b^3-c^3+3abc$. 3. $8x^3-y^3+z^3+6xyz$. 4. $27x^3-8y^3-64z^3-72xyz$. 5. $1-x^3+8y^3+6xy$. 6. $-x^3-8y^3-18xy+27$. 7. $27a^3-64b^3-72ab-8$. 8. $x^3+y^3+3xy-1$.

115. [p. 107]

5. 1. 6. 484. 7. 256. 8. 112. 9. 194. 10. 1171.
11. 65. 12. $(a+b+c)^2=$ &c. 13. $(a+b)^2=$ &c. 14. $\frac{3}{2}$.
15. 5. 16. 5. 17. $(x-y-z)^2$. 18. $(1+a-b)^2$.
19. $(x^2-ax+1)^2$. 20. $(x^2+ax-a^2)^2$.
23. $x(y-z)+y(z-x)+z(x-y)=0$, see Ex. 22.

116. [p. 110]

1—10. 0.

117. [p. 112]

5. $(b-c)(c-a)(a-b)$. 6. 0. 7. $(y-z)(z-x)(x-y)$.
8. $-(y-z)(z-x)(x-y)$. 9. $-(b-c)(c-a)(a-b)x^2$.
10. $-(b-c)(c-a)(a-b)$. 11. $(b-c)(c-a)(a-b)$.
12. $-(b-c)(c-a)(a-b)x$. 13. $-4(b-c)(c-a)(a-b)$.

119. [pp. 114—117].

1. x^2. 2. $4a^2-16a+16$. 3. $8a^4$. 4. x^2. 5. $4c^2$.
6. $x^2+y^2+z^2-yz-zx-xy$. 7. $125x^3$. 16. $(a^2-4)^2=$ &c.
17. $a^2(a+2b)^2=$ &c. 18. $(a+b)^3+1=$ &c. 19. $(x+1)^3+27y^3=$ &c.
20. $(2a-b)^3-(a+b)^3=7a^3-15a^2b+3ab^2-2b^3$.
21. $(a-b)^3+(c-d)^3=$ &c. 22. $6x$. 23. $a^3+a^2c+abc+bc^2$.
24. 16. 25. -12. 26. $a^2+2ab+b^2$. 27. $x^4-2a^3x-2ax^3-a^4$.
28. $(a^2-b^2)x^2-4abxy-(a^2-b^2)y^2$. 29. $8y^2$. 30. $4bc$.
31. $4x^2y+2y^3$. 44. $a^3+b^3+c^3-3abc$. 45. $a^3+b^3+c^3-3abc$.
46. $3abc-a^3-b^3-c^3$. 53. $4abc^2$. 54. $-2b^2c$. 66. $\dfrac{m^2-n^2}{3m}$.

67 $\frac{b^3-a^3}{3a}$ 70 $2a^3+6ab^2$ 71 $\sqrt{(p^2+4qr)}$, p^3+3pqr
72 $\sqrt{(a^4-4b^2)}$, a^3-3ab^2 73 2540 74. -2044.
75 5 76 3

119*a*. [pp 118—119]

5 $x^3+(a+b-c)x^2+(ab-ac-bc)x-abc$ 6 ax^3+bx^2-bx-a
7 $a^3+2a^2c-a(b^2-bc-c^2)-bc(b-c)$ 8 $x^3-2bx^2-(a^2-ab-b^2)x+ab(a-b)$ 9 $(a^3+1)x^3+(a-2)x^2+2x-1$ 10 $(x^3-1)a^3-(x^3+x^2-2)a^2+(4x^2+3x+2)a-3(x+1)$ 11 $a^3-b^3+c^3+3abc$
12 $x^3-3x^2+3x+y^3-1$ 13 $(m+1)a^5-(m^2-2)a^4+(m^3+2)a^3+(m^2-2m-1)a^2-(m^2+m+2)a+2m$ 14 $(a^3-b^3)x^4+(a^2+2ab+3b^2)x^3-(a^2b-ab^2+2a+2b+a^2+ab+b^2)x^2+(a+b-2ab)x+ab$.
15 $2(a-b)x^5-(a^2-2ab-b^2)x^4+2(a+b-ab^2)x^3-(2a^2+2b^2-a^2b^2)x^2+2ab^2x-(a^2-b^2)$ 16 $a^3(b^2-c^2)-a^2(b^3-c^3)+b^2c^2(b-c)$

119*b* [p 120]

5 $a^2-ab+b^2-2a+b+1$ 6 $a^2+b^2+c^2+bc+ca+ab$
7 $x-a-b$ 8 $a^2+b^2+c^2$ 9 $x-c$ 10 $a+b+c$
11 $x^2+x+(a-b)$ 12 *Sym of ref* is a, $a^2(b-c)-a(b^2-c^2)+bc(b-c)$ 13 *Sym of ref* is a, a^2-xa+z 14. $x-a-b$
16 $(a^2+a+1)x^2+(a+1)x+1$ 17 $a^2+b^2+c^2-bc-ca-ab$
21. *Sym of ref* is a, $(x-1)a+x^2$ 22 x^2-xy+y^2
23. *Sym. of ref* is a, $(2x-y)a^2+(x+y)xa-x^3$

121. [pp 122—123]

4. $a(b+x)$ 5 $m(1+x)$ 6 $3r(r^2-5s)$ 7 $ab(a+b)$
8 $b(a-c+ac)$ 9 $2a(4a-3b-2c)$ 10 $5p(4q-3pr+11r)$.
11. $3ab(a^2-b^3)$. 12 $ab(2ab+3)$ 13 $p(p^2-3q^2)$.
14 $2x^2(4-5mx^3)$ 15 $6m^5(3m^3-n)$ 16 $xy(3x+2y+1)$
17. $4ab(a^2-4a+5b)$ 18 $xy(2x^2y^2-3rx+2ry)$ 19 $3x^2y(2x-3y+4xy)$ 20 $9x^4y^4(9x^2+7y^4)$ 21. $6m^2x^3(4m^2x^3z^2-7y^6)$.
22 $7a^2xy^2(a+2x-3y)$ 25 $(a+c)(b+d)$ 26 $(m+n)(m+1)$
27 $(a+1)(x+1)$ 28 $(2a-b)(x-2y)$ 29 $(1-x)(1-y)$
30 $(a-b)(a-c)$ 31 $(x-m)(x+y)$ 32 $(a+1)(a^2+1)$
33 $(1-x)(1+x^2)$. 34 $(f+h)(fg-ch)$. 35 $(a+bc)(b+ac)$

36. $(ax-b)(cx-d)$ 37. $(a-cq)(bq-a)$. 38. $(yz-x^2)(zx-y^2)$.
39. $(a+b)(x-y+z)$. 40. $5m(p-q)(p+2q-5m)$ 42. $x(y+b)(z+c)$. 43. $a(a+3)(b-4)$. 44. $m(my-x)(mx-y)$
45. $8mn(m-2n)(n-3p)$ 46. $2a(aq-3bf)(2ah-cf)$.
48. $(x-a)(x-b)$. 49. $(x-1)(x+ma)$ 50. $(b+c)(a-bc)$
51. $(a+bc)(b-ac)$ 52. $(x+c)(ax-b)$ 53. $(a-bq)(a-cq)$.

122. [pp. 123—124].

4. $(1+a)^2=(1+a)(1+a)$ 5. $(3x-1)^2$. 6. $a(3a+5x)^2$.
7. $(4a^2-b)^2$ 8. $(6xy-1)^2$ 9. $4a^2(3ax-2)^2$ 10. $(5ax+2by^2)^2$.
11. $3xy(x-3y)^2$ 12. $(2a+7)^2$ 13. $(p^2-10)^2$ 14. $(13m+2)^2$.
19. $(2x+y-z)^2$ 20. $(a+x-y)^2$ 21. $(2m+2q+1)^2$.
22. $(2ax+3bx-y)^2$. 23. $(1-x)(1-x-a)$ 24. $(ax-y)(ax-y-1)$.
25. $ax(1+y)(3x+3y-1)$. 26. $(4a^2-b)(4a^2x^2-bx^2-1)$
27. $(a+b)(a+b+2c)$ 28. $(2a)^2=4a^2$. 29. $(p+q-r+s)^2$.
30. $(1+a)^2(1+x)^2$. 31. $(x+y)^2$

123 [p. 125]

5. $(2a^2+b^2)(2a^2-b^2)$. 6. $(6a^3+x^2)(6a^3-x^2)$.
7. $(5a^4+3b^5)(5a^4-3b^5)$. 8. $(m^2+4)(m+2)(m-2)$
9. $(4a^2+1)(2a+1)(2a-1)$ 10. $(1+x^4)(1+x^2)(1+x)(1-x)$
11. $(a^4+x^4)(a^2+x^2)(a+x)(a-x)$ 12. $(x^8+a^8)(x^4+a^4)(x^2+a^2)(x+a)(x-a)$ 13. $x(x+2y)(x-2y)$ 14. $2a(5a+1)(5a-1)$.
15. $3(1-4ax)(1+4ax)$ 16. $2(4mn+1)(4mn-1)$ 17. $3a(a+x)(a-x)$. 18. $5ax(2a+b)(2a-b)$ 19. $2x(5x+4y)(5x-4y)$.
20. $3x(1-2x)(1+2x)$ 21. $2a^2x^2(5ab+2xy)(5ab-2xy)$.
22. $x(9x^2+4y^2)(3x+2y)(3x-2y)$ 23. $3x(x+\frac{1}{3}y)(x-\frac{1}{3}y)$.
24. $2a(a^2+\frac{1}{4}x^2)(a+\frac{1}{2}x)(a-\frac{1}{2}x)$. 25. $2a^3b^2(9a+4b^2)(9a-4b^2)$
26. 60800. 27. 758000 28. 81933 29. 19455
30. 3393600 31. 4096 32. 7225 33. 21025.
34. 37249. 35. 173056. 36. 978121. 37. 99980001
38. $(x-y+z)(x-y-z)$ 39. $(m+2n+q)(m-2n-q)$
40. $(3y-4b+6x)(3y+4b-6x)$. 41. $(6x-8y+1)(6x-8y-1)$
42. $(2a+3b-9c)(2a-3b+9c)$ 43. $(4+15z+18u)(4-15z-18u)$.
44. $(ax+by+1)(ax+by-1)$. 45. $(15a+5b-2m)(15a+5b+2m)$
46. $(5p+6q-2)(5p-6q+2)$. 47. $(5x-2y+3z)(5x-2y-3z)$

48 $(2lx-2my+9xy)(2lx-2my-9xy)$ 49 $25(r-s+t)(r-s-t)$
50. $(a+b+c+d)(a+b-c-d)$ 51. $(3x-2y+2z+1)(3x-2y-2z-1)$.
52 $96x$ 53 $-60ab$ 54 $13(a-b)(a+b)$ 55 $4(2ax-by)(ax+2by)$
56 $ab^2(a-b)$. 57 $-4ax(a^2+x^2)$ 58 $(3a-2b-c)(a+4b-5c)$
59. $-4(2a+c)(a+2b+5c)$ 60 $(x+y)^3(x-y)$
61. $(1+x)(1-x)(1+y)(1-y)$ 62 $y(y+x)(y^2-xy+2x^2)$
63 $(m+1)(m-1)(x+y)(x-y)$ 64 $(p+q)(p-q)(r+s)(r-s)$
65. $8ax(a^2+x^2)$ 66 $(a^2+b^2)(x^2+y^2)\{a(x-y)+b(x+y)\}\{a(x+y)-b(x-y)\}$. 67 $(x+y+z)(x+y-z)(x^2-2xy-y^2-z^2)$
69 $(a+b+c)(a-b+c)(a+b-c)(a-b-c)$
70, 71 $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$
72. $(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)$

124 [pp 126—128]

3 $(2a-b+c)(2a-b-c)$ 4 $(1-y+z)(1-y-z)$
5. $(1+ax+x)(1+ax-x)$ 6 $(1+a+b)(1-a-b)$
7 $(x-y+z)(x+y+z)$ 8 $(2p-q+2)(2p-q-2)$
9 $(a+y-z)(a-y+z)$ 10 $(ay+ax-1)(ay-ax+1)$.
11. $(d-3c+2a)(d-3c-2a)$ 12 $(3x+5y+8)(3x-5y-8)$
13 $(x^2+pxy-y^2)(x^2-pxy-y^2)$. 14 $(ax^2+cxy+by^2)(ax^2-cxy+by^2)$
15 $(x+y+3)(x-y-1)$ 16 $\{(a+b)x-(a-b)y\}\{(a-b)x+(a+b)y\}$.
17 $(a+2x+y+3)(a+2x-y-3)$. 18 $(x+y+z+1)(x+y-z-1)$
19 $(a+b+x-y)(a+b-x+y)$ 20 $(a-b+c-d)(a-b-c+d)$.
21 $(2a+b+3c+d)(2a-b-3c+d)$ 22 $x(x+y-z)(x-y+z)$
26. $(x^2+2x+2)(x^2-2x+2)$ 27 $(1+2x+2x^2)(1-2x+2x^2)$
28 $(2a^2+2ab+b^2)(2a^2-2ab+b^2)$ 29 $(2x^2+6x+9)(2x^2-6x+9)$
30 $4(x^2+2ax+2a^2)(x^2-2ax+2a^2)$ 31 $(9x^2+12xy+8y^2)(9x^2-12xy+8y^2)$ 32 $(x^2+6ax+18a^2)(x^2-6ax+18a^2)$ 33 $(x^2+x+1)(x^2-x+1)$ 34 $(x^4-x^2+1)(x^2+x+1)(x^2-x+1)$. 35 $(a^4-a^2x^2+x^4)(a^2+ax+x^2)(a^2-ax+x^2)$ 36 $(x^2+2x+3)(x^2-2x+3)$.
37. $(x^2+3x-3)(x^2-3x-3)$ 38 $(x^2+4x-5)(x^2-4x-5)$
39 $(x^2+ax-a^2)(x^2-ax-a^2)$ 40 $(x^2+ax-4x^2)(x^2-ax-4x^2)$.
41 $(x^2+xy+3y^2)(x^2-xy+3y^2)$ 42 $(2a^2+2ab+3b^2)(2a^2-2ab+3b^2)$
43 $(2a^2+ab+3b^2)(2a^2-ab+3b^2)$ 44 $(2a^2+ab-3b^2)(2a^2-ab-3b^2)$
45. $(3x^2-2ax-5a^2)(3x^2+2ax-5a^2)$ 46. $(3x^2+3ax+5a^2)(3x^2-3ax+5a^2)$ 47 $(13a^2+3a-1)(13a^2-3a-1)$

48. $(a^2+4a-1)(a^2-4a-1)$. 49. $(8a^2+7ab+b^2)(8a^2-7ab+b^2)$.
50. $(4x^2+8xy+9y^2)(4x^2-8xy+9y^2)$.
51. $(a^2+2x^2)(a^2-2x^2)(a^2+2ax+2x^2)(a^2-2ax+2x^2)$
52. $(5x^2-2x+1)(x^2-2x+5)$. 53 $(3x^2+a^2)(x^2+3a^2)$.
55 $(x-y)(x+y-3z)$ 56. $(x-y)(x+y+a+b)$ 57 $(x+y)(x-y)^3$.
58 $(b-d)(a-b+c-d)$. 59 $(1-ax)(1+ax+bx^2)$.
60 $(ax^2-y)(ax^2+y-z^2)$ 61 $(a+3b)(a+b-3)$.
62. $(p+qr)(p+qr-m)$ 63. $(x-y-z)(x-y+z+2)$.
64. $(x+y-z)(x-y+z+1)$. 65. $(a-b-c)(a+b+c+1)$
66. $(2b+3a-3x)(2b-3a+3x-1)$.

124*a*. [*p*. 128].

3. $(a+b)(a-b)^2$. 4 $(x+y)^3(x-y)^3$. 5. $(x+2)^3(x-2)^3$
6. $(1-x)^3(1+x)(1-4x+x^2)$ 7. $(x-a)^3$. 8. $(x+1)^3$.

124*b*. [*p*. 130].

8 $(x+\frac{1}{2}y)(x^2-\frac{1}{2}xy+\frac{1}{4}y^2)$. 9. $(a-6x^2)(\frac{1}{4}a^2+\frac{3}{2}ax^2+9x^4)$.
10 $(ax+by+1)\{(ax+by)^2-(ax+by)+1\}=$&c
11. $\{x-2(y+z)\}\{x^2+2x(y+z)+4(y+z)^2\}=$&c.
12. $\{4(a+b)+y\}\{16(a+b)^2-12(a+b)c+9c^2\}=$&c.
13. $\{(a+b)x-2cy\}\{(a+b)^2x^2+2(a+b)cyx+4c^2y^2\}=$&c.
14. $(x-y)(x^2-5xy+7y^2)$ 15 $(a^2+bc)(a^4-4a^2bc+7b^2c^2)$.
16 $2(x+a)(x-a)(x^4-2a^2x^2+4a^4)$. 17 $-(x-1)^2(x^4+2x^3+6x^2+2x+1)$ 18 $(x^2+2ax+2a^2)(x^2-2ax+2a^2)(x^8-a^4x^4+a^8)$.
19. $xy(x+2y)(x^2-2xy+4y^2)$ 20. $2a^2(a+2x)(a^2-2ax+4x^2)$. $(a-2x)(a^2+2ax+4x^2)$ 21 $(a+b)(a^2-ab+b^2)(a^6-a^3b^3+b^6)$.
22. $2(x+z)(x^2+2xz+z^2+3y^2)$ 23. $(x-y-z)(x^2-2xy+xz+y^2-yz+z^2)$ 24. $(x-y+1)(x^2+xy+y^2-x-2y+1)$ 25. $(a+2b)(a^2+ab+b^2)$ 26 $(x-2a)(2x^2-5ax+17a^2)$ 27 $(2x-y)(4x^2+2xy+y^2+4)$. 28 $(a-b)(a^2-ab+b^2)$. 29. $(x-y)(2x^2-xy+2y^2)$
30 $(a+b-1)(a^2+2ab+b^2+a+b+2)$ 31. $2(x+2y)(x^2+xy+y^2)$.
32 $-3(a+b)x\{(a+b)x+1\}$. 33. $(a+b+c)(a^2+b^2+2bc+c^2)$.

125. [*pp*. 132—133].

11 $(x+3)(x+4)$ 12 $(a+2)(a+5)$ 13. $(x+3)(x+7)$.
14. $(p+4)(p+20)$. 15. $(z+14)(z+3)$ 16. $(x+4)(x+5)$.

17	$(a-1)(a-3)$.	18	$(a-4)(a-5)$	19.	$(x-2)(x-4)$
20	$(a-4)(a-3)$	21	$(x-2)(x-3)$	22	$(x-17)(x-10)$
23	$(x-4)(x+8)$	24.	$(a+3)(a-7)$	25	$(a+9)(a-8)$.
26.	$(x+4)(x-8)$	27.	$(x-4)(x+10)$.	28	$(x-2)(x+21)$
29	$(x-6)(x+7)$	30	$(x-9)(x+6)$	31	$(x+3)(x-18)$
32	$(m-8)(m+12)$	33	$(m+3)(m-32)$	34	$(m+4)(m-24)$
35	$(a-10)(a+12)$	36	$(a+5)(a-24)$	37.	$(p-9)(p+16)$
38	$(x-4)(x+12)$	39	$(x-6)(x+8)$	40	$(x+3)(x-16)$
41.	$(k-1)(k+20)$	42	$(l+3)(l-26)$	43	$(l+6)(l-13)$
44	$(x-12)(x+13)$.	45	$(x+6)(x-26)$	46	$(ab+3)(ab+12)$.
47	$(ab-2)(ab-18)$	48	$(xy+3)(xy-6)$	49.	$(pq-2)(pq+16)$
50	$(mn+2)(mn-8)$	51.	$(ax-4)(ax+9)$.	52	$(a+6b)(a-9b)$.
53	$(x+5y)(x-20y)$	54	$(x-4y)(x+6y)$	55	$(x+2y)(x-12y)$
56	$(a-6b)(a+10b)$	57	$(l+3m)(l+15m)$	58	$(a-3b)(a+20b)$
59	$(m+16n)(m-8n)$	60	$(m+4n)(m-32n)$	61	$(p-19q)(p+20q)$
62	$(6+5x)(6-x)$	63	$(8+13a)(8-15a)$	64	$(1-4m)(1+6m)$
65	$(k+20)(k-21)$	66	$(x+26)(x-25)$	67	$(q+25)(q+80)$
68	$(1-3x)(1+6x)$	69	$(1+2x)(1-9x)$	70	$(3-4x)(3+8x)$
71.	$(3+2x)(3-16x)$	72.	$(3-2x)(3-14x)$	73	$(5-12m)(5-6m)$
74	$(5-2x)(5+8x)$	75	$(5+x)(5-16x)$	76	$(m-20)(m+15)$
77	$(m+5n)(m-10n)$	78.	$(x-8a)(x+10a)$	79	$(1-xy)(1+4xy)$
80	$(1+30xy)(1+25xy)$			81	$(k+16)(k-12)$
82	$(x+4y)(x-10y)$	83	$(a-18)(a+27)$	84.	$(x+18)(x-20)$
85	$(x-14)(x+40)$.	86	$(x-8)(16-x)$	87	$(x+6)(12-x)$
88	$(x+20)(4-x)$	89	$(x+3\frac{1}{4})(x+\frac{1}{6})$	90	$(x-3)(x-\frac{1}{3})$
91	$(x+2)(x-\frac{5}{3})$	92	$(a+2)(a+x)$	93	$(x+y)(x+ay)$
94	$(y+x)(y+x-1)$	95	$(x-a+2)(x-a-3)$		
96	$(x-y+1)(x-y+3)$	97	$(x+2y-1)(x-y+2)$		
98.	$(x+a^2-2ab+b^2)(x-a^2-2ab-b^2)$				

126 [*pp* 135—136]

5	$(x+2)(4x+3)$	6	$(a+4)(5a-1)$	7.	$(2x+3)(3x-4)$
8	$(2x-9)(3x-4)$	9	$(x-5)(3x+5)$	10	$(x+8)(4x-9)$.
11	$(2y-5)(4y+7)$.	12	$(3x+8)(5x-12)$	13	$(4x+1)(3x-5)$.
14	$(2x-1)(5x+4)$	15	$(3m+7)(3m-4)$	16	$(x+8)(8x-3)$.

17. $(5x+3)(3x-7)$ 18. $(4x-3)(2x+5)$. 19. $(3x-8)(4x+5)$
20. $(2x-1)(4x+7)$ 21. $(3x-2)(2x+5)$. 22. $(4x+3y)(x-4y)$.
23. $(2+3x)(5-9x)$. 24. $(3x+2)(4x-3)$ 25. $(2x-5)(6x-1)$
26. $(4x+7)(x-3)$. 27. $(6p-5q)(4p-7q)$ 28. $(1-5x)(3+2x)$
29. $(9xy-2)(xy+6)$. 30. $(x+4a)(8x-3a)$ 31. $(3+2x)(2-5x)$
32. $(5a-3b)(2a-5b)$ 33. $(3x+4y)(4x+5y)$ 34. $(2a-5b)(6a+b)$.
35. $(3m+5n)(6m-7n)$ 36. $(4x-3a)(3x-8a)$
37. $(3x-4)(14x+5)$ 38. $(7y+15z)(8y-3z)$
39. $(8x+9y)(3x-8y)$ 42. $(2x+3)(9x+2)$.
43. $(3x+5)(5x+3)$. 44. $(3x+4)(4x+3)$ 45. $(3x-8)(8x-3)$.
46. $(x-12)(12x-1)$ 47. $(5x-8)(8x-5)$. 48. $(x+8)(8x-1)$
49. $(3m-4)(4m+3)$. 50. $(5x+3)(3x-5)$. 51. $3(4x+1)(x-4)$.
52. $(x+16)(16x-1)$ 53. $(x+11)(11x-1)$ 54. $(x+9)(9x-1)$.
55. $(x-6)(6x+1)$ 56. $(5l-9)(9l+5)$ 57. $(m-8n)(8m+n)$.
58. $3(3x+4)(4x-3)$ 59. $(3x+2y)(2x-3y)$. 60. $7(x-2y)(2x-y)$.
61. $(x-7y)(7x+y)$ 62. $(5x-8y)(8x+5y)$.

127. [p. 138]

1. $(x+6)^2-6^2$ 2. $(x+\frac{5}{2})^2-(\frac{5}{2})^2$ 3. $(x-10)^2-(10)^2$. 4. $(x-\frac{3}{2})^2-(\frac{3}{2})^2$. 5. $(x-9)^2-9^2$ 6. $(x+\frac{13}{2})^2-(\frac{13}{2})^2$. 7. $(x+2a)^2-(2a)^2$. 8. $(x-a)^2-a^2$ 9. $(x+15)^2-5^2$ 10. $(x-\frac{15}{2})^2-(\frac{3}{2})^2$ 11. $(x-10)^2-(14)^2$. 12. $(x-\frac{1}{2})^2-(\frac{7}{2})^2$ 13. $\left(x+\frac{m}{2}\right)^2-\left(\frac{5m}{2}\right)^2$. 14. $(x+p-q)^2-q^2$. 15. $(x-2y)^2-(3y)^2$ 16. $(x+q-\frac{1}{2})^2-(\frac{1}{2})^2$.
17. $(x^2+3x+1)^2-1$. 18. $(x^2+ax-a^2)^2-a^4$

128. [pp. 140—141]

7. $(x+12)(x+16)$ 8. $(l-6)(l-15)$. 9. $(a+18)(a-9)$
10. $(x-8)(x-10)$ 11. $(a+19)(a-20)$ 12. $(3x+4)(x-8)$.
13. $(3a-2)(4a-5)$ 14. $(5-x)(6-5x)$ 15. $(4+7x)(3-x)$
16. $(5x+3y)(2x-5y)$. 17. $(3x-2a)(2x+9a)$
18. $(4x-3a)(x-12a)$ 19. $(3-20x)(4+3x)$
20. $(5x-31)(3x+25)$ 21. $(4z-35)(6z-49)$
22. $(y+94)(y-95)$. 23. $(x+81)(8x-135)$
24. $(x+y+3)(x-y-1)$. 25. $(x+y-4)(x-y-2)$.
26. $(a-b-c)(a-5b+c)$. 27. $(x+y+z)(x+9y-z)$.

28. $(a-c)(a-2b-3c)$
29. $(a-b)(a+3b-2c)$
30. $(x-y+3)(x-3y+1)$
31 $(x-a+c)(3x+a-3c)$.
32. $(x+2y+3)(y+1)$
33 $(2x+3y+4)(y-3)$
34. $(x+y+z)(x-2y-z)$.
35 $(x+5y+4)(3x-1)$
36 $(2y-x+a)(y-2x-a)$
37 $(x+3y+3a)(x+y+a)$
38. $(3x-1)(x+5y-2a)$.

129. [pp 143—144]

10 $(x-1)(x+1)(x^2+12)$
11 $(2x^2-1)(3x^2+4)$
12 $(5x^2-3y^2)(3x^2-5y^2)$
13 $(x+2)(x-2)(3x^2-2)$.
14 $(x-y)(x+y)(2x^2+3y^2)$.
15. $(x+1)(2x-1)(x^2-x+1)(4x^2+2x+1)$
16 $(x-2)(x+3)(x^2+2x+4)(x^2-3x+9)$
17. $(x-y)(x^2+xy+y^2)(3x^3+2y^3)$
18 $(a+2)(a-2)(a^2+4)(2a^2+2a+1)(2a^2-2a+1)$.
19 $(a-x)(a+x)(a^2+x^2)(a^2+2ax+2x^2)(a^2-2ax+2x^2)$
20 $(a+1)^2(a^2+2a+2)$
21 $(x+1)(x+2)(x+3)(x+4)$
22 $(x+1)(x+2)(x^2+3x-3)$
23 $(x-1)(x+5)(x+1)(x+3)$
24 $(x-1)(x-5)(x^2-6x-1)$
25 $(x-1)(x+3)(x-2)(x-4)$
26 $(x-1)(x-2)(x+1)(x-4)$
27 $(x-1)(x-8)(x+1)(x-10)$.
28 $(x-4)(x-6)(x+1)(x-11)$
29 $(x+1)^2(x-1)^2(x^4-2x^2+3)$
30 $(x-3)(x-4)(3x^2-21x-8)$
31 $(x-2)(3x+4)(3x^2-2x-6)$.
32. $(x+1)(x+2)(x+3)(x+4)$
33 $5(x+y)(8x+7y)$
34 $(5x^2-3y^2)(13y^2-11x^2)$
35. $(2a+7x)(a+10x)$.
36 $(8a-15b)(5b-2a)$
37 $2(9b-29a)(26a-3b)$
38 $(x-2y)(x+4y)(x^2-4xy-8y^2)$
39 $(a+b)(a-b)(a^2-3ab+4b^2)$
40. $(a+b)(a-6b)(a^2-12ab+15b^2)$
41. $(x-3)^2(4x^2-3x+36)$
42 $(x+y)^2(x+4y)(10x-11y)$
43 $(a-1)(a+1)(a+5)(a+7)$
44 $(x-2)(x-3)(3x^2-15x-5)$
45 $(x+2)(x+7)(x^2+9x+4)$.
46 $(x+5)(x+6)(x^2+11x+8)$
47 $(x+1)(x-6)(x^2-5x+16)$
48. $(2x+a)(x-3a)(2x^2-5ax+8a^2)$

130 [p 146].

7 $(a+b+1)(a^2-ab+b^2-a-b+1)$.
8. $(a+2b-1)(a^2-2ab+4b^2+a+2b+1)$
9 $(x+y-a)(x^2-xy+y^2+ax+ay+a^2)$
10 $3(y-z)(z-x)(x-y)$

11. $3abc(b-c)(c-a)(a-b)$. 12. $3(2x-y)(x+y)(x-2y)$

15, 16, 17. Apply COROLLARY. 18. 0 19. $2x^3=\frac{137842}{421875}$.

21 Apply COROLLARY

131 Ex (i) [p. 147]

2 $(a+b)(x+c)$ 3. $(a+b)(a^2+b^2)$. 4 $(2x-y)(3x^2-4y^2)$.

5. $(x-2y)(x+y)(x^2-xy+y^2)$ 6. $(a-c)(a+b+c)$

7. $(ax+b)(cx^2+d)$ 8. $(a+b)(a-b)(a-c)$.

9. $(x-a)(x^2+ax+a^2)(x+b)$ 10. $(ax+1)(bx+1)$

11. $(ax+b)(bx+a)$ 12. $(ax+by)(ay+bx)$ 13. $(ab+xy)(ax^2+b^2y)$.

14. $(x-y)(x^2+3xy+y^2)$. 15. $(x-y)(x+y)^3$.

16 $(x-a)(x^2+ax+a^2)(ax+1)(a^2x^2-ax+1)$. 17. $(a^2+b^2)(x^2+y^2)$

18. Enclose the first 3 terms and the last 3 terms in brackets.

131. Ex (ii) [pp. 148—149]

7. $(x-a)(x-b-c)$ 11 $(x-b)(x+2a+b)$ 12. $(x+a)(x^2-bx+1)$.

13. $(x-2c)(x^2-2ax+3a^2)$. 14. $(1-ax)(1-ax-cx^2)$

16. $(x-y)(x-5y+2z)$. 17. $(a-b)(a+2b-3c)$

18. $(a-4b)(a-9b+5c)$ 19. $(x-a)^2(x-b)$.

20. Put $n^2=a$, $(x+m)(x+m+n)(x+m-n)$.

21. $(x^2+px-q)(x^2-ax-1)$ 22. Put $c^2=m$; $(a-b^2)(a-4b^2)(a-c^2)$

23. Put $q^2=a$; $(x+p)(x+2p)(x+q)(x-q)$.

131 Ex (iii). [p 150]

8. $(x-2y)(x+3y-1)$. 10. $(x-y+1)(x+y-2)$.

11. $(a+2)(a+6b-5)$ 12. $(a+3b)(a-b+2c)$.

13. $(x-3a)(x+5a+2)$ 14. $(x+a+ab)(x-b-ab)$

15 $(x-3a-6b)(x-2a+6b)$ 16 $(x-ab-a^2)(x+ab-b^2)$

18. $(x-3a)(3x-2a+5b)$

131. Ex (iv). [p 154]

12 $(x+1)(x+2)(x+4)$ 13 $(x+1)(x+4)(x+5)$.

14. $(x-1)(x+2)(x+8)$. 15. $(x-1)(x-3)(x+5)$.

16 $(x+1)(x^2+3x+8)$ 17. $(x-2)(x+3)(x-4)$

18. $(x+3)(x-4)(x+5)$ 19. $2(x+2)(x-2)^2$

20. $(a-1)^2(a+2)$ 21 $(x+1)(x-2)^2$. 22 $(x+1)(x+5)(x-6)$
23 $(2a-b)(2a^2+ab+b^2)$ 24 $(x+1)(x+3)(4x-3)$
25 $(2x-3)(4x^2-2x-3)$ 26 $(a-3b)(a^2+2ab+6b^2)$
27 $(a+b)(a^2+8ab-8b^2)$ 28 $(x-2y)(x-4y)(x+6y)$.
29. $(a-3b)(a^2+ab+3b^2)$ 30 $(a-b)(2a-b)(3a+b)$
31 $(x+1)(5x-3)(5x-2)$ 32 $(x+1)(x+2)(x+3)(x+4)$.
33 $(x-1)(x+2)(x-3)(x+4)$ 34 $(x+1)(x+2)(x^2+2x+3)$.
35. $(x-1)(x+2)(x-4)(x+6)$ 36 $(x-1)(x-3)(x^2-3x-5)$
37 $(x-2)(x+6)(x^2-2x+3)$ 38 $(x+2)(x-3)(x^2+x+7)$
39 $(x-2)(x+4)(x^2-2x+12)$ 40 $(2x+1)(2x-1)(3x^2+1)$
41 $(x-1)(x+3)(x^2-2x+7)$ 42 $(x-3)(x^3-2x^2-6x-18)$
43. $(x+1)(2x-1)(4x^2-2x+3)$ 44 $(x^2+4x+1)(x^2-4x+1)$
45 $(x+1)(x-2)(3x^2-2x+4)$ 46 $(x-1)(x+3)(7x^2+6x+9)$
47 $(x+1)^2(5x-3)$ 48. $(x+1)(x-2)(x^2-2x-2)$.
49 $(x-1)(x+5)(x^2-4x-4)$ 50 $(x+y)(x-3y)(3x-5y)$.
51 $(x-y)(2x+y)(4x+y)$ 52 $(x^2-xy+y^2)^2$
53 $(a^2+b^2)(a-b)^2$ 54 $(12x-5)(2x^2-x+1)$
55 $(x+3)(2x-5)(x^2+3x-1)$

132 [p 149]

12 $-(b-c)(c-a)(a-b)$ 13 $(b-c)(c-a)(a-b)$.
14 $-(b-c)(c-a)(a-b)(a+b+c)$ 15 $(b-c)(c-a)(a-b)(a+b+c)$.
16 $(b-c)(c-a)(a-b)(bc+ca+ab)$ 17 Put a^2, b^2, c^2 for a, b, c respectively in Ex 1, $-(b^2-c^2)(c^2-a^2)(a^2-b^2)=$&c
18 Substitute as in Ex 17, $(b^2-c^2)(c^2-a^2)(a^2-b^2)=$&c
19 $3(b+c)(c+a)(a+b)$ 20 $(a+b+c)(bc+ca+ab)$
21 $(b+c)(c+a)(a+b)$ 22 $(a+b+c)(bc+ca+ab)$
23 Expanding we get $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$ $-3abc\{(b-c)+(c-a)+(a-b)\}$, $(b-c)(c-a)(a-b)(a+b+c)$ [Ex 2].
24, 25. $-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$
26 Ex 24 with sign changed 27 Expanding we get $abc\{(b-c)+(c-a)+(a-b)\}-\{a^3(b-c)+b^3(c-a)+c^3(a-b)\}$, $(b-c)(c-a)(a-b)(a+b+c)$ [Ex 2] 28. Expanding we get $bc(b-c)+ca(c-a)+ab(a-b)-abc\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$, $(b-c)(c-a)(a-b)(abc-1)$ [Ex 1]

29. We have $(b^2-c^2)(b+c)^2+(c^2-a^2)(c+a)^2+(a^2-b^2)(a+b)^2$
$=(b^4-c^4)+2bc(b^2-c^2)+(c^4-a^4)+2ca(c^2-a^2)+(a^4-b^4)+2ab(a^2-b^2)$;
$-2(b-c)(c-a)(a-b)(a+b+c)$ [Ex. 2]
30. Proceed as in Ex 29, $2(b-c)(c-a)(a-b)(a+b+c)$

133 [pp. 160—161].

7 a^2-ab+b^2. 8. $x-y$. 9. $a-b+c-d$ 10. $2(x-y)$.
11. $x^2-xy+y^2-3x-3y+2$ 13 $a+b$. 14 $a^2-ab+b^2-2a+b+1$.
15. x^2-ax+a^2 16 $(1+a)^2+x^2$ 17 $8ab$ 18 $x^2-xy+y^2+x+y+1$.
19. x^4-x^2-1 20 $3(b+c)$ 21 $x^2-2xy+4y^2+x+2y+1$.
22 $a-b+c$ 23 $(x^2+ax+a^2)(x^4-a^2x^2+a^4)=$&c 24. x^2+3x-1.
27. The other factor $=x^2+y^2+z^2+2xyz-1$ 28 $x+y+z+xyz$

Miscellaneous Examples III. [pp. 161—166]

12 $(a+1)x^2+(a^2+1)x+a^3$ 16 0 18 abc. 20 $a^2+2ab+b^2$ $-5(a+b)+4$ 22 $bc+ca+ab$ 27. $(b+c)(c+a)(a+b)=$&c.
29 (1) $(f+h)(fg-ch)$, (2) $p(p-1)(p^2+2q+2)$ 31 0 32 $\frac{1}{2}$.
33. $\frac{203}{72}$ 34. $b-ap+q$ 35 $\frac{1}{16}$ 36 3 37 (1) $(lx+my)(mx-ly)$; (2) $2(x-y)(1-xy)$ 47. (1) $(x-a-b)^2-(a-b+ab)^2$, (2) $(x^2+5ax+5a^2)^2-a^4$ 50 $2(a+b)x$. 51 Multiply successively by a, b, c, and add 56 Multiply successively by $2a$, $2b$, $2c$, and add.
60. $16x^4-8x^2(2y^2+a^2)+(4y^2-a^2)^2$ 63 $a(b^2-ca)+b(c^2-ab)$ $+c(a^2-bc)=0$ identically, apply Ex 4, § 132. 64 $(ab+a-b+1)$ $(ab-a+b+1)$ 65 $(x+a)^2(x-a)^3$. 66 $(2x+a)^2(x-4a)$
67 $(a-b)^2(a+2b)$. 68 $(x+a)^2(x^2+2ax+b^2)$ 69. $(x+a+1)$ $(x+a-1)(x^2+2ax-2)$ 70. $(x^2+x+a+1)(x^2-x-a+1)$
71 $\{(a+b)x+(a-b)y\}\{(a-b)x+(a+b)y\}$ 72 $(x-a)(x-b)(x^2-ab)$.
73 $(a-b)(a+b+c)(a^2+ab+b^2)(a^2+b^2+c^2-bc-ca-ab)$
74. $2m^2n(m+n)$ 75 $(b-c)(c-a)(a-b)(bc+ca+ab)$.
76 $(u+x+y+z)(u-x-y+z)(u+x-y-z)(x-u-y+z)$
77 $(3x^2+1)(7x^2-4x+1)$ 78 $\{a(x+y)-(x-y)\}\{b(x+y)-(x-y)\}$.
82; 83; 84 0 86. $\{2x-(a-1)y\}\{3x+(a+2)y\}$.
87 $(2x+3y+z)(x-y+z)$. 89 $4(a^2+b^2+c^2)$. 90 $8ac$.
91 $2(a^2+b^2+c^2)$ 93 $4(a^2+b^2+c^2+d^2)$ 94 $8(ac+bd)$
97. $(x-y+2z)(x^2+xy-z^2)$. 98 After reduction it is identical with Ex 7, § 132 99 Apply Examples 3 and 25 of § 132

100 $(a+1)(b-c)+(b+1)(c-a)+(c+1)(a-b)=0$ identically; apply Ex 4 of § 132.

146. [p. 171]

9 8 10 8. 11 $\frac{1}{4}$ 12 3 13. -1 14. 5 15 12
16 1 17 5 18 6 19 $5\frac{1}{3}$ 20. 4 21 3 22 $\frac{5}{6}$
23 3. 24 10 25 15 26 $4\frac{1}{3}$. 27. $-\frac{4}{13}$ 28 29. 29 4
30. $-3\frac{1}{2}$. 31. $3\frac{13}{16}$ 32. $\frac{23}{101}$ 33 $-\frac{13}{a}$. 34. $\frac{c}{2m-1}$
35. $\frac{a+1}{3p}$ 36 $\frac{a-d}{c+3}$ 37 $\frac{c-d}{a-1}$ 38 $\frac{ma}{2b+1}$ 39 $\frac{c}{a-b}$
40. $-\frac{2}{3}$ 41. 5 42 $\frac{m^2}{n}$. 43 $\frac{1}{a}$

147. [pp 172—174]

1 $3x-a$ 2 $20-x$ 3 $\frac{63}{x}$ 4 $(x+34)$ yrs 5 $(50-3x)$ miles.
6 A, $(6x+3)$ miles ; B, $(6x-6)$ miles 7 $(5x+20)$s 8 $5x$.
9 $(x-6)$ yrs ; $(x+6)$ yrs 10. $12x$ miles 11 $\frac{25}{24x}$ 12. $\frac{30}{x}$ hrs
13 $\frac{3x}{a}$ miles. 14 $(y-x+3)$ Rs 15 $(x+y-20)$ Rs. 16 $5(x+3)$ yrs 17 $\frac{a}{x-\frac{1}{4}}$ miles per hr 18 A has $(x-y)$ Rs , B has $(x+y)$ Rs
19 $2m$, $8(m-3)$ 20 $10x+y$, $10y+x$. 21 $100y+10z+x$; $100z+10x+y$ 22 $x=a+15$ 23 $y=5-x$ 24 $x-3=y+4$.
25 If x represent the price of horse in rupees, then $x-m$=price of the saddle

148 [pp 175—179]

3 114 4. 71 5 28 6 31 7 4 8 20 9 24 10 9
12 98 , 85 13 38 , 53 14 16 , 10 15 19 , 12 16 35 , 29
17 22 , 14 18 48 , 40 19 38 , 2 21 15 , 21.
22 Rs 875 , Rs 125 23 153 , 54 24 A 750 , B 250 , C 125
25 A 695 , B 570 26 276 , 245 27 Rs 53 , Rs 19
28. 117 , 102 , 130 29 579 ; 186. 31. 15 yrs after

32 41, 11 33. 23 yrs 34 A's age = 35 yrs, B's = 30 yrs, C's = 27 yrs. 36 Rs 48 37. Rs 250 39 30d.
40 22 days 42 3 sov, 12 shill, 60 six-pences
44 $3\frac{1}{2}$ miles per hour 45. 4 miles per hr 46. 7 miles per hr.
47 $3\frac{4}{7}$ hrs after the second pipe is opened

Miscellaneous Examples IV [pp 180—184]

1. $2d-2b-2c$. 2. $\frac{3}{4}$ 3 $2bx+2cy$. 4 $4x^2y(y-x)$ 5 $(a^2+c^2)(a^2+2ac-c^2)$ 6. $a+b+c+d$ 8 $y^4-9y^3+32y^2-52y+29$ 10 a^3-b^3 11 -13. 12 40 years 13. $2a-6b$ 14 $\cdot 002$ 15 $2ax-2by$, $(a-1)x-y$ 16 $x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2-(abc+abd+acd+bcd)x+abcd$; $x^4+8x^3+24x^2+32x+16$ 17 $-(y-z)(z-x)(x-y)$ 18 $a^6-a^5x+a^3x^3-ax^5+x^6$ 20 $4a^2b^3$. 21 Diff. $=16x(x^2+1)$ 22 -9 23 18 24 $6x$ 25 $x^4+32x^2-320x-1024$ 26 $2, x^2$ 27 $a^5+3a^3b^2+4ab^4$ 28 a^2+b^2 30 $3abc-a^3-b^3-c^3$. 33 1. 34 225, 169 35 $(a^2-ab-b^2)x+(a^2+ab+b^2)y-(a^2+ab-b^2)z$ 36 $3x+8a$ 37 $x^3+5x^2y+7xy^2+8y^3$, $-x^3+8x^2y+7xy^2+7y^3$. 38 $a^3+(x+\frac{1}{2}y-\frac{1}{2}z)a^2+(\frac{1}{2}xy-\frac{1}{2}xz-\frac{1}{4}yz)a-\frac{1}{4}xyz$. 39 $a^3+b^3+c^3-3abc$ 40 $(y+z)(z+x)(x+y)$ 41 $a^4+2a^3b-3a^2b^2+b^4$; $a^4+27a^2+108a+81$. 45 $2\frac{1}{2}$ 46 5 at 19s, 7 at 17s 47. $3c$ 48 $\frac{1}{8}$ 49 Diff $=(x-y)^2$ 50 $(b+c)(c+a)(a+b)$ 51 $(a+1)\times(a-1)(b+1)(b-1)(1-x)(1+x)$ 52 $(x-y)^2$, $4a^2b^2$ 55 0. 56. $-2\frac{3}{5}$. 57. In 20 days 58 $a+4b$, 0 60 $(a+b-c)x+(b+c-a)y+(c+a-b)z$ 61 $-35x^2+36xy+52xz-9y^2-18yz+55z^2$.

62. $\{x(a+1)+y(b+1)\}(x^2-xy+y^2)$. 63 13. 66. $\dfrac{4ab}{b-a}$

67. 42, 26 68. 59 69 48 70. $5x+13y+6z$, 6 306x.
71. $12abc$ 72. $a^4+b^4-c^4-2a^2b^2+4abc^2$.

73 $2ab+3ac+6bc$ 74. $(a-b)(x-a)(x-b)(x+a+b)$ 78 $a-\dfrac{3c}{2}$.

79 36; 12 80 $8a-3b$, 0 81. $8a^2+6ab-2b^2$, $8a-2b$
82. $(1+x-y)^3-1$, $(x-y)^2+3(x-y)+3$. 83 $3-29x-01x^2$, 2709.

84 193; 83 85. $(a^2-bc)(b^2-ca)(c^2-ab)$ 86 $-\dfrac{23x^4}{32}$

92. $\dfrac{2c^2-a^2-b^2}{2a+2b-4c}$ 93 13

151 [pp 186—187]

5. $ab.$ 6. ax 7 a^2 8 $a^2b^2.$ 9 $ab^2c^2.$
10 $15a$ 11 $21ax$ 12 ab 13 4 14 $16m^2.$
15 $7a^2b^2x^2y^2$ 16 $3xyz$ 17 a^2c^2 18. $4m^2x^4$
20 $2mp^2$ 21 $ax.$ 22 $18m^2n^2.$ 23 $19a^2mx$

152 [pp 187—190]

3. $a+x$ 4 a^2-ax 5 $2ax.$ 6 $x+2c.$
7. x 8 $a-1$ 9 $x-2y$ 10 $4(x+y-z)$
12 $(a+b)^2x^3.$ 13 $a(a-b)^2$ 14 $a(p+q)x$ 15 $4(a-1)^2(x-a)^2.$
16 $6b(x-a)^2(x^2-y^2)$ 19 $x-y$ 20 $x-y$ 21 $x+y$ 22 $x+2$
23 $a^2x^2(a+x)$ 24 $a^2x(x-1)$ 25 $1+4x$ 26 a^2+x^2
27 $8(a+x)$ 28 $3(a+b)$ 29 $4(ax+2).$ 30 $4x^2(x-4a)$
31 $8(x^2-x+1)$ 32 $3a(a-b)(a+b)^2$ 33 $5(x+1)(x+2).$
34 $4a(a-x)$ 35 $3(a+b)(a-b)^2$ 36 $4(x+1)(x+2)$ 41 $x+2.$
42. $2x-1.$ 43 $x+2$ 44 $x+1$ 45 a^2+x^2 46 $3x+1$
47. a^2+a-2 48 $b-c$ 49 $2a+5$ 50 $2x-a$ 51 $x+3$
52 $3x-2$ 53 $x-1$ 54 $x+3$ 55 c^2-4b^2 56 $ab-b^2.$
57 $1+x$ 58 $x+3$ 59 $a-c$ 60 $1+x$ 61 $x-2$
62 $x-2$ 63. $x-1$ 64 $x-y$ 65 x^2-y^2 66 $5x^2-1.$
67 x^2-1 68 x^2+ab 69 $1-ax$ 70 $2a+3b+c$

155 [pp 192—198]

3 $x-3$ 4 $x-1$ 5 $3x+1$ 6 $2x+3$ 7 $x+5$ 8 $3x-2.$
9. x^2+x+1 10 x^2+2x-1 13 $x-5$ 14 $x+1$ 15 $x+1$
16 $2x^2+5x-3$ 17 $x+6$ 18 $x-4$ 19 $x+5$ 20 $x+3$
21 $x-7$ 22 $2x-1$ 23 $x-3$ 24 $x-7$ 25 $x+2$
26 $3x-7$ 27 x^2+x+1 28 $x-2$ 29 $2x^2-3x+1$
30 $x+4$ 31 $2x^2+1$ 32 x^2+7x+5 33 $x^2-x+1.$
36 $3(2x+3)$ 37. $2(x^2+ax-2a^2)$ 38 $x(x-2).$ 39 $2x(2x-1)$
40 $x(x+2)$ 41. $2ab(2a-3b)$ 44 $x-2$ 45 $2x-1$ 46 $x+5$
47. $2x^2-xy+y^2$ 48 $a^2+2ab+3b^2$ 50 $x-2a$ 51 $3x^2+y^2.$
52 $a-2b$ 53 $x-3$ 54 x^2+3x+5 55 $2x^2-6x$ 56. $2x-y.$
57 a^2x-3ay^2 58 $9a^2b^2(a-1)$ 59 x^3-2x^2+x
60. $x^2+7xy-y^2$ 61 $2x-9$ 62 $x-a$ 63 x^2-1 64 $x^2-a.$

65. $x^2-(a-b)x+b^2$ 66. x^2-ax-1. 67. $3x^2-2xy+y^2$.
68. $(x+1)^2$. 69. $x-a$ 70. $2x-1$. 71. x^3-4x^2+x-1.
72. x^2-2x+3 73. x^2+2x+3 74 $a-2b$.

158. [pp. 199—200].

2 $x-2$. 3 $2x+1$. 4. $x+y$. 5. a^2-ab+b^2. 6. $x+7$. 7. $a-2$.
8. $a-b$. 9 $x-2a$. 10. $1+2x$. 11 $x-1$. 12 $2x-9$.

159. [p. 203]

4. $10a^2b^2$. 5. $24xy^2$. 6 $48a^2b$. 7. $63a^2x^4$. 8. $168x^2y^5$.
9 xyz 10. $2a^2x^2$. 11. $60x^2b^2x^2$ 12. $50m^2n^3$. 13. $570a^2bx^2y^5$.
14. $24a^3x^3$. 15. $60a^3x^4$. 16. $960x^2x^2y^3z^4$.

160. [p. 204]

5. $3ax^2(2+x)$ 6. $axy(x-y)$ 7. $xy(x+y)$. 8. $6a(x^2-y^2)$.
9. $12x(x^2-1)$ 10. $60xy(a^2-x^2)$ 11 $210(a^4+a^3b-ab^3-b^4)$.
12 $(2x-1)(8x^3+1)$. 13 $10(x^2-y^2)(x+y)$. 14. $(a^2-x^2)(a^2+ax+x^2)$
15 $42x(x^2-y^2)$ 16. $2ax(a^2-x^2)$ 17. $(x-y)(x^3+y^3)$.
18. $(x^2+1)(x^3+1)$ 19. $(1+a)(1-a)^2$. 20. $12x^2y^2(x^2-y^2)$.
21 $(y-z)(z-x)(x-y)$. 22. $-ax^2(a^2-x^2)$. 23. $x^2(x+1)(x+2)$.
24 $120xy(x^2-y^2)$. 25. $(x+2)^2(x-2)^2$. 26. $75(x+y)(x+2y)(x+3y)$.
27. $36xy^2(x^3-y^2)$. 28. $12xy(1-a^2)$. 29. $a^2x^2(a^2-y^2)$ 30 x^6-1
31. $60(x+y)(x-y)^2(x^3+y^3)$. 32 $x(x^2+x+1)(x^2-1)$
33 $x^2y^2(x^2-y^2)(x^6-y^6)$

161. [p 206]

4 $(3x-1)(x-3)(x-6)$ 5 $(x-3)(2x-7)(3x-14)$ 6. $(x-1)^3(x+1)$.
7 $(x-a)(x-b)(x-c)$. 8 $(3x-2y)(x-y)(4x^2-y^2)$ 9 $(x-1)(x^2-4)$
(x^2-9) 10. $x(x-2)(x+3)(x-4)$. 11 $mn(x-1)(x-5)(x+6)$
12 $(a-x)(2a+3x)(3a^2+2ax+x^2)$ 13 $ab(a^2-b^2)(ax-by)(x^2-y^2)$.
14 $(x^2-1)(x^4-4)(x+3)$. 15. x^6-a^6. 16 $a^2x(2a-3x)(a+3x)(3a-x)$
$(a-x)$ 17. $(x+3a)(x-a)\{x^2+(2a-3)x-6a\}$. 18. (x^2-2x+3)
$(x^2-2x-4)(x^2-2x-5)$. 19 $(2x^2-1)(x-6)(x+8)(3x-1)(x+20)$.

163. [p 207]

2 $(x+2)(2x-1)(3x+1)$. 3. $(x-3)(2x-1)(4x+5)$. 4. $(x+9)(x-8)^2$
$(3x+10)(3x-10)$ 5. $(1+2x)(1-2x)(1+2x+4x^2)(1+2x-4x^2)$
6. $(9x^2-1)(9x^4-1)(x^2-3)^2$.

Miscellaneous Examples V [pp 210—212]

1 $2a^2-3x^2$ 2 a^2-b^2 3 $3x-2$ 4 $2x-y$ 5 $a+(b-c)x+cx^2$
6 $x^2+(a+y)x+y^2$ 7 $(a-b)x-(a-2b)$ 8 $x+2a$ 9 x^2-2x+3
10 $x-3$ 11 x^2+2x+3 12 $4a^2-3ab+b^2$ 13. $(x^2+2x-3)(x^2+3)$ $\times(x^2+2x+3)$ 14 $6(x-2a)(x^2-9a^2)(x^2-16a^2)$ 15 $(x+6)(x-7)$ (x^2-25) 16 $(x+a)(x^2-bx+b^2)(x^2+bx+b^2)$ 17. $x(x-1)(3x+1)(4x^2+$ $2x-1)$ 18 $(x-1)^2(7x-5)(2x^2+3x-5)$ 19 $bx+a$, $(bx+a)(a^2x^2-b^2)$
20 $x+y-1$; $(x+y+1)(x^3+y^3+3xy-1)$ 21 $8(x^2+y^2)$, $48(x^4-y^4)$.
22 $x-a$, $(x-a)^2(x-3a)(3x-7a)$ 23 $a+x$, $(a+x)(4a^2-9x^2)$ $\times(9a^2-4x^2)$
24 $2x^3-4x^2+x-1$, (H C D) $\times(2x+5)(3x^2+4x+1)$.
25 $(a+2)x+(a-1)$, (H C D) $\times\{(a-1)x+(a+1)\}\{(a+2)x+(a-2)\}$.
26 $a+b+c$, $(a+b-c)(bc+ca+ab)(3abc-a^3-b^3-c^3)$ 27 $x-8$; $(x+9)(x-8)^2(9x^2-100)$ 28 $x-y$ 29 $(x-1)^2$ 30 $x^2-3ax+2a^2$
31 $3x-1$, $\frac{1}{3}$ 32 2 33 8

170 [p 216]

3 $6b$ 4. $10x+\frac{x}{5}$ 5 $5a+\frac{3b}{4}$ 6 $a-\frac{x^2}{a}$ 7 $1+\frac{2x}{a-x}$

8 $x+2+\frac{2}{x-1}$ 9 $a+2b+\frac{3ab^2-3b^3}{(a-b)^2}$ 10 $a-\frac{a^2-b}{x+a}$ 11 $a^2-ax+x^2-\frac{2x^3}{a+x}$ 12 $a+\frac{3x^3}{2a^2-x^2}$ 13 $a^2-x^2-\frac{ax-2}{a^2+x^2-1}$

171 [p 217]

1 $\frac{a^2+2x}{a}$. 2 $\frac{33x-y}{2x}$ 3 $\frac{y^2}{ax}$. 4 $\frac{(a-b)^2}{4ab}$. 5 $\frac{a^2-(b-c)^2}{2bc}$

6 $\frac{2(x-a)^2}{x-2a}$. 7 $\frac{2x^2}{x-a}$ 8 $\frac{a^3-a-1}{a+1}$.

172 [pp 218—219]

6 $\frac{3a}{bc}$ 7. $\frac{4y}{5x}$ 8 $\frac{2a^2bx}{3y}$ 9 $\frac{7n^2}{45m^2}$. 10. $\frac{a-x}{a+x}$

11 $\frac{2x-3y}{y^2(4x+5y)}$ 12 $\frac{ay}{a-y}$ 13 $\frac{2m-n}{2a}$ 14 $\frac{3m}{p-q}$

15 $\frac{c}{d}$ 16 $\frac{x}{a^2(c-x)}$. 17 $-\frac{x}{y}$. 18 $-\frac{4x^2}{3a(2x+3a)}$

19. $\frac{b(a-b)}{4a(a^2-ab+b^2)}$ 20 $\frac{2a^2(x+a)}{3x^2(x^2-ax+a^2)}$ 21 $a(m+n)$

22. $\frac{x-b}{x+c}$ 23. $\frac{1-a-b}{1-b}$. 24. $\frac{a+b+c}{a-b+c}$. 25. $\frac{2x-3a}{4x^2+6ax+9a^2}$

26. $\frac{ax+by}{ax-by}$. 27. $\frac{1}{1-a+a^2}$ 28. $\frac{a+b-c}{a+b+c}$. 29. $\frac{1-a}{1-b}$.

30. $\frac{1}{a+b+c}$ 31. $\frac{x-1}{x+1}$ 32. $\frac{x+4}{x-4}$ 33. $\frac{x-1}{x+2}$ 34. $\frac{3a-2x}{3a+2x}$.

35. $\frac{2a-3x}{2a+3x}$ 36. $\frac{2+3x}{1+5x}$ 37. $\frac{1+4x}{3+2x}$. 38. $\frac{3y+4}{5y-4}$

39. $\frac{x+(a-b)y}{x-(a-b)y}$. 40. $\frac{ax-by-cz}{ax+by+cz}$. 41. $\frac{(a+b)x-ab}{(a-b)x-ab}$.

42. $\frac{1}{x^2-2x+2}$ 43. $\frac{a+b}{a-b}$ 44. $\frac{2y^2+3y-5}{7y-5}$. 45. $\frac{x^2-x+1}{x^2-3x+1}$.

46. $\frac{x^2-x+1}{x^2+x+1}$. 47. $\frac{x^2-x+1}{x-4}$ 48. $\frac{3x^2+4x+2}{4x^2+x+2}$.

49. $\frac{2x^2-6x+5}{3x^2-5}$ 50. $\frac{x(x+5)}{9x^2-x-3}$. 51. $\frac{6x^2-4ax-a^2}{6(x-a)}$.

52. $\frac{x^2-xy+y^2}{x^2+xy+y^2}$ 53. $\frac{a-bx}{a+cx}$ 54. $\frac{a^2-ab+b^2}{a^2+ab+b^2}$. 55. $\frac{3a^2+2b^2}{5a(2a+3b)}$.

56. $\frac{2x^2+3x+5}{2x^2+3x-5}$ 57. $\frac{x^2+x-12}{x^2-x-12}$ 58. $\frac{3a^2+ax+2x^2}{2a^2+ax+3x^2}$.

59. $\frac{3a(a^2-7ab+12b^2)}{2b(a^2+7ab+12b^2)}$. 60. $\frac{2(x^2-2ax+3a^2)}{3a(2x^2+5ax-3a^2)}$.

173 [p 221]

4. $\frac{20a}{180}$, $\frac{75a}{180}$, $\frac{84a}{180}$. 5. $\frac{24x}{36}$, $\frac{27y}{36}$, $\frac{10z}{36}$ 6. $\frac{a^3}{abc}$, $\frac{b^3}{abc}$, $\frac{c^3}{abc}$.

7. $\frac{5x^2}{xyz}$, $\frac{4y^2}{xyz}$, $\frac{6z^2}{xyz}$, $\frac{x^2+2y^2}{xyz}$. 8. $\frac{4x(a+x)}{36ax}$, $\frac{3a(a-x)}{36ax}$.

9. $\frac{5(4x-5)}{50}$, $\frac{20x}{50}$, $\frac{2(7x+6)}{50}$. 10. $\frac{6+6a}{30}$, $\frac{15-5a}{30}$, $\frac{3a-24}{30}$

11. $\frac{cx^2-abc}{abc}$, $\frac{ay^2-abc}{abc}$, $\frac{bz^2-abc}{abc}$. 12. $\frac{a(a-b)}{a^2-b^2}$, $\frac{b(a+b)}{a^2-b^2}$, $\frac{c(a-b)}{a^2-b^2}$.

13. $\frac{4x(1-x)}{12(1-x)}$, $\frac{15a(1-x)}{12(1-x)}$, $\frac{6(1+x)}{12(1-x)}$ 14. $\frac{1+x}{1-x^2}$, $\frac{1-x}{1-x^2}$, $\frac{1}{1-x^2}$

15. $\frac{2(x^2-4)}{(x-1)(x^2-4)}$, $\frac{-3(x+2)(x-1)}{(x-1)(x^2-4)}$, $\frac{4(x-1)}{(x-1)(x^2-4)}$. 16. $\frac{x(x-y)}{2y(x-y)}$, $\frac{2xy}{2y(x-y)}$, $\frac{2(1-y)}{2y(x-y)}$. 17. $\frac{x^2(a-x)^2}{a^2x^2(a^2-x^2)}$, $\frac{a^2(a+x)^2}{a^2x^2(a^2-x^2)}$, $\frac{ax(1-x^2)}{a^2x^2(a^2-x^2)}$.

173 [p 221]

18 $\frac{ab(b-c)(c-a)}{(a-b)(b-c)(c-a)}$, $\frac{bc(c-a)(a-b)}{(a-b)(b-c)(c-a)}$, $\frac{ca(a-b)(b-c)}{(a-b)(b-c)(c-a)}$

19 $\frac{a^2(a+b)}{ab(a^2-b^2)}$, $\frac{b^2(a-b)}{ab(a^2-b^2)}$, $\frac{ab(a+b)}{ab(a^2-b^2)}$

20 $\frac{3(4x^2-1),\ 5(2x-1)^2,\ 3x(2x-1),\ 4x(2x+1)}{(2x+1)(2x-1)^2}$

174 [pp 223—225]

9 $\frac{29x}{30}$ 10. $\frac{ax}{315}$ 11 $\frac{44a-7x}{75}$ 12 $\frac{41x}{24y}$ 13 $\frac{2axy^2+3bx^2-aby}{x^2y^2}$

14 $\frac{96x^3-9x^2y+10y^3}{24xy}$. 15 $\frac{a^2+b^2+c^2}{abc}$ 16 0.

17. $\frac{8a^3+6a^2x+ax^2+10x^3}{30a^2x^2}$ 18 $\frac{x^2-y^2}{xy}$ 19 $\frac{acx+b^2y}{bc}$

20 $\frac{x^2-2xy-y^2}{y(x-y)}$ 21 $\frac{2}{1-x^2}$ 22. $-\frac{1}{1+x}$ 23 $\frac{a^2+b^2}{a^2-b^2}$

24 $\frac{2mq}{q^2-p^2}$ 25 $\frac{a^2+x^2}{a^2(a+x)}$ 26 $\frac{a+bx}{c+dx}$ 27 $\frac{2ax+3by}{6}$ 28 1.

29 $\frac{x^2+a^2}{2a(x+a)}$ 30 $\frac{1}{2}$ 31 $\frac{29x}{50}$ 32 $\frac{2x^2+(m-n)x+8a^2}{2x^3}$

33. $\frac{2(1+x^2)}{x(1-x^2)}$ 34 $a-x$ 35 1 36 $\frac{2a^m}{a^{2n}-1}$ 37. $\frac{2a^2}{x(a^2-x^2)}$

38 $\frac{4ab}{a^2-b^2}$ 39 $\frac{2a^2-2ab+b^2}{a^2-ab}$ 40 $\frac{x+3}{x^4-1}$ 41 $\frac{4}{(9x^2-4)(3x-1)}$.

42 $\frac{4(2-x)}{x^4-1}$ 43 $\frac{y}{x+y}$ 44. $\frac{1}{1-x^4}$ 45 $\frac{18}{x^2-9}$

46 $\frac{x^2}{(x-1)(x-2)(x-3)}$ 47 $\frac{3n^2}{(3m+2n)(9m^2-n^2)}$. 48 $\frac{1+2x+3x^2}{4(1-x^4)}$.

49 $\frac{2}{x(x-2)}$ 50 $\frac{1}{(x-1)(x-3)(x-5)}$ 51 $\frac{12}{x^8-5x^4+4}$.

52 $\frac{4(x^4+a^4)}{x^4-a^4}$ 53 $\frac{1}{x^2-1}$ 54. $\frac{3x-4}{(x+1)(x^3-1)}$ 55 $\frac{a+bx}{b+ax}$

56 $\frac{3+2x+x^2}{(1-x)(1-2x)(1-3x)}$ 57 $\frac{-2x+2x^2-x^3}{(1-x)^4}$.

58. $\frac{a^3}{a^4-b^4}$. 59. $\frac{x^2+1}{(x+1)^3(x+3)}$. 60 $\frac{x+c}{(x-a)(x-b)}$.

61 $\dfrac{23+16x-30x^2-3x^3}{6-11x-21x^2-x^3+3x^4}$. 62 $\dfrac{1}{(x+1)(x+3)}$.

63 Take the first and third, and the second and fourth, terms together, thus $(x^{2n}+x^n+1)-(x^n-1)=x^{2n}+2$.

176. [pp. 226—227]

5 $\dfrac{3ay}{10bx}$ 6 $\dfrac{3}{2}$ 7 $\dfrac{a^2(a+x)}{y^2(a-x)}$ 8 $\dfrac{ax+x^2}{ab-bx}$ 9 $\dfrac{a^4-x^4}{a^2x}$

10. $\dfrac{a-x}{a+2x}$ 11 $\dfrac{3a^2(a-b)}{b}$. 12 $\dfrac{x}{a}+\dfrac{bx^2}{a^2y}+\dfrac{ay^2}{b^2x}+\dfrac{y}{b}$. 13 $\dfrac{1}{y}+\dfrac{x}{y^2}+\dfrac{x}{yz}$ $+\dfrac{y}{xz}+\dfrac{1}{z}+\dfrac{y}{z^2}$ 14 $\dfrac{bx}{ay}-\dfrac{ay}{bx}$ 15 $\dfrac{8ab}{9x^2}+2+\dfrac{9x^2}{8ab}$ 16. $\dfrac{x}{x+a}$.

17. $\dfrac{a^4}{a^2+x^2}$ 18 $\dfrac{ax+ay}{cy-y^2}$. 19. $\dfrac{1}{(x-y)^2}$ 20 $\dfrac{a^4x(ax-1)}{a-b}$.

21 $rs+(qs+rt)x+qtx^2$. 22. $x^2+1+\dfrac{1}{x^2}$. 23. $1-x$

24 $a^2x^2+1+\dfrac{1}{a^2x^2}$ 25 $\dfrac{3a^4}{x^4}-\dfrac{19a^3b}{10x^3y}+\dfrac{21a^2b^2}{5x^2y^2}+\dfrac{9ab^3}{10xy^3}+\dfrac{b^4}{y^4}$

178 [pp 228—229]

5 $\dfrac{1}{4a}$. 6. $\dfrac{8x^2}{3ab}$. 7 $\dfrac{a^2c}{mb}$ 8 $a-x$. 9 $\dfrac{a}{a^2+b^2}$.

10. $\dfrac{cx-x^2}{c^2+cx+x^2}$. 11 $\dfrac{2ay^2(1-x)}{c}$ 12. $\dfrac{x-1}{x^2}$ 13 $\dfrac{x^2+x}{3x^2+3}$.

14 $-\dfrac{a}{b}$. 15. $\dfrac{1}{x^2+y^2}$ 16 1. 17 $\dfrac{x^2}{y^2}$ 18 $\dfrac{(x-1)(x-2)}{x^2}$.

19. $\dfrac{1}{(1+x)(1-x)^2}$. 20. y^2+y+1. 21. $z-1+\dfrac{1}{z}$ 22. $\dfrac{3x}{4b}-\dfrac{2a}{x}$.

23 $x+\dfrac{1}{x}$ 24 $x+\dfrac{1}{x}$ 25 $\dfrac{acx^2}{bd}+\dfrac{bx}{cd}+\dfrac{a}{b}$

179. [pp 230—231]

6. $\dfrac{3x-4}{6x}$. 7. $\dfrac{2a}{4-a}$ 8 $2m-1$ 9. $5x-3$ 10. $\dfrac{2x}{3x+2}$.

11 $\dfrac{ay+bx}{ab}$ 12 $\dfrac{m-n}{x+y}$. 13 $\dfrac{x}{z}$ 14 $\dfrac{(a-b)x}{ab}$. 15 $\dfrac{2a}{a-x}$

16 $-\dfrac{b}{a}$ 17 $\dfrac{xy-y^2}{x^2+xy}$. 18. $x-1$ 19. $\dfrac{y^2}{xy-x^2}$ 20. $\dfrac{1}{m^2+n^2}$.

21 y 22 $\frac{a-b}{a+b}$ 23 $\frac{3(x^2+1)}{x+2}$ 24 $\frac{a^2+x^2}{2ax}$ 25 x
26 $\frac{y+z-x}{y+z+x}$ 27 $\frac{y-x}{y}$ 28 1 29 $\frac{1}{x+1}$ 30 $\frac{a^2-b^2}{a^2b}$.
31 $\frac{2bc}{(b+c-a)^2}$ 32 $\frac{3x+1}{2x+1}$ 33 $\frac{ab-1}{a^2b-ab^2-a+2b}$
34 $\frac{4}{3x}$ 35 $\frac{61-42x}{11-12x}$

180 [pp 233—234]

5 1 6 0 7 0 8 $\frac{1}{xyz}$ 9 1 10 1 11 x^2 12 $\frac{1}{xyz}$
13 1 14 1 15 1 16 1 17. 1 18 -1.
19 1 20 $\frac{1}{(x+a)(x+b)(x+c)}$ 21 $\frac{x}{(x-a)(x-b)(x-c)}$
22 $\frac{x^2}{(x+a)(x+b)(x+c)}$ 23 See examples 20, 21 and 22

181. [p 235]

1. 0 2. 0 3 0 4. 0 5 $\frac{3x^2-(a^2+b^2+c^2)}{(x-a)(x-b)(x-c)}$ 6 0 7 0 8. 0.
9 0 10 $\frac{2bc^2(a-c)+2ca^2(b-a)+2ab^2(c-b)}{(a^2-b^2)(b^2-c^2)(c^2-a^2)}$ 11. $\frac{(b^2-ac)y}{(ax-by)(by-cz)}$.

Miscellaneous Examples VI [pp 240—246]

1 1 2 $b^2-a^2+\frac{b^4}{a^2}-\frac{a^4}{b^2}$ 3 $\frac{ax+by}{ax-by}$ 4 Take the second form [§ 130] 6 0 7 $\frac{2a^2}{(a-x)^2}$ 8 $\frac{(a-b+c)(a-b-c)}{2ab}$.
11 $-\frac{b}{a}$. 12 $\frac{a}{a+b}$ 13 $\frac{a+b}{c^2-a^2-b^2+2ab}$. 17 $\frac{4b(a^2+b^2)}{(a^2-b^2)^2}$.
18 a^4+b^4 19 1 . 20 $\frac{2(b+c)}{a+b+c}$ 24 0 25 $\frac{x-y}{x+y}$.
26 $a^2+2ab+b^2-c^2$ 27 1 30 $\frac{1}{1-x^2}$, $\frac{(m+n)^2}{4mn}$ 31 $\frac{x-3y}{x+3y}$.
33 $(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)-4(ab+cd)^2$
35 $\frac{x^2-b^2}{x^2-c^2}$ 36 $\frac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$ 37 1. 40 $\frac{x^2-x+1}{x^2-1}$

41. $3x-5y$. 44 $\left(\frac{a+b}{a-b}\right)^2$. 45. $a^3+\frac{1}{a^3}$.

46. $\frac{9(a-b)(a-c)(b-c)}{(b+c-2a)(c+a-2b)(a+b-2c)}$ 48 $\frac{x^2-bx+b^2}{x^2+bx+b^2}$. 51. $\frac{x-z}{1+xz}$

52. a^4-b^4 53. $a+b+c$ [See § 180] 54 $\frac{3x-a-2b}{(x-a)(x-b)}$.

57. $\frac{5x}{2(x^2-1)}$. 58. $\frac{(2a-3b)^2-1}{(2a-3b)^3}$. 59 $\frac{(1-a)(1-b)}{(1+a)(1+b)}$

60 $\frac{2(x^2-1)}{x^2-y^2-2x+1}$. 63. $\frac{x^3+4m^2x-8m^3}{x^3-4mx^2-4m^2x+8m^3}$.

64. $\frac{(1-x)(3-x^2+x^3+x^5)}{(1+x)(1+x^2)(1+x^5)}$. 65. $\frac{(x^2-x+1)(e^x-1)}{(x-1)(e^x+1)}$.

67 $\frac{ab(2-x^2)}{2(b^2-a^2)x}$. 73 1. 75 $\frac{4xy^3}{x^4-y^4}$

77 $\frac{2x^2z^2+2y\{(y+z)x^2-(z+x)y^2+(x+y)z^2\}}{x^2y^2z^2}$.

78 $\frac{(a+b+c+d)(a+b-c-d)(a-b+c-d)(b+c-a-d)}{4(ab-cd)^2}$.

187. [p. 248]

7. a^6; a^{21}, $x^6y^9-243a^{15}x^{10}$, $-a^{18}x^{30}y^{45}$ 8. $\frac{x^4}{y^6}$; $-\frac{27a^3}{x^3}$, $-\frac{a^{10}b^{15}c^5}{32}$; $\frac{x^{24}y^{32}}{a^{16}c^8z^{40}}$. 9. $-81a^{16}b^8c^4$; $x^{2mp}y^{2np}z^{p2}$; $(-1)^na^nx^{mn}y^{np}$.

10. a^{2m}; $(-1)^{m+1}a^{m+1}$; $-\frac{a^{2m-1}}{b^{2m-1}}$, $\frac{a^{4m+2}}{b^{2m+2}}$

188. [pp. 248—249]

1. $\frac{x^2}{4}+\frac{xy}{3}+\frac{y^2}{9}$. 2 $\frac{a^2}{b^2}-10+\frac{25b^2}{a^2}$. 3 $\frac{x^2}{4}-\frac{2}{3x}+\frac{4}{9x^4}$. 4. $1+x-\frac{5x^2}{1}-\frac{x^3}{3}+\frac{x^4}{9}$. 5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{2xy}{ab}-\frac{2xz}{ac}-\frac{2yz}{bc}$ 6. $\frac{a^4}{x^2}-\frac{4a^2b^2}{xy}+\frac{6a^2c^2}{xz}+-\frac{12b^2c^2}{yz}+\frac{9c^4}{z^2}$. 7. $2a^2x^2+8b^2y^2-24bcyz+18c^2z^2$. 8 $a^{2m}+2.+b^{2n}$ 9. $a^{2m}+4b^{2n}+c^2-4a^mb^n+2a^mc-4b^nc$. 10 $e^{4x}+3e^{2x}+2e^x+1$. 11. $(a^2+2ab+b^2)x^2-2(a^2-b^2)xy+(a^2-2ab+$
12. $\frac{x^4}{a^4}+\frac{2x^2y^2}{a^2b^2}+\frac{y^4}{b^4}-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}+1$ 13. $\frac{a^2x^2}{m^4}+\frac{2abxy}{m^2n^2}+\frac{b^2y^2}{n^4}-\frac{2ax}{m^2}-\frac{2b}{?}$.

16 $(x-y)^{\frac{3}{4}}$ 17. $(m+n)^{\frac{m}{n}-\frac{3}{2}}$. [201

20 b^{2m+n-p} 18 a^{2m} 19. x^{2a-c}

23 $20b^{-2m-n}p$ 21 $y^{\frac{4a}{3}-\frac{c}{4}}$

29. 2 24 $(a-b)^{-1}$ 22. $c^{\frac{m}{a}+\frac{2n}{b}-\frac{1}{3}}$.

30 1. 25 $(a^m+a^n)^3$.

34 $280v^{-\frac{3}{m}-1}$. 31 x^{m+1} 32 $30a^{m-2}$ 28. 64.

35 $120p^{-1}$ 33 $3a^{\frac{c}{5}}$

202 [p 264].

6 a^{-2y}

10. m^{1-x} 7. $3x^{-a-5c}$

14 $x^{2m-6n}y^{1?n-7m}$ 11. x^{a+1} 8 $5a^{m+1}$.

15 $a^{x-4p+q-6}b^{2x+y+p+2q-2}$ 12 $3x^{2c-2a}$. 9. $3xy^{m+2}$

16 $(x+y)^a(x-y)^{cp}$. 13 $(a+b)^{-1}$

203 [p 264]

5 729 6 249 7. 4096 8. x 9 $2a^{\frac{1}{2}}$ 10 x^4 11 $-x^6$.

12 x^{4m} 13 $(-1)^{p+1}x^{2p+2}$ 14 $(-1)^{\frac{1}{4}}x$ 15. x^{20}.

16 $a^{6m}b^{4m}$ 17 $4^6a^{-4}b^3$ 18 $5^{-\frac{1}{2}}a^{2-m}x^{-n}y^2$

19. $2^{2m}a^{-2m^2}b^{2mn}$ 20 $a^6+6a^3b^3+9b^6$ 21 $a^{4m}-3a^{2m}b^{\frac{n}{2}}$

$+3a^mb^{-\frac{2n}{3}}-b^r$. 22 $a^8-2a^4b^4+b^8$

204 [p. 265]

6 $(12)^4$ 7 6 8 $(70)^{-5}$ 9 24 10 $(1950)^8$.

11 $1296a^4b^4$ 12 $(15ab)^{m+1}$ 13 $(90y^{1-m})^a$. 14 $a^4-2a^2b^2+b^4$

15. $(x^3+a^3)^3$ 16 $\{x^2-(a+b)x+ab\}^4$ 17 $(4x^2-9y^2)^{-4}$

18 $(ab+a+b+1)^{-2}$ 19 $(x^2+xy-x-y)^{-2m}$ 20 x^{3m} 21. 1.

205 [p 265]

4 $\dfrac{125}{216}$ 5 $\dfrac{5^ma^mx^my^m}{8^mb^m}$ 6 $\dfrac{8}{b^{12}}$ 7 x^3. 8. 1

9 $-\left(\dfrac{a}{2}\right)^{2n-1}$. 10 $\dfrac{2xy^6}{15ab^6}$ 11. $\dfrac{(a-b)(ab+b^2)^{2m-1}}{2^{2m-2}a^{2m}}$.

207 [p 268].

7 216 8 $-\dfrac{4}{25}$. 9 -1152 10. -61 11. $\dfrac{y^6}{x^5}$ 12 $\dfrac{x^2}{cy}$

13 $\dfrac{x^{m+1}}{y^m}$ 14 $16a^4$ 15 $-\dfrac{1}{27x^6}$ 16 $\dfrac{b^2}{4a^4}$ 17 $\dfrac{b^{nx}c^x}{a^{mx}d^{2x}}$

18 $-\dfrac{b^{3x-2y}c^{3x}}{a^{3m}}$ 19 1 20 mx

Miscellaneous Examples VII [*pp* 269—272]

1 $3a^{\frac{2}{5}}-4a^{\frac{3}{5}}$. 2. $3y^{\frac{2}{3}}z^{\frac{2}{3}}-6y^{\frac{4}{3}}+2z^{\frac{4}{3}}$ 3. $x^{-4}y^{\frac{1}{3}}-2x^{-3}y^{\frac{1}{2}}$.

4 $\frac{1}{4}a^{-\frac{5}{2}}-\frac{1}{8}a^{\frac{3}{2}}x^{-\frac{1}{2}}-\frac{2}{5}a^{\frac{4}{3}}x^{-\frac{1}{2}}+\frac{1}{6}y^{\frac{1}{4}}-\frac{1}{3}y^{\frac{3}{4}}$.

5. $(4-m)x^{-\frac{4}{5}}+(a+d-n)x^{-\frac{3}{5}}+(3b-5c)x^{-\frac{2}{5}}$.

6. $8(a+1)x^{\frac{m}{3}}-2(b+3)y^{-\frac{m}{2}}+(8c-5)z^{-\frac{m}{4}}+cz^{-\frac{3m}{4}}$.

7. $6x^{-\frac{2}{3}}-5a^{-\frac{1}{2}}x^{-\frac{1}{3}}+6\frac{1}{2}$ 8 $-\frac{1}{4}a^{\frac{m}{n}}-1\frac{1}{10}a^{-\frac{m}{n}}b^{-3}+\frac{1}{10}a^{-m}+\frac{3}{5}a^{-2m}$

9. $x+y$ 10. $x^{\frac{5}{2}}-x^{2}-4x^{\frac{3}{2}}+6x-2x^{\frac{1}{2}}$ 11. $6a^{-8}b-9a^{-9}b^{2}$ $-4a^{-7}b^{-1}+6a^{-8}$. 12 $\frac{1}{2}a^{-2}x^{-6}-\frac{1}{60}a^{-1}b^{-2}x^{-3}y-\frac{2}{15}b^{-4}y^{2}$.

13 $a^{m}-2a^{\frac{m}{2}}b^{-2}+b^{-4}$. 14 $a^{2}x^{\frac{3}{2}}-b^{2}y^{-1}$ 15 $a^{m}+a^{\frac{m}{2}}b^{\frac{n}{2}}+b^{n}$.

16 $x^{\frac{5}{2}}+x^{-\frac{5}{2}}+\frac{11}{6}\left(x^{\frac{3}{2}}+x^{-\frac{3}{2}}\right)+\frac{11}{6}\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)$.

17 $2x^{-\frac{5}{p}}-4x^{-\frac{4}{p}}+2x^{-\frac{3}{p}}-5x^{-\frac{2}{p}}+8x^{-\frac{1}{p}}+6x^{\frac{1}{p}}-3x^{\frac{2}{p}}$.

18. $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ 19. $9+3a^{-\frac{1}{6}}+a^{-\frac{2}{6}}$ 20 $x^{\frac{2}{3}}+2x^{\frac{1}{2}}+3x^{\frac{1}{3}}+2x^{\frac{1}{6}}+1$.

21 $x^{2}+x^{-2}-1$ 22. $a^{\frac{2}{3}}-a^{-\frac{2}{3}}$. 23 $x-x^{\frac{1}{2}}$ 24. $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}$ $-x^{\frac{1}{3}}y^{\frac{1}{3}}+x^{\frac{1}{3}}z^{\frac{1}{3}}+y^{\frac{1}{3}}z^{\frac{1}{3}}$. 25. $a^{-1}x^{m}-2bx^{m+1}y^{-1}+3cx^{m+2}y^{-2}$ 26 $5a^{2}b^{-2}$ $-a^{-2}b^{2}(1+3b)$ 27. $a^{\frac{1}{2}}b^{\frac{1}{6}}$ 28. $-\frac{x}{y}$. 29. $\left(\frac{y}{x}\right)^{\frac{1}{6}}$ 30 $\left(a^{-\frac{1}{n}}b^{\frac{1}{m}}\right)^{pq}$.

31. $\left(\frac{a}{b}\right)^{mn+qr}$ 32 x. 33 $a^{\frac{1}{x}}x$ 34 $\frac{2mn}{a^{m^{2}-n^{2}}}$ 35 $\frac{a^{2}x^{2m-1}}{y^{2m+1}}$.

36 $\frac{x(1-y)}{y^{m}}$ 37. $\frac{x^{m}+yx^{m-1}}{y^{m}}$ 38. $\frac{ba^{n-1}}{(a-b)^{n}}$ 39. $\left(\frac{4y^{-2}}{b^{-4}}-\frac{9x^{2}}{a^{2}}\right)^{-2}$.

40. $\frac{b-a}{x-y}$ 41. $\frac{(3y)^{2m}}{(z-y)(x-a)^{2m-1}}$. 42 1 43 $\frac{1}{4}x^{-1}y^{-2}-\frac{3}{4}x^{-\frac{3}{2}}y^{-\frac{3}{2}}$ $+\frac{9}{16}x^{-2}y^{-1}$ 44. $ax^{-3}-3a^{\frac{1}{3}}x^{-1}+3a^{-\frac{1}{3}}x-a^{-1}x^{3}$ 45 $(x^{m}-y^{n})(x^{m}+y^{n})$.

46 $$\left\{\left(\frac{x}{a}\right)^{\frac{2}{m}}+\left(\frac{x}{a}\right)^{\frac{1}{m}}\left(\frac{y}{b}\right)^{\frac{1}{m}}+\left(\frac{y}{b}\right)^{\frac{2}{m}}\right\}\left\{\left(\frac{x}{a}\right)^{\frac{1}{m}}-\left(\frac{y}{b}\right)^{\frac{1}{m}}\right\}.$$

47. $(x^{-3}+y^{-1})(x^{-3}-y^{-1})$ 48 $a^{-1}x^{-1}-1+ax$ 49. $x^{\frac{m}{2}}-\frac{1}{2}b^{\frac{1}{n}}x^{\frac{2}{p}}$.

50. -17. 51. $\frac{a^{4}b^{4}}{(b+a)^{3}(b-a)}$. 52 $\frac{x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}$. 53 $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$.

Miscellaneous Examples VII [pp 269—272]

54 $\dfrac{x^{\frac{m}{3}}-2y^{\frac{m}{3}}}{x^{\frac{m}{3}}-y^{\frac{m}{3}}}$ 55. $\dfrac{x^{\frac{2}{m}}+2x^{\frac{1}{m}}+3}{3x^{\frac{2}{m}}+2x^{\frac{1}{m}}+1}$ 56 $\dfrac{e^x-1}{e^x+1}$

57. $\dfrac{1}{x^{\frac{2}{3}}+3x^{\frac{1}{3}}y^{\frac{1}{3}}+2y^{\frac{2}{3}}}$ 58. $\dfrac{ab^{-1}+yx^{-1}}{ab^{-1}-yx^{-1}}$ 59 $\dfrac{a+x}{a-x}$.

60 $\dfrac{4a^2b^{-2}-4ab^{-1}+1}{4a^2b^{-2}-3ab^{-1}-1}$ 61 $x^{2^{n-1}}-1$ 62 $\frac{1}{18}a^{\frac{5}{4}}$ 64 a^n

211. [p 273]

1 $\sqrt{50}$ 2 $\sqrt[3]{108}$. 3 $\sqrt{100ax^3}$. 4 $\sqrt[3]{a^{10}x^7}$

5. $\sqrt[m]{q^{2m+3}r^{4m+2}}$

212 [p 274]

1 $6\sqrt{2}$ 2 $6\sqrt[3]{5}$ 3 $72\sqrt{2}$ 4 $a^2\sqrt[5]{3b^3c^4}$, 5 $a(a+x)\sqrt{2}$

214 [p 275]

1. $3\sqrt{5}$ 2 $\sqrt{5}-4$ 3 $ab\sqrt[4]{b^3}$ 4 $\sqrt[12]{\dfrac{b^4c^3x}{a}}$ 5 x.

6 $(a+c)\sqrt{(ab)}$ 7 $\sqrt{bx(a^2-x^2)}$. 8 $3a^2$ 9 $\dfrac{\sqrt{a}}{x^2}$

10 $\dfrac{1}{a}\sqrt{x}-\sqrt{y}$ 11. $\dfrac{a}{b}$ 12. $\dfrac{9a}{x}\left(\dfrac{a^3}{2x^3}\right)^{\frac{1}{4}}$ 13 $3\sqrt{\dfrac{x+}{x-}}$

217 [pp 277—278]

6 $3-\sqrt{3}$. 7 $2\sqrt{7}+3\sqrt{3}$ 8 $\frac{1}{7}(3+\sqrt{2}+6\sqrt{3}+2\sqrt{6})$ 9 $7-4\sqrt{}$

10 $\dfrac{x}{a-\sqrt{a^2-x^2}}$. 11 $5+2\sqrt{2}+\sqrt{3}$. 12 $\dfrac{2\sqrt{b}}{a^2-b}$. 13 $\dfrac{2(x+y)}{x-y}$

14 2 15 $\frac{1}{3}\sqrt{6}$ 16 $5\sqrt{6}$ 17 $\dfrac{2a}{\sqrt{a^2-x^2}}$ 18. $\dfrac{2}{x^2}$ or $\dfrac{2\sqrt{1-x^2}}{x^2}$

19 $\dfrac{4x\sqrt{(r^2-a^2)}}{a^2}$ 20 $2x^2$ 21 $\dfrac{2(1-x)}{\sqrt{1-4x}}$ 22 $\dfrac{2a\sqrt{a^2-x^2}}{x^2}$; 1

23. 9 24 $\dfrac{2a^3}{a^4-x^2}$ 25 0 26 0 27

29 Given expn $=(x^3+y^3+z^3-3xyz)+2xyz=2xyz$ [$x+y+z=0$

$2(q-r)\sqrt{p}+2(r-p)\sqrt{q}+2(p-q)\sqrt{r}$ [§§ 114 and 117]

218. [*p.* 279].

4. $3+\sqrt{2}$ 5 $3+\sqrt{5}$. 6 $\sqrt{3}+\sqrt{5}$ 7. $4-\sqrt{3}$. 8. $2-\sqrt{5}$.
9 $2\sqrt{3}+3\sqrt{2}$. 10 $2\sqrt{2}-1$. 11. $2-\frac{1}{2}\sqrt{3}$. 12 $\frac{\sqrt{2}}{2}(\sqrt{3}+1)$.
13 $\sqrt{a}+\sqrt{a+2b}$ 14. $\sqrt{x+1}-\sqrt{y-1}$. 15 $\sqrt{\frac{1}{2}(1+x)}+\sqrt{\frac{1}{2}(1-x)}$ 16. $\sqrt{x}-\sqrt{y+z}$ 17. $a-\sqrt{ax-a^2}$.
18 $x+\sqrt{a^2-x^2}$. 19. 4 20. $\frac{1}{2}\left(xy+\frac{1}{xy}\right)$.

219 [*pp* 279—280]

13 1 ; 1 ; $\frac{1}{x^3}$; 1 ; 0. 14. $x^{\frac{3}{2}}$; $x^{\frac{5}{2}}$; $x^{\frac{8}{3}}$; $a^{\frac{3}{4}}b^{\frac{1}{2}}$, $x^{\frac{10}{3}}$.

15. $\frac{x^{-2}}{2^{-1}}$; $\frac{1}{4^{-1}a^{-2}x^{-3}}$; $\frac{y^{-1}z^{-3}}{3^{-1}x^{-3}}$; $\frac{x^{-1}y^{-\frac{1}{2}}}{4^{-1}z^{-1}}$; $\frac{x^{-1}z^{-\frac{1}{2}}}{5^{-1}y^{-\frac{3}{2}}}$

16. $5\sqrt[5]{a^3}$ or $5(\sqrt[5]{a})^3$; $3\sqrt[3]{x^5y^2z^4}$ or $3(\sqrt[3]{x})^5(\sqrt[3]{y})^2(\sqrt[3]{z})^4$; $\frac{1}{3}\sqrt[6]{\frac{a^6x}{y^4}}$.

17 $\frac{3a}{b}$, $\frac{x^2}{ay^3}$; $\frac{b^{\frac{1}{3}}c^4}{a^{\frac{1}{5}}}$

221. [*p.* 282]

1. Second. 2. Third. 3 Second 4. First if x, and second if x and y, are variables

222 [*p* 282]

1. $4x+3=0$. 2 $(a-c)x+(b-d)=0$ 3 $\frac{1}{2}x+9=0$.
4 $\left(1-\frac{a}{c}\right)x+\frac{b^2}{c}=0$ 5 $\left(\frac{b}{c}-\frac{d}{c}\right)x+a=0$ 6 $\frac{9}{20}x-\frac{9}{10}=0$

224. [*pp* 283—292].

3 12 4. $\frac{3}{5}$ 5. 120 6 10 7 $6\frac{1}{2}$. 8 8 9 $3\frac{5}{21}$ 10. 2.
11. 3 12 $1\frac{1}{8}$ 13. 2 14 16. 15 13. 16. 5 17. $\frac{2}{3}$ 18 7.
19. 2. 20 9 21 2 22 9 23. 8 24 5 25 3. 26. 8
27 $\frac{80}{67}$ 28 3 29. 35 30. 72 31 $\frac{8a}{25}$. 32 51. 33. 7
34 8 35. 11. 36 $25a+24b$. 40 $\frac{b^3}{a-c}$. 41. $\frac{ace}{cd-be}$.
42. $\frac{1}{ab}$ 43. $\frac{d}{c}$ 44. $\frac{a^2+b^2}{a+b}$. 45 1 46. $\frac{b(h-g)}{a-bq}$.

224 [pp 283—292]

47 $\frac{ab}{a-b}$. 48. $\frac{3ac-b}{a-3b-c}$. 49 $a+b$. 50 $\frac{2npb-2mpc+6mn}{p(2na-2mb+nd)}$.

51 $\frac{4a^2(a^2+ab-b^2)}{3a^3-6a^2b+ab^2+6b^3}$ 53. 8 54. 20 55 1 56. 4

57 7 58 8 59. 4 60 $\frac{3}{2}$. 63 7 64. $\frac{ad-bc}{(a-c)n-(b-d)m}$.

65 $\frac{b}{5a}$ 66 $\frac{br-cq}{cp-ar}$ 67 $\frac{1}{3}$ 68 7 69 $\frac{5}{a}$ 70. 7 71 $3\frac{1}{2}$.

72 $2\frac{1}{2}$ 73 $-3\frac{1}{8}$ 74 $-2\frac{1}{2}$ 76 $6\frac{3}{5}$ 77 $36\frac{1}{11}$ 78 $\frac{a(m+n)-2ap}{p(m+n)-2mn}$

79 5 80 $\frac{-ab}{a+b+c}$ 81. $\frac{a^2(1-b^2)+b^2(1-a^2)}{ab(a^2+b^2-2)}$ 82. $\frac{(m-n)(n^2-m^2)}{m^2-mn+n^2}$

88 12 89 $\frac{1}{2}$ 90 $\frac{b-1}{a+2c}$ 91 2 92 $\frac{d-c}{a-b}$ 93 12 94 3

95 3 96 2 97 6 98 $3\frac{3}{8}$ 99 4 100 3 101 14. 102 3.

103 4 104 110 105 $-\frac{2a}{3}$. 106. 1. 107 $\frac{7}{8}$. 108 15.

109 $\frac{b}{a}(a-b+c)$ 110 $\frac{70xb-3ac}{320c}$ 111 $\frac{bf(h-q)}{af+2bc-bfq}$. 112 $\frac{ac}{b}$

113 $\frac{4ab^2-10a}{4a-3b}$. 114. 4 115. 6 116 2 117. $\frac{a+b+c+d}{m+n}$

118. $\frac{ab(c+d)-cd(a+b)}{cd-ab}$ 119 $\frac{ab(c+d-cd(a+b)}{ab-cd}$ 120 $\frac{1}{b-a-1}$.

121. $\frac{ad-ce}{cf-bd}$ 122. $\frac{c^2-ab}{a+b-2c}$ 123 $\frac{1}{2}(a+b)$ 124. $4\frac{1}{4}$ 125 $\frac{a^2+b^2}{a+b}$

126 $-3a$. 127 $1-a$ 130 6 131. 5 132. 2 133. 4.
134 11

225 [pp 294—295]

8 10 9 a 10 $\frac{1}{5}$ 11 19 12. $(2^m-3)^2$ 13. $\frac{1}{5}(a+1)$.

14 $-11\frac{5}{13}$. 15 2 16 $\frac{9}{20}$ 17 16 18 25 19 $\frac{\sqrt{a}}{\sqrt{a}+2}$

20 $a^{12}-m$ 21. $3\frac{6}{7}$ 22 $\frac{1}{2}a$ 23 $\frac{16}{25}$. 24 4 25 $\frac{4}{7}\sqrt{3}$

26. $\frac{(a-1)^2}{2a-1}$ 27. 5 28 $6\frac{1}{4}$ 29 $\frac{25}{27}$ 30 $\frac{1}{a}\left(b-\frac{c^2}{c-1}\right)^2$

31 $\frac{9a}{16}$ 32 $\frac{4ab-9(a+b)^2}{8(a+b)}$ 33 $\frac{25}{16}$ 34 $2\sqrt{ab-b^2}$

35. $\dfrac{ac^{\frac{n}{n+1}}}{a^{\frac{n}{n+1}}-c^{\frac{n}{n+1}}}$ 36. $\dfrac{\sqrt{3}}{2}$ 37. 5 38. $\left(\dfrac{2a}{1+a}\right)^2$. 39. $\dfrac{2ab}{1+b^2}$.

40. $\left\{\dfrac{2ac}{b(c^2+1)}\right\}^{\frac{1}{n}}$ 41. $2a$. 43. 49. 44. $\dfrac{8a^3+15a^2b+6ab^2-b^3}{27b}$.

45. $\left\{a^2+\left(\dfrac{2a}{3c}-\dfrac{c^2}{3}\right)^3\right\}^{\frac{1}{2}}$ 46. 37.

226 [p 296].

3 $-\frac{1}{3}$. 4. $2a-1$. 5. $m-3n$ 6 8 7 $\frac{2}{3}m$. 8 7. 9. 3.

10. $\dfrac{a}{2}-2$ 11. 2 12 $\sqrt{(\frac{3}{2}a)}$. 13 $\left(\dfrac{m}{2}\right)^{\frac{1}{m-2}}$.

227. [pp 298—299].

4 $\frac{4}{9}$ 5 $\frac{35}{24}$ 6. $b-\dfrac{a}{c^2}\left(\dfrac{c-1}{c+1}\right)^2$. 7. $a\dfrac{25b^2-1}{25b^2+1}$. 8 $\frac{7}{5}(b-a)$.

9 $\sqrt{ab}$ 10. $\dfrac{2}{\sqrt{4ab-b^2}}$. 11. $\dfrac{1}{a}\sqrt{\dfrac{2a}{b}-1}$. 13. 1 or 9.

14 $a\left\{1-2\sqrt{\dfrac{b}{c}}\right\}$. 15. $\dfrac{a(1-\sqrt{b})^2}{1+b}$ 16. $\dfrac{a}{\sqrt{2b-b^2}}-a$.

17. $2a\sqrt{1-a^2}$ 18. 8 19 $\dfrac{a}{(\sqrt{b}-1)^2}$. 20. $\sqrt{1+4(a-1)^2}$.

21. $\sqrt[4]{\frac{1}{2}+\frac{1}{2}\sqrt{2}}$ 22 $\sqrt{\dfrac{a^2-1}{a^2+3}}$ 23 $a\sqrt{\left(\dfrac{3a+1}{2}\right)^4-1}$

24 $1-\sqrt{1-a^2}$. 25 $\dfrac{1}{2}\left(a+\dfrac{1}{a}\right)-1$. 26. a or $\dfrac{1}{a}$

228 [p 300].

4. 17 5. $\dfrac{b^2-a^2}{4a-b}$. 6 $\dfrac{c^2(a-b)^2}{\{(b+c)\sqrt{b}+(a+c)\sqrt{a}\}^2}$.

7 $\dfrac{b-q}{a-p}$. 8. $\dfrac{b}{a}$.

229 [p. 301].

4 (1) $x(\sqrt{a}+\sqrt{b}-\sqrt{c})^2=d$, first: (2) $25x^4+4x^3+6x^2-28x+9=0$; fourth: (3) $x^9=x^4-2x^2+1$; ninth: (4) $x+y^2=\dfrac{xy}{a}$; second:

229. [p 301]

(5) $\frac{x^3}{a}+\frac{y^3}{b}=ax^2y$, third. (6) It is already in the required form; fourth (7) $yx^4+1=x^3-3x^2y+3xy^2-y^3$, fifth (8) $x\sqrt{a}+\frac{ax^3}{b}+\frac{1}{\sqrt{a}}=0$, third 5. $\frac{a^2+c^2}{2a}$ 6 $a=11, b=-3$ 7 $\frac{3abc}{2(bc+ca+ab)}$.

230 [pp 302—304]

3 18 4 120 5 $\frac{35}{37}$ 6 $\frac{15}{36}$ 7 110, 90 8 A, Rs 20 as 8 B, Rs 10, C, Rs 18 as 8, D, Rs 13 9 A, Rs 30, B, Rs 27, C, Rs 23 10 56 11 A, 320, B, 280, C, 200. 12 A, Rs 55, B Rs 20 13 £2 10s 14 12 gals 15 44, 45 16 256, 257. 17 ·036 18 36 years 19 24000 20 £1480 21 A, £24, B, £36 22 Rs 7 as 2 23. 1512000 sq miles 24 21 tolahs 25 5, 10, 15 26 500 27 5000 28 4550 29 36 sq in. 30 70, 35 and 14 respectively 31 20 32 180 33 $3\frac{3}{4}$ cubits.

231 [p 305]

3 86 or 68 4. 57 5 46 6 84. 7 305.

232 [p 308]

4 15 days 5 $17\frac{1}{7}$ hrs 6 25 days 7 300 cub ft 8 In 30 hrs. 9 In 25 and 100 hrs 10 A in 32 hrs, B in $53\frac{1}{3}$ hrs 11 24 days.

233 [pp 310—311]

4 $3\frac{3}{4}$ miles per hour 5 4 miles per hour 6 6 hrs 25 min. 7 3 miles an hr 8 $15\frac{1}{5}$ miles, 4 hrs 8 m, and 3 hrs $52\frac{1}{2}$ m. 9. $1\frac{1}{2}$ miles, $4\frac{1}{2}$ miles 10. 5 miles 11 24 miles per hr 12 $3\frac{1}{2}$ min.

234 [p 312]

2 77 miles 3 5 miles per hr, 45 miles 4 $9\frac{1}{7}$ miles from Ely 5 6 miles per hr, 32 miles 6 Constable's speed at first $8\frac{1}{4}$ miles per hr., thief's $9\frac{3}{4}$ miles, $71\frac{1}{2}$ miles 7 30 hrs. after they first started; 20 miles from the place whence the quicker walker started

235. [p 314]

2 (1) $5\frac{5}{11}$ min. past 1 ; (2) $21\frac{9}{11}$ or $54\frac{6}{11}$ min. past 1 3. (1) $32\frac{8}{11}$ min. past 6 ; (2) exactly at 6 ; (3) $16\frac{4}{11}$ or $49\frac{1}{11}$ min. past 6. 4 (1) 12 min past 2 ; (2) $3\frac{3}{11}$ min past 2 5 $5\frac{5}{11}$ min. after 2.

236 [pp 318—321]

1 $20\frac{1}{2}$ 2 $12\frac{58}{253}$ 3. 420 4 *A*, 48 ; *B*, 28. 5. *A*, £60; *B*, £50; *C*, £30 6. $x=2$, £5. 5*s*. 7. 13*s*. 8 Each man 70*s*. ; each woman 35*s* ; each child 11*s* 8*d* 9 *A*, Rs 61 ; *B*, Rs. 40 10. 72 lbs. 11. 48. 12 50 gals 13. 1819 14 183 15 $4\frac{1}{2}$ miles per hour. 16 4 sov, 59 shill , 55 six-pences 17. 10 from the first ; 5 from the second. 18 Rs 19200 19. $2\frac{1}{2}$ and $3\frac{1}{4}$ yds 20 17 and 15 chhataks. 21 30 miles per hour ; $19\frac{3}{8}$ miles 22 $1\frac{1}{59}$ min past 12. 23. $131\frac{1}{4}$ miles 24 Up to 14 lbs., 3*d* , every additional 7 lbs , 2*d*. 25 36. 26 6*d* 27 Rs. $3\frac{1}{2}$ per maund 28. 20 mds. ; 30 bighas 29. 4500. 30 36 and 64 tolahs. 31 100 gallons 32 1008. 33 180000

34 $c\left(1+\frac{ab}{a+b}\right)$ 35. $\frac{a(c-b)}{a-b}$, $\frac{b(a-c)}{a-b}$ 36 $\frac{nb+m}{a+b}$ and $\frac{na-m}{a+b}$ days.

37 $\frac{abc}{b+c}$ miles 38 $\frac{2c(a+b)}{a-b}$ miles, or $\frac{2c(a+b)}{b-a}$ miles.

39. $\frac{(100)^2c}{(b+d)(100+a)}$ rupees.

239 [p 325]

N. B —*The values of* x *and* y *are given in order.*

7. 44, 15. 8 1; 3. 9 12, 9. 10 1; 5. 11. 3; 2. 12 3, 3 13 4, 10 14. 2; 3. 15 4; 7 16 12, 20 17 5, 10

18 24, 12 19 $\frac{ac+b^2}{a^2+b}$; $\frac{ab-c}{a^2+b}$. 20 1; 0. 21. $a\frac{abc}{a^2+b^2}$; $b\frac{abc}{a^2+b^2}$.

22 $2b-a$, $2a-b$ 23 $x=y=\frac{ab}{a+b}$

241. [pp 326—329]

N B —*The values of* x *and* y *are given in order.*

2 1; −1 3 10; 11. 4 10; 4. 5 8, 10. 6 9, 8. 7 8, 10 8. 3, 2. 9. 5; 2 10. 2, 3 11. 6, 8 12 144; 216 13. 24, 12 14 5, 6. 15. 7, 10.

16 15, 3 17. $(bc'-b'c)\div(ab'-a'b)$, $(ca'-c'a)\div(ab'-a'b)$

18 $\dfrac{(a+b)m+(a-b)n}{2(a^2+b^2)}$, $\dfrac{(a-b)m-(a+b)n}{2(a^2+b^2)}$ 19 $\dfrac{a+b-c-d}{bc-ad}$, $\dfrac{a-b-c+d}{bc-ad}$ 20 $\dfrac{ac(bm+dn)}{ad+bc}$, $\dfrac{bd(cn-am)}{ad+bc}$ 21 $\dfrac{a^2-b^2}{a^2+b^2}$, $\dfrac{2ab}{a^2+b^2}$

22 1, 1 23 $\dfrac{a}{b}\,\dfrac{a^2+ab+b^2}{a+b}$; $-\dfrac{a^2}{a+b}$ 24. $\dfrac{2ab}{a+b}$, $\dfrac{2ab}{b-a}$.

25 $\frac{1}{3}$; $\frac{1}{4}$. 26 $\dfrac{1-ab}{n-bm}$, $\dfrac{1-ab}{m-an}$ 27 $\dfrac{bc-ad}{nb-md}$; $\dfrac{bc-ad}{mc-na}$. 28 $\frac{1}{2}$; 1.

29 5, 6 30 $13\frac{1}{2}$, $1\frac{1}{2}$ 31 11, 8 32 3, $\frac{1}{2}$ 33 $\dfrac{1}{a}$, b

34 5, 12 35 5, 4 36 $\dfrac{(a^2-b^2)c+(c^2-d^2)b}{ac-bd}$; $\dfrac{(a^2-b^2)d+(c^2-d^2)a}{bd-ac}$

37 12, 9 38 7, 8. 39 $2\frac{2}{15}$; $1\frac{1}{3}$ 40 $5(m+n)$, $3(m-n)$

41 $\dfrac{abc(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$, $\dfrac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}$ 42 $\dfrac{(a^2+b^2)c}{a^2-b^2}$; $\dfrac{(a^2+b^2)c}{2ab}$

43. $\frac{1}{5}$, -17 44 $\dfrac{a^2+ab+b^2}{a+b}$, $\dfrac{a^2-ab+b^2}{a-b}$ 45 $b+c$, $a+c$.

46 21, 20 47 5, 6 48 4, 3 49 3, $-\frac{5}{2}$ 50 2, 4

51 2, 2 52 7, $\frac{6}{7}$ 53 02, 29 54 $\frac{17}{37}$, $\frac{17}{117}$. 55 10, 5.

56 1, 1 57 3, 1 58 $\frac{9}{4}$, $\frac{27}{28}$

243 [pp 330—332]

N B—*The values of x, y and z are given in order.*

2 2, 1, 3 3 11, 15, 17 4 2, 4, 6 5 8, 6, 12 6 5, 8, 12 7 $\frac{1}{2}(b+c-a)$, $\frac{1}{2}(c+a-b)$, $\frac{1}{2}(a+b-c)$ 8 5; 12, 4 9 64, 80, 100 10 15, 13, 18 11 5, 7, 6 12 5, 7, -3

13 $\dfrac{b^2+c^2-a^2}{2bc}$, $\dfrac{c^2+a^2-b^2}{2ca}$; $\dfrac{a^2+b^2-c^2}{2ab}$ 15 2, 3; 6, 16 $\dfrac{2a}{m+p-n}$; $\dfrac{2b}{m+n-p}$, $\dfrac{2c}{n+p-m}$. 18 Each $=2$ 19 $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. 20 1, 2, 3

21 3; 5; 8. 22 $\dfrac{1}{x}=\dfrac{1}{2}\left(\dfrac{m}{b}+\dfrac{n}{c}-\dfrac{l}{a}\right)$, $\dfrac{1}{y}=\dfrac{1}{2}\left(\dfrac{n}{c}+\dfrac{l}{a}-\dfrac{m}{b}\right)$, $\dfrac{1}{z}=\dfrac{1}{2}\left(\dfrac{l}{a}+\dfrac{m}{b}-\dfrac{n}{c}\right)$ 23. $\dfrac{2}{x}=\dfrac{1}{b}+\dfrac{1}{c}$, $\dfrac{2}{y}=\dfrac{1}{c}+\dfrac{1}{a}$; $\dfrac{2}{z}=\dfrac{1}{a}+\dfrac{1}{b}$

244. [p 336].

N B—*The values of* x, y *and* z *are given in order.*

5 $4, -20; 22$ 6 $4, 5; 2$ 7 $\frac{11}{28}, -\frac{32}{15}; -\frac{3}{4}$

8 $\frac{1}{(a-b)(a-c)}, \frac{1}{(b-a)(b-c)}; \frac{1}{(c-a)(c-b)}$ 9 $(b^2-c^2)(c+a)(a+b)$, $(b+c)(c^2-a^2)(a+b)$, $(b+c)(c+a)(a^2-b^2)$ 10. $b+c-a, c+a-b$, $a+b-c$ 11 $k(a^2-bc); k(b^2-ca), k(c^2-ab)$; where $k=\frac{2}{a+b+c}$.

12 a, b, c 13 Each$=a^2+b^2+c^2-bc-ca-ab$ 14 $-a, -b, -c$.

15 $ka(b^2-c^2); kb(c^2-a^2), kc(a^2-b^2)$; where $k=\left\{\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right\}$ $\div 2(bc+ca+ab)$

245 [pp 339—340].

5 $x=1, y=3, z=4, u=2$. 6 $u=1\frac{1}{2}, x=1, y=4, z=\frac{1}{2}$ 7 $x=4$, $y=9, z=16, u=25$ 8 $x=5, y=2, z=3, u=4, v=1$. 9. $x=5$, $y=4, z=3, u=2, v=1$ 10 $x=abc, y=ab+bc+ca, z=a+b+c$.

11 $x=0, y=z=\frac{1}{2}$ 12 $(b+c-a)x=(c+a-b)y=(a+b-c)z=k$, where $k=(a+b+c)^2$ 13 $x=b-c, y=c-a, z=a-b$.

14 $u=abcd, x=-(abc+abd+acd+bcd), y=ab+ac+ad+bc+bd+cd$, $z=-(a+b+c+d)$

15 $x=\frac{b+c-a}{b+c+a}, y=\frac{c+a-b}{c+a+b}, z=\frac{a+b-c}{a+b+c}$ 16. $x=2, y=3, z=4, u=6$.

17 $x=\frac{1}{2}(\sqrt{a^2+2bc}+\sqrt{a^2-2bc}), y=\frac{1}{2}(\sqrt{a^2+2bc}-\sqrt{a^2-2bc})$.

18. $x=12, y=7$. 19 $x=\sqrt{\frac{(a+b-c)(c+a-b)}{2(b+c-a)}}$,

$y=\sqrt{\frac{(a+b-c)(b+c-a)}{2(c+a-b)}}, z=\sqrt{\frac{(b+c-a)(c+a-b)}{2(a+b-c)}}$

20 $x=\sqrt{\frac{(a^2+b^2)(c^2+a^2)}{2(b^2+c^2)}}, y=\sqrt{\frac{(a^2+b^2)(b^2+c^2)}{2(c^2+a^2)}}$,

$z=\sqrt{\frac{(b^2+c^2)(c^2+a^2)}{2(a^2+b^2)}}$. 21. $x=a\sqrt{\frac{2(b+c)}{(c+a)(a+b)}}$,

$y=b\sqrt{\frac{2(c+a)}{(a+b)(b+c)}}, z=c\sqrt{\frac{2(a+b)}{(b+c)(c+a)}}$. 22 4; 9; 11.

246 [pp 343—347]

8 98; 85 9 63, 21 10. 10, 16 11 24; 36 12 $\frac{2}{15}$, 13 $\frac{8}{13}$ 14 58 or 85 15 22, 26 16 75, 100, 125 17 8 18 35 and 14 years 19 18, 11 20 40, 21 21 16 and 12 ft 22 315, 280 and 304 rupees 23 12, 5*s* 24. 31, 7. 25 £288, 5 per cent 26 1*s* 4*d*, 2*s* 10*d* 27 91 28 48 29 17. 30. 17. 31 631 32 *AB*, 35 miles, *BC*, 29 miles; *AC*, 45 miles 33 24 bales or 72 casks 34 £100 at 4, £400 at 2 35 3 36 39*s*, 21*s*, 12*s* 37 Each equal cock in 32 hrs, the other in 24 hrs. 38 3 miles per hour 39 15 and 2 miles respectively 40. 12 and 4 miles per hr 41 1080 yds, 16½ min 42 24 miles per hr, 96 miles 43 4 and 5 yds 44 *A* in 5 min., *B* in 5 min 20 sec 45 Rate of faster train $=\frac{(a+b)(m+n)}{2mn}$ ft, and of slower train $=\frac{(a+b)(n-m)}{2mn}$ ft. per sec, 44 ft and 36 ft per sec. 46 Time of going $=\frac{ct}{b+c}$ hrs, time of returning $=\frac{bt}{b+c}$ hrs, stream's vel $=\frac{a(b^2-c^2)}{2bct}$ miles per hr; 3 hrs, 13 hrs; 5 miles per hr 47 30*d*, 15 48 $AB=31\frac{1}{2}$ miles, $BC=63$ miles; 21 and 42 miles per hr 49 72 weeks

250 [p 350]

1 3 2 2 $d-b : a-c$ 3 3 . 4 4 $\frac{\sqrt{a}+1}{\sqrt{a}-1}$ 5 16 . 25, 1 · 8, $\sqrt{10}$ 19, 13 14

252 [p 351]

2 15 3 3 4 $\frac{(a-b)mc}{na-mb}$, $\frac{(a-b)nc}{na-mb}$. 5. $b(m+1) : a(m-1)$ 6. $x=4$ 7 $x=17\frac{1}{2}$

261 [p 354]

1 24, x^2y^2 2 $87\frac{1}{2}$ 3 $1\frac{1}{3}$

263 [pp 357—358]

5 $\frac{ac}{b}$ 8. 5 · 6 is greatest, 2 · 3 least 10 5 : 2 11 2 5 12 1 · 2 16 1 . 5, $3x : 4y$, $1-y : x$ 19 $\frac{a^2+bc}{ac}$ 20 7 6 21. 6, 45 23 16 25 9 years 26 2*c* 27. 2 3 . 4

Miscellaneous Examples VIII. [pp 361—364]

29 $x=3$ 30 $2ax=(a+b+c)(a^2+2b^2+c^2)$, $2by=(a+b+c)(a^2+b^2+2c^2)$, $2cz=(a+b+c)(2a^2+b^2+c^2)$.

31. $\dfrac{1}{a+b+c}=\dfrac{\frac{2a}{x}}{b+c-a}=\dfrac{\frac{2b}{y}}{c+a-b}=\dfrac{\frac{2c}{z}}{a+b-c}$.

32 $x=\frac{1}{2}k(b+c-a)$, where $k^2=1-(a^2+b^2+c^2)$, &c 36 12 ; 3.

37. 3 : 13 38 Length 30 and breadth 25 yds. 39. 300.

40 11 : 24 41. 22 miles. 42 $6\frac{2}{3}$ inches 43 6 . 7.

44 6 inches 45 Val of gold : val of silver $=20n^2q : m^3p$.

264 [p 365]

4. $a=b=c$. 6 $x=a$, $y=b$. 7. $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{1}{a^2+b^2}$.

266. [p 368]

2 $3a^2$. 3 3 4 $\dfrac{y+2}{y}$

272 [pp 373—375]

2 $a=0$, $b=3$, $c=2$, $d=0$, $e=5$ 3. $p=2$, $q=8$, $r=2$, $s=-12$.

5. $a=2$, $b=0$, $c=-7$, $d=-2$ 6 $l=-6$, $m=12$, $n=-8$.

8 $2x+3$; -15 10. $p=2a$, $q=a^2$ 12. $a=-25$; x^2-3x-4

13 $l=0$, $m=-1$, $n=-12$. 15 $a=-29$, $b=-3$, $c=27$.

16. $a=14$, $b=20$.

273 [p. 376]

3. 16 4. -3. 5 $\dfrac{-d+q\{b-q-p(a-p)\}}{c-q(a-p)-p\{b-q-p(a-p)\}}$.

7 $p^2-4q=0$.

275. [pp 377—378]

4 12 5 9 6. $\dfrac{b^6}{a^6}-\dfrac{3b^3}{a^3}+\dfrac{b^2}{a^2}-\dfrac{b}{a}+8$ 12. $k=16$.

276, Ex. (ii) [p 380].

1 $a^3\mp a^2b+ab^2\mp b^3$ 2 Ex. 67, § 95.

3. $a^5\mp a^4b+a^3b^2\mp a^2b^3+ab^4\mp b^5$

4 $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$

5 $x^7 \mp x^6y + x^5y^2 \mp x^4y^3 + x^3y^4 \mp x^2y^5 + xy^6 \mp y^7$

6 $x^8 - x^7y + x^6y^2 - x^5y^3 + x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8$

7 $a^9 \mp a^8x + a^7x^2 \mp a^6x^3 + a^5x^4 \mp a^4x^5 + a^3x^6 \mp a^2x^7 + ax^8 \mp x^9$.

8. $a^{10} - a^9x + a^8x^2 - a^7x^3 + a^6x^4 - a^5x^5 + a^4x^6 - a^3x^7 + a^2x^8 - ax^9 + x^{10}$

9 $x^{15} \mp x^{14}a + x^{13}a^2 \mp x^{12}a^3 + x^{11}a^4 \mp x^{10}a^5 + x^9a^6 \mp x^8a^7 + x^7a^8 \mp x^6a^9$
$+ x^5a^{10} \mp x^4a^{11} + x^3a^{12} \mp x^2a^{13} + xa^{14} \mp a^{15}$.

278 [pp. 384—385]

4 $(x-1)^2(x^2+5x+2)$ 5 $(x+1)(12x^2+x+3)$

6 $(x-1)^2(4x+3)(5x-1)$ 7 $(x^2-1)(x+1)(x+12)(3x-5)$

8 $(x+1)^3(3x^2-9x+8)$ 9 $(x^2-1)^2(x-5)(5x-2)$

10 $(a-b)(a+2b)(3a+b)$ 11 $(x+y)(2x+5y)(4x+3y)$

12 $(a^2-x^2)(a-2x)(3a+4x)$ 13 $(x+a)^3(x-3a)$

17 $(x-3)^2(2x^2+3x-4)$ 18 $(3x-y)(x-y)(2x^2+3xy+3y^2)$

19 $(a-x)(a-2x)(a-3x)(a-4x)$

280 [pp 388—390]

3 $n(nc+bd)+a(mc-ad)=m(mb+na)$ 4 $ab(c+d-e-f)$
$+cd(e+f-a-b)+ef(a+b-c-d)=0$ 5 $a^3+b^3+c^3-3abc=0$

6 $x^2+y^2+z^2+2xyz=1$ 12 $a^2+ab+b^2=3$ 13 $a^2=b^2-2c^2$

14 $(p-q)^2=(p+q)^3(p^2+pq+q^2)$ 15 $a^3+b^3+c^3+abc=0$

16 $m^{\frac{2}{3}}-n^{\frac{2}{3}}=4$ 17 $(ab'-a'b)^2=(bc'-b'c)^2+(ca'-c'a)^2$

18. $(a+b)c^2=(ab)^{\frac{1}{4}}\{(ab)^{\frac{1}{2}}+2(1-c^2)(ab)^{\frac{1}{4}}+1\}$

281 [pp 392—395]

7 4 8 $2\sqrt{6}$ 9 12, 16, 24 11 x^2-2x+1 14 $a=3, b=4,$
$c=12$ 16 5, $5x+4$ 23 See *App*, Ex (1), and § 264

286 [p 399]

1 ± 4 2 ± 3 3 ± 5 4 ± 2 5 $\pm\frac{\sqrt{85}}{2}$

6. ± 4 7 0, 4 8 $\pm\frac{2}{5}\sqrt{10}$ 9 $\pm\sqrt{6}$ 10 0, $a+b$

11 $\pm\frac{7}{2}\sqrt{2}$ 12 0, $\frac{2ab}{a+b}$. 13 0, 3. 14 $\pm 2\sqrt{-1}$.

15 ± 6 16 ± 9 17 $\pm\sqrt{3}$ 18 0, 3 19 5, 5

20 $\pm\frac{1}{2}$ 21 $\pm\frac{24a^2}{25b}$. 22. $0, \pm\sqrt{3}$ 23 $\pm\frac{\sqrt{3}}{2}$.

24 $0, \pm\frac{2}{a}\sqrt{\left(1-\frac{1}{a^2}\right)}$. 25. $\pm\sqrt{\left(\frac{a-2}{a+4}\right)}$ 26. ± 5

290, Ex (i). [*p*. 402]

1 $-1, -4$. 2 $-1, -2$ 3 $-3, -12$. 4. $-3, -8$.
5. $-3, -7$ 6 $-5, -7$. 7. $-4, -20$ 8 $-25, -80$.

290, Ex (ii) [*p*. 402]

1. 4, 5 2 3, 4 3 10, 17. 4 5, 7 5. 4, 20. 6. 11, 30.

290, Ex (iii) [*p* 403.]

1. $3, -\frac{8}{3}$. 2 $-\frac{7}{4}, -2$ 3. $\frac{3}{4}, -4$. 4 $6, -\frac{27}{2}$. 5. $\frac{9}{2}, \frac{4}{3}$ 6 $1, -\frac{1}{4}$

290, Ex. (iv) [*p*. 403].

1 $-\frac{3}{4}, -2$ 2. $\frac{1}{3}, -4$ 3 $\frac{2}{5}, -6$ 4. $\frac{5}{2}, -\frac{2}{3}$. 5 $\frac{4}{3}, -\frac{5}{11}$.
6. $\frac{5}{2}, \frac{3}{5}$. 7. $\frac{4}{3}, -\frac{1}{4}$ 8. $2, \frac{7}{24}$. 9 $13, -\frac{13}{3}$. 10 $6, -\frac{9}{2}$
11 $12, -\frac{105}{8}$. 12 $\frac{4}{3}, \frac{5}{4}$.

290. Ex (v) [*pp* 404—406]

1. $2, -3$ 2 $\frac{1}{4}, -2$. 3 $\frac{p}{4}, \frac{3p}{4}$ 4 $\frac{a}{3}, -\frac{3a}{4}$. 5 $\frac{4a}{3}, -\frac{5a}{2}$.

6 $\frac{5a}{3}, -\frac{7a}{4}$. 7 $6, -13\frac{1}{2}$ 8. $\frac{1}{7}, -1$. 9. $3, \frac{1}{2}$. 10. $2\frac{2}{5}, 2\frac{2}{3}$.

11 $\frac{1}{2}(11\pm\sqrt{233})$ 12 $6, -1$ 13. 2, 6. 14. $3, 8\frac{1}{3}$.

15 $2, -\frac{1}{2}$ 16. $2\pm\sqrt{3}$ 17. $a\pm b$. 18. $\frac{-b\pm\sqrt{b^2-ac}}{a}$.

19 $\left(\frac{a+b}{a-b}\right)^{\pm 1}$ 20 $-b, b-2a$ 21 7, 16 22. $1\frac{1}{3}, -1\frac{1}{2}$.

23. $a, m+n$. 24. $-a, -b$. 25 $5, -2$. 26 5, 13
27. $5a, -2a$ 28 $\frac{1}{2}(11\pm\sqrt{33})$. 29. $4, 1\frac{2}{5}$. 30. $a-c, c-b$

31. $a+b, 2b$ 32. $1, \frac{2b}{a-b}$. 33. $3, -\frac{1}{2}$ 34 $3, 1\frac{10}{11}$.

35. $3, -1\frac{11}{13}$ 36 $5, -\frac{1}{3}$ 37. 12, 11. 38 $6, -\frac{1}{5}$.
39 $4, 2\frac{2}{3}$ 40. $5, -6$ 41 $4, -2$. 42 $3, 1\frac{1}{3}$. 43 $3, -\frac{4}{5}$
44. $2\pm\sqrt{6}$. 45. $1, -1\frac{2}{3}$ 46 $4, -2\frac{2}{3}$ 47 $6, 3\frac{1}{7}$

48 $6\pm\sqrt{601}$ 49 $3, 1\frac{1}{5}$ 50 $1, -2\frac{4}{7}$ 51. $\frac{1}{6}(5\pm\sqrt{61})$.

52 $4, \frac{10}{11}$ 53 $7, 3\frac{14}{15}$ 54 $4, \frac{12}{17}$ 55 $1\frac{7}{25}, -1$ 56 $\frac{b^2}{ac}$

57. $3a, \frac{3a}{2}$ 58 a, b 59. $a+b, \frac{1}{2}(a+b)$ 60 $a, \frac{1}{a}$

61 $1, -\frac{a+b+c}{b+c}$ 62. $1, \frac{a-b}{b-c}$ 63 $1, \frac{c(a-b)}{a(b-c)}$ 64 $\frac{b}{a+b}, -\frac{a}{a+b}$

65 $-1\pm\frac{\sqrt{a^2-1}}{a}$ 66 $\frac{1}{2}(-3\pm\sqrt{3})ab$ 67 $\frac{a}{b}, -\frac{b}{a}$ 68 $-a, -b$

69 $4, -1$ 70 $5, -\frac{5}{4}$ 71 $4, \frac{3}{2}$ 72 $3, -\frac{1}{8}$.

73 $a^2+b^2+c^2\pm\sqrt{2(a^4+b^4+c^4)}$ 74. $1, \frac{b+c-2a}{c+a-2b}$.

75 $0, \frac{1}{2}\{-(a+b+c)\pm\sqrt{a^2+b^2+c^2-2bc-2ca-2ab}\}$.

76. $\{bc+ca+ab\pm\sqrt{b^2c^2+c^2a^2+a^2b^2-2abc(a+b+c)}\}\div2(a+b+c)$

77 $\frac{1}{3}\{a+b+c\pm\sqrt{b^2+b^2+c^2-bc-ca-ab}\}$

78 $\{bc+ca+ab\pm\sqrt{b^2c^2+c^2a^2+a^2b^2-abc(a+b+c)}\}\div(a+b+c)$

79. $\pm a, \frac{a}{2}, -2a$ 80 $a+b, \frac{ab(a+b)}{a^2+b^2}$

81 $0, \pm\sqrt{\frac{1}{2}(b^2+bc+ab+2ac)}$. 82 $a, -\frac{b^2+c^2}{b+c}$

83 $1, -\frac{2ab}{a^2+2ab-b^2}$ 84 1, 3. 85. 15, 7 86 13, 193

87 $5, -\frac{4}{3}$ 88 $9, -3\frac{3}{5}$ 89 $4, -3\frac{1}{5}$ 90 a, b 91 $\frac{7}{5}a, -a$.

Appendix [*pp* 419—434]

1 8 2 $(x^2-4)^2(x+3)^2$ 3 $5(x^2-13x-1)$. 4 $(a-1)x+a$

10 $x+4-8x^{-1}$ 11 $\sqrt{2x-3}+\sqrt{x+2}$. 12 $2\frac{1}{2}$ 13. $x=8\frac{4}{5}$, $y=-11$ 15 £10 16 94 17. $2x-\frac{1}{2x}$ 18 $x(x+1)\div(x^2+4x+1)$

23. 18 24 $3\sqrt{5}-2\sqrt{3}$ 25 4. 26 $x=\frac{ac}{a+b}$; $y=\frac{bc}{a+b}$ 28 35

29. 1. 31 $2a^6-6a^5, 0$ 32 $2\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}+\frac{1}{abc}\right)$ 33 $bx-ay$

37 $\dfrac{2a}{\sqrt{(x+a)}}$. 39. $\frac{5}{9}$. 40. $x=b-c$, $y=c-a$, $z=a-b$. 42. 90; 670, 55 43 $2\frac{1}{2}$ 44. $4a^2-9b^2+24bc-16c^2$ 45 $(1+x)(1-x)$ $\{1+y+(1-y)x\}\{1+y-(1-y)x\}$ 51. $\frac{1}{6}$. 52 $x=\dfrac{(d-b)(d-c)}{(a-b)(a-c)}$; similar values for y and z 54. 100 lbs; 2 cwt., 3 cwt 55 a. 56. $(a-b)(x-c)^2(2x-a-b)$. 59 $\frac{3}{4}(3^{\frac{4}{3}}+3^{\frac{2}{3}}+1)$. 60 1 61. 1. 62 $\dfrac{1}{abcdf}$. 64. 6 65 $x=a$, $y=b$, $z=c$. 67 72; 60; 30 68 2 213...69 $(ax+by-1)(bx+ay+1)$ 70. $(2a)^4$. 72. $\dfrac{x^2-3x+1}{x^2-4x+1}$. 73. $\dfrac{a^3(a+x)}{x(a^2+ax+x^2)}$ 74. 1 77 $a^x b^y$ 78 $1\frac{3}{4}$ 79. $x=2\frac{1}{2}$; $y=1\frac{1}{2}$. 81. 5 miles per hr.; 15 miles 82 162. 83. $4(ax+by+cz)$. 84. $x(x+y)(x+2y)$ 85. $(x^6-1)^2=$&c 86. $(x-1)(x-2)(x-3)$. 88. $\dfrac{1}{(\sqrt{x}+1)\sqrt{(x+1)}}$. 89 a^3-3ab^2, $a^4-4a^2b^2+2b^4$. 92. 1 93 $x=3$, $y=4$ 95. $10\frac{5}{13}$ hours 96. 3 97. $(a-1)x^3-3ax^2-a^2(a-1)x-(a^3+2a^2+2a+1)$ 99 $(a+b)(a-b)(a^2+b^2)(x+y)\times(x-y)(x^2+y^2)$ 100. See § 180 101. $\dfrac{a}{bc}+\dfrac{b}{ca}+\dfrac{c}{ab}$ 103. $\dfrac{x^2}{(x-a)^n}$. 104 1. 106 $\dfrac{\sqrt{(a-x)}}{\sqrt{a}+\sqrt{x}}$. 108 $\dfrac{3a}{a-1}$ 109 $x=\frac{1}{2}(2a+b)$, $y=\frac{1}{2}(2a-b)$. 112 ·5 113. $\frac{11}{5}a-2b$ 114. $4z(x+y-xy)(xyz^2+xyz+1)$ 115. $x^2+x(y+z)+yz$ 116 $\frac{7}{2}\sqrt{3}$ 117. $\dfrac{1-x^2}{2x^2-1}$. 121. $\frac{1}{3}(a+b+c)$ 122. $x=y=1$, $z=0$ 124 7 : 5 125. 68. 126 $a^2-3ab+b^2$. 127. $3(x-a)(x-b)(x-c)(c-a)(a-b)(b-c)$. 128. a. 129 $\frac{1}{2}(3x-y)$. 130 $x^2-\dfrac{3a^2x}{4b}$. 131 0. 133 b^4. 134. $\dfrac{ac^2}{b^2}$. 135. $x=\dfrac{a}{m}\dfrac{a^2+b^2}{a^2-b^2}$, $y=\dfrac{b}{m}\dfrac{a^2+b^2}{a^2-b^2}$. 137. 240 apples; 60 oranges, worth $1\frac{1}{2}d$ each. 138 $(m^2-4n^2)(2x-y)$; 9 139 $\dfrac{3x^2-ax+a^2}{2x^2+a^2}$. 140. $\frac{1}{4}x^5-\frac{4}{3}x^4+\frac{77}{24}x^3-\frac{43}{12}x^2-\frac{11}{4}x+9$ 141. $(2x-11y+1)(x+2y-3)$. 142 1 144 $\dfrac{a^{-m}}{b^{-3p}}-\dfrac{c^{-2n}}{d^{-6q}}$. 146. $-\frac{5}{8}$ 147. $x=bc(b-c)$, $y=ca(c-a)$, $z=ab(a-b)$.

149 $50, 120$. 150 550 151 $x^8-2x^7a+2x^6a^2-2x^4a^4+2x^3a^5-2xa^7+a^8$

152 $9a^2+6ab+4b^2$ 153 $x^3-14x^2y+49xy^2-36y^3$.

154 $3\sqrt{\frac{x}{y}}-4+3\sqrt{\frac{y}{x}}$ 156 $\frac{2a}{b}, 2\sqrt{\left(\frac{a^2}{b^2}+1\right)}$.

158 $\frac{1}{2}$ 159 $x=a-b,\ y=b-c,\ z=c-a$

161 A, 4s 8d, B, 6s 3d 163 $x^4+\frac{x^3}{y}+\frac{x^2}{y^2}+\frac{x}{y^3}+\frac{1}{y^4}$ 164 10

166 $9x^2+33x+19$ 167 $a+b+c$ 168 $\frac{5}{3}\sqrt{3}-2$ 171 $c-b-a$

172 $\frac{2x}{a+2b+c}=\frac{2y}{a+b+2c}=\frac{2z}{2a+b+c}=a+b+c$ 174 $2\frac{1}{2}$ miles per hour

175 $(4a^2-9c^2)y^4-2acy^3-(a^2-16ac-c^2)y^2+2(2a^2+c^2)y-(4a^2-c^2)$

176 $\frac{5}{18}$ 177 $-9(b-c)(c-a)(a-b)$ 178 $a^2+b^2+c^2+bc+ca+ab$

179 $\frac{xe^y+ye^x}{xe^x+ye^y}$ 180 y^5-5y^3+5y. 183 $5\frac{1}{2}$ 184. $x=\frac{4}{5}a,\ y=\frac{5}{4}a$

185 30 minutes 186 $1+(a+b+c)x+(a^2+b^2+c^2+bc+ca+ab)x^2+\ldots$

187 $\frac{x-q}{qx+p}$ 191 $\left\{\frac{mn(a+b)}{ab(m+n)}\right\}^{12}$ 193 $a+b$ 194 $x=8,\ y=13$

195 36 196 $ad-bc$ 197 $(a^2-b^2)x^4+2ab^2x^3y+(2a^2+2b^2-a^2b^2)x^2y^2$
$-2ab^2xy^3+(a^2-b^2)y^4$ 198 1 199 $(2x-y-z)(2y-z-x)(2z-x-y)$

200 $x^2+y^2+z^2+yz+zx+xy$ 201. 0.

202 a^3 or $\frac{1}{a^2}$ 204 (1) $\frac{a+1}{2b-1}$, (2) 2

205 $x=a^{\frac{n-2m}{n-m}}b^{\frac{n}{n-m}},\ y=a^{\frac{n}{n-m}}b^{\frac{n-2m}{n-m}}$ 207 16 and 20 tons.

209 $(\sqrt{x}+\sqrt{y}+\sqrt{z})(\sqrt{y}-\sqrt{z}-\sqrt{x})(\sqrt{z}-\sqrt{x}-\sqrt{y})(\sqrt{x}-\sqrt{y}-\sqrt{z})$

210 $\frac{1-x^2}{1+x^2}$. 211 $\frac{(b+c)(c+a)(a+b)}{bc+ca+ab}$ 212 $\frac{3}{2}x^{\frac{2}{m}}-\frac{1}{3}a^{-\frac{1}{n}}x^{-\frac{1}{n}}$.

214 $\left\{\frac{\sqrt{(x+y)}-\sqrt{(x-y)}}{\sqrt{(z+y)}-\sqrt{(z-y)}}\right\}^{\frac{3}{2}}$ 215 $\frac{a\sqrt{(b-4\sqrt{b})}}{\sqrt{b}-2}$

216 $x=a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}c^{\frac{m}{m^2-n^2}}d^{\frac{n}{n^2-m^2}},\ y=a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}c^{\frac{n}{n^2-m^2}}d^{\frac{m}{m^2-n^2}}$

218 1 mile 220 $(a^2-bc)(b^2-ca)(c^2-ab)$ 221 $1+xyz$.

222 $\frac{1}{abc}$ 223 $\frac{x}{1-x^2}$ 226 $-a$ 227 $-1\frac{1}{3}$ 229 $\frac{7}{8}(x^2+xy+y^2)$

231 $\frac{a^2+2ac-c^2}{2\sqrt{(2ac-c^2)}}$ 232 Since left side $=3(x-a)(x-b)(x-c)(b-c)$
$\times(c-a)(a-b)$ [see *App*, Ex 127], we get $x=a$ or b or c

234 1, 2, or 3, 4

UNIVERSITY PAPERS.

CALCUTTA UNIVERSITY.

Selections from the University Questions [1858—1890].

1. Explain the rule for the signs in algebraical multiplication, and multiply $7x^{\frac{1}{2}}-3y^{\frac{1}{2}}+2x^{\frac{1}{3}}y^{\frac{2}{3}}$ by $6x^{\frac{1}{3}}-2y^{\frac{2}{3}}+7x^{\frac{2}{3}}y^{\frac{1}{3}}$

[*Ans* $42x^{\frac{5}{6}}-18x^{\frac{1}{3}}y^{\frac{1}{2}}-9x^{\frac{2}{3}}y^{\frac{2}{3}}-14x^{\frac{1}{2}}y^{\frac{2}{3}}+6y-4x^{\frac{1}{3}}y^{\frac{4}{3}}+49x^{\frac{7}{6}}y^{\frac{1}{3}}+14xy$.

2 *A* and *B* can do a piece of work in 30 days, *A* and *C* in 40 days, and *B* and *C* in 50 days All three work together for 10 days If then two be taken away, how long will each of the others take to finish it? [See § 232] [*Ans* *A* $31\frac{17}{23}$, *B* $42\frac{16}{17}$ and *C* $104\frac{2}{7}$ days

3 If $m : n :: p : q$, prove that

$$\frac{(m-n)(m-p)}{m}=(m+q)-(n+p)$$ [*Mis Ex.* viii, *Ex* 3]

4 In a right-angled triangle, the base is 8, and the sum of the hypotenuse and perpendicular is 12, it is required to find them

[*Ans* Hyp $=8\frac{2}{3}$, perp $=3\frac{1}{3}$

5 A person has two horses, and a saddle worth 75 rupees if the saddle be put on the *first* horse, his value becomes *double* that of the *second*, but if the saddle be put on the *second* horse, *his* value will not amount to that of the *first* horse by 350 rupees What is the value of each

[*Ans* Val of first horse = Rs 925, val of second horse = Rs 500.

6 There are three numbers, such that the *sum* of the first and second divided by their product is $\frac{1}{2}$, the sum of the second and third divided by their product is $\frac{1}{3}$, and the sum of the first and third divided by their product is $\frac{1}{4}$ Find the numbers [*Ans* $4\frac{4}{5}$, $3\frac{3}{7}$, 24

7 Shew that

$$\{(ax+by)^2+(ay-bx)^2\}\times\{(ax+by)^2-(ay+bx)^2\}=(a^4-b^4)(x^4-y^4)$$

8 Divide $x^6+2x^3y^3+y^6$ by $(x+y)^2$. [*Ans.* $(x^2-xy+y^2)^2$.

9 Solve the equation

$$\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c}.$$ [*See* § 224, *Ex.* 80]. [*Ans* $\frac{ab}{a+b-c}$.

10 Divide $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{1}{2}}+y$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$, [*Ans* $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.
and simplify the expressions

$$\frac{a+c}{(x-a)(b-a)}+\frac{b+c}{(x-b)(a-b)},$$ [*Ans* $\frac{x+c}{(a-x)(x-b)}$.

and $$\frac{a^4-b^4}{a^2-2ab+b^2}\times\frac{a-b}{a(a+b)}.$$ *Ans.* $\frac{a^2+b^2}{a}$.

11 Solve the following equation —

$$\frac{2x+11}{x+5}-\frac{9x-9}{3x-4}=\frac{4x+13}{x+3}-\frac{15x-47}{3x-10}.$$ [*Ans.* $-\frac{5}{6}$.

12. Reduce $\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}$ to the form $\frac{1}{a^4-x^4}$.

13 Multiply $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$ [*Ans* $x^{\frac{3}{2}}-2xy^{\frac{1}{2}}+2x^{\frac{1}{2}}y-y^{\frac{3}{2}}$.

14 Square $a^{\frac{1}{5}}-b^{\frac{1}{7}}+c^{\frac{1}{3}}$, and divide 1 by $(a+b)^2$ giving three terms of the quotient

[*Ans* $a^{\frac{2}{5}}+b^{\frac{2}{7}}+c^{\frac{2}{3}}-2a^{\frac{1}{5}}b^{\frac{1}{7}}+2a^{\frac{1}{5}}c^{\frac{1}{3}}-2b^{\frac{1}{7}}c^{\frac{1}{3}}$, $a^{-2}-2a^{-3}b+3a^{-4}b^2$.

15 What fraction is that which, if 1 be added to the numerator, becomes 1, and if 1 be added to the denominator, becomes $\frac{1}{2}$? [*Ans* $\frac{2}{3}$

16 Prove that $\frac{x+y}{y}-\frac{x}{x+y}-\frac{x^3-x^2y}{x^2y-y^3}=1$

17 A post is a fourth of its length in the mud, a third of its length in the water and 10 feet above the water, what is its length? [*Ans* 24 ft

18 Add together $x^2-(x-y+z)(x+y-z)$, $y^2-(y-x+z)(y+x-z)$, and $z^2-(z-x+y)(z+x-y)$ [*Ans* $2(x^2+y^2+z^2-yz-zx-xy)$.

19 Multiply $x+y+z-\sqrt{(xy)}-\sqrt{(yz)}-\sqrt{(xz)}$ by $\sqrt{x}+\sqrt{y}+\sqrt{z}$, and divide $x^8+a^4x^4+a^8$ by x^2-ax+a^2

[*Ans* $x^{\frac{3}{2}}+y^{\frac{3}{2}}+z^{\frac{3}{2}}-3x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}$, $(x^4-a^2x^2+a^4)(x^2+ax+a^2)=$&c

20 Simplify the expression

$$\frac{1}{2}\,\frac{1}{x-1}-\frac{x-5}{x^2-7x+10}+\frac{1}{2}\,\frac{x-6}{x^2-9x+18}$$ [*Ans* $\frac{1}{(x-1)(x-2)(x-3)}$.

21 Divide the continued product of $1+x+y$, $1+x-y$, $1-x+y$, and $x+y-1$ by $1+2xy-x^2-y^2$, and resolve into four factors $4(uz-xy)^2-(u^2-x^2-y^2+z^2)^2$

[*Ans.* $(x+y)^2-1$, $(u+x+y+z)(u+x-y-z)(u+z-x-y)(x-y+z-u)$.

22 Find the Greatest Common Measure of

$2x^5-11x^2-9$ and $4x^5+11x^4+81$ [§ 155, *Ex* 73],

and reduce $\frac{x^3-6x^2-37x+210}{x^3+4x^2-47x-210}$ to its lowest terms [*Ans* $\frac{x-5}{x+5}$

23 Simplify as much as possible *any one* of the following—

(1). $\frac{x^3}{(x-y)(x-z)}+\frac{y^3}{(y-x)(y-z)}+\frac{z^3}{(z-x)(z-y)}$ [*Ans* $x+y+z$.

(2). $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-z)(y-x)}+\frac{1}{z(z-x)(z-y)}$ [*Ans* $\frac{1}{xyz}$.

(3) $\frac{x^2-yz}{(x-y)(x-z)}+\frac{y^2+zx}{(y+z)(y-x)}+\frac{z^2+xy}{(z-x)(z+y)}$ [*Ans* 0.

24 Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$ [§ 183, *Ex.* 1]

25 Solve *any two* of the following equations :—

(1) $\frac{x-a}{b}+\frac{x-b}{c}+\frac{x-c}{a}=\frac{x-(a+b+c)}{abc}$. [*Ans* $\frac{ab^2+bc^2+ca^2-a-b-c}{bc+ca+ab-1}$.

(2) $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0.$ [*Ans.* $\frac{8a}{25}$.

(3) $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$ [*Ans* $4\frac{1}{2}$

(4) $\left(\frac{a^2}{x}+b\right)^{\frac{1}{2}}-\left(\frac{a^2}{x}-b\right)^{\frac{1}{2}}=c^{\frac{1}{2}}$ [*Ans.* $\frac{4a^2c}{4b^2+c^2}$.

26 Divide x^6-x^{-6} by $x-x^{-1}$, and find the continued product of $ax-1$, $x^2+\frac{1}{a^2}$ and $ax+1$ [*Ans.* $x^4+x^2+1+x^{-2}+x^{-4}$, $a^2x^4-\frac{1}{a^2}$.

27 Simplify the following :—

(1). $\frac{x+3y}{4(x+y)(x+2y)}+\frac{x+2y}{(x+y)(x+3y)}-\frac{x+y}{4(x+2y)(x+3y)}$. [*Ans* $\frac{1}{x+y}$.

(2). $\frac{a^2+3a+2}{a^2+2a+1}\times\frac{a^2+5a+4}{a^2+7a+10}$. [*Ans* $\frac{a+4}{a+5}$.

28 Extract the cube root of

$x^6+6x^5+21x^4+44x^3+63x^2+54x+27$. [*Ans.* x^2+2x+3.

29 *A* is twice as old as *B* and 4 years older than *C*. The sum of the ages of *A*, *B* and *C* is 96 years Find *B*'s age. [*Ans.* 20 years

30 Divide $(x+y+z)(xy+xz+yz)-xyz$ by $x+y$ [*Ans.* $(y+z)(z+x)$.

31. Reduce to its simplest form

$\frac{x^2-(y-z)^2}{(x+z)^2-y^2}+\frac{y^2-(x-z)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}$ [*Ans* 1.

32. Extract the square root of

$$\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12 \quad [\S\ 198,\ Ex\ 6]$$

33 Simplify $\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)-\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$; [*Ans.* $\frac{xy}{x^2+y^2}$.

or shew that

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

34 Prove either of the identities [*See App Ex* 1]

$(ay-bx)^2+(cx-az)^2+(bz-cy)^2=(x^2+y^2+z^2)(a^2+b^2+c^2)-(ax+by+cz)^2$;

$16s(s-a)(s-b)(s-c)=2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$, where $2s=a+b+c$.

35 Solve the equation

$$\frac{\sqrt{x}+a}{(\sqrt{x}-b)(\sqrt{x}-c)}+\frac{\sqrt{x}+b}{(\sqrt{x}-a)(\sqrt{x}-c)}+\frac{\sqrt{x}+c}{(\sqrt{x}-a)(\sqrt{x}-b)}=0$$

[*Ans* $\frac{1}{3}(a^2+b^2+c^2)$

36 Extract the square root of —

$x^6+8x^4-2x^3+16x^2-8x+1$ [§ 198, *Ex* 18],

or $a^4+b^4+c^4+d^4-2(a^2+c^2)(b^2+d^2)-2a^2c^2+2b^2d^2$

[*Ans* $a^2-b^2+c^2-d^2$.

37 Extract the square root of

$a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd$ [§ 197, Ex (ii) 2]

38 Simplify $\left(\frac{a+b}{a-b}+\frac{a^2+b^2}{a^2-b^2}\right)-\left(\frac{a-b}{a+b}-\frac{a^3-b^3}{a^3+b^3}\right)$ [*Ans.* $\frac{a^4+a^2b^2+b^4}{-ab(a-b)}$.

and shew that $1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2=\frac{(a+b+c)(a+c-b)(b+c-a)(a+b-c)}{4b^2c^2}$.

39. Solve the equations —

(1) $\frac{4x+3}{9}+\frac{29-7x}{12-5x}=\frac{8x+19}{18}$ (2) $\frac{8x+4}{\sqrt{(x+5)}}=4\sqrt{(x+5)}$

[*Ans* (1) 6, (2) 4.

40 There is a number, the sum of whose digits is 5, and if 10 times the digit in the place of tens be added to 4 times the digit in the place of units, the number will be inverted What is the number? [*Ans* 23

41. Divide $x^3+y^3+3xy-1$ by $x+y-1$ [§ *119b*, *Ex. 2*]

42. Simplify $\frac{x}{x-a}-\frac{x}{x+a}-\frac{\frac{x+a}{x-a}-\frac{x-a}{x+a}}{\frac{x+a}{x-a}+\frac{x-a}{x+a}}$. [*Ans* $\frac{4a^3x}{x^4-a^4}$

43 Solve the equation

$\frac{a}{x}+\frac{b}{y}=m$, $\frac{b}{x}+\frac{a}{y}=n$ [*See* § 241, *Ex* 27] [*Ans* $\frac{a^2-b^2}{am-bn}, \frac{a^2-b^2}{an-bm}$

44 A labourer is engaged for 30 days, on condition that he receives 2*s* 6*d* for each day he works, and loses 1*s* for each day he is idle: he receives 2*l* 7*s* in all How many days does he work, and how many days is he idle? [*See* § 148, *Ex* 38] [*Ans* 22 days, 8 days

45 Divide

$x^8+x^6y^2+x^4y^4+x^2y^6+y^8$ by $x^4-x^3y+x^2y^2-xy^3+y^4$

[*Ans.* $x^4+x^3y+x^2y^2+xy^3+y^4$

46 Prove that

$$\frac{1}{1+\frac{1}{a-x}}+\frac{1}{1-\frac{1}{a+x}}+\frac{2}{1+\frac{1}{a^2-x^2}}=\frac{4a^4}{a^4-x^4};$$

and shew that the notation $\frac{\frac{a}{b}}{c}$ is of ambiguous meaning.

47. Simplify the expressions—

$$\frac{x^{a+b}\,x^{a-b}\,x^{c-2a}}{x^{c-a}};\quad \frac{\frac{1+x}{1-x}+\frac{4x}{1+x^2}+\frac{8x}{1-x^2}-\frac{1-x}{1+x}}{\frac{1+x^2}{1-x^2}+\frac{4x^2}{1+x^4}-\frac{1-x^2}{1+x^2}},$$

[*Ans.* x^a; $(2+x^2)(1+x^4)-x$.

and reduce $\frac{2x^4-x^3-9x^2+13x-5}{7x^3-19x^2+17x-5}$ to its lowest terms

[*Ans.* $\frac{2x^2+3x-5}{7x-5}$.

48. AB is a railway 220 miles long; and three trains (P, Q, R) travel upon it at the rates of 25, 20 and 30 miles per hour respectively, P and Q leave A at 7 A.M. and 8-15 A.M. respectively, and R leaves B at 10 30 A.M. When and where will P be equidistant from Q and R? [§ 236, *Ex* (*vii*)]

49 Find the square root of $x^4-2x^2-\frac{2}{x^2}+\frac{1}{x^4}+3$ [*Ans* $x^2-1+\frac{1}{x^2}$.

50 Reduce $\frac{10x^3+19x^2-9}{25x^3-19x+6}$ to its lowest terms, and find the least common multiple of $2(x-2)^2$, $2x^2-8$, x^3+2x, $2x^2-4x$

[*Ans.* $\frac{2x+3}{5x-2}$, $2x(x-2)(x^2-4)(x^2+2)$

51 Simplify :—

$$\left(1-\frac{1}{1+x}\right)\left(x+\frac{1}{2+x}\right)\times\frac{\frac{1}{x^2}-x}{1+\frac{1}{x}}-\left(1+x+\frac{1}{x}\right)$$ [*Ans.* $\frac{x(1-x)}{2+x}$

52 A and B compared their monthly incomes and found that A's income was to that of B as 7 to 9, and that the third of A's income was Rs 30 greater than the difference of their incomes. Find what each received [*Mis. Ex* *viii. Ex* 4 (i)]

53 Simplify

(1) $\left\{\frac{2a}{x^2-a^2}-\frac{1}{x-a}+\frac{2}{x+a}\right\}\times\frac{x^2}{x-a+\frac{a^2}{x}}$. [*Ans.* $\frac{x^3}{x^3+a^3}$.

(ii) $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-x)(y-z)}+\frac{1}{z(z-x)(z-y)}$. [§ *180, Ex 8*]

54 Rs 1100 are so divided among A, B and C, that if A were to give B Rs 200, B would then have twice as much as A, and three times as much as C How many rupees did A, B and C each receive originally?

[*Ans* A, Rs 500 ; B, Rs 400 , C, Rs 200

55 Find the L C.M of $3ax^2-3a^2x$, x^2-a^2, x^2-ax, $\sqrt{(3ax)}$, $\sqrt{x}-\sqrt{a}$

[*Ans* $3ax(x^2-a^2)$

56 Assuming $\frac{a+b-c}{a+b}=\frac{b+c-a}{b-c}=\frac{-ca-b}{c-a}$, and that $a+b+c$ is not$=0$, shew that $a=b=c$ [*App Ex 8*]

57 Two persons started at the same time from A One rode on horse-back at the rate of $7\frac{1}{2}$ miles an hour and arrived at B 30 minutes later than the other, who travelled the same distance by train at the rate of 30 miles an hou Find the distance between A and B. [See § 233]

[*Ans.* 5 miles.

58 Simplify

(i) $\frac{x^{m+2n}x^{3m-8n}}{x^{5m-6n}}$ (ii) $\frac{a}{a+b}-\frac{a+b}{2b}+\frac{a^2+b^2}{2b(a-b)}$.

(iii) $\frac{\frac{a^3}{b^3}-\frac{b^3}{a^3}}{\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}-\frac{b}{a}-1\right)}\times\frac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{ab}}$. (i) x^{-m}. (ii) $\frac{a^2+b^2}{a^2-b^2}$ (iii) $a-b$

59 The expression $ax-3b$ is equal to 30 when x is 3, and to 42 when x is 7 , what is its value when x is $4\frac{2}{3}$ and for what value of x is it zero?

[*App Ex. 22*]

60 A certain number consists of two digits ; the left-hand digit is double the right-hand digit, and if the digits be inverted the ratio of the number thus formed to 60 is 4 · 5 Find the number [*See* § 231] [*Ans* 84.

61 Find the G C.M of $6x^4+x^3-6x^2-5x-2$ and $2x^4+3x^3+2x^2-7x-6$ [*Ans.* $2x^2-x-2$

62 Shew that if $\frac{a-b}{c}+\frac{b-c}{a}-\frac{c-a}{b}=1$ and $a-b+c$ is *not*$=0$,

then $\frac{1}{a}=\frac{1}{b}-\frac{1}{c}$ [*App Ex* 9]

63 How many bundles of hay at Rs. 5 per thousand must a *ghaswala* mix with 5600 bundles at Rs. 6 per thousand in order that he may gain 20 per cent by selling the whole at 11 annas per hundred [*Ans* 2080

64. Simplify —

$$\frac{\frac{a}{a-b}-\frac{a}{a+b}}{\frac{b}{a-b}-\frac{b}{a+b}}\cdot\frac{\frac{a+b}{a-b}-\frac{a-b}{a-b}}{\frac{a+b}{a-b}-\frac{a-b}{a+b}}\times\frac{a^2}{a^2+b^2}.$$ [*Ans.* $\frac{2a^4}{(a^2-b^2)^2}$

65 Find the square root of $4-4c+2b+c^2-bc+\dfrac{b^2}{4}$ [§ 197, *Ex* (*ii*), 4]

66. A can do a piece of work in 9 days, B in twice that time, C can only do $\frac{3}{4}$ as much as A in a day; how long would A, B and C, working together require to do the same piece of work? [*See* § 232] [*Ans* 4 days

67 Find the Greatest Common Measure of $x^4+x^3-11x^2-9x+18$ and $x^4-10x^3+35x^2-50x+24$ [*Ans* x^2-4x+3

68 Find the first four terms of the square root of a^2+x^2, and from the result deduce the square root of 101 correct to six places of decimals [*App*, *Ex* 12]

69 Two passengers have together 5 cwt of luggage, and are charged for the excess above the weight allowed 5*s* 2*d* and 9*s* 10*d* respectively; but if the luggage had all belonged to one of them he would have been charged 19*s* 2*d* How much luggage is each passenger allowed to carry free of charge, and how much luggage had each passenger? [*App* *Ex.* 54]

70 Divide $x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)+4xyz$ by $1+xy+yz+zx$. [*Ans* $x+y+z+xyz$

71 Extract the square root of [*App*, *Ex* 1.]

$$(a^2+b^2+c^2)(x^2+y^2+z^2)-(bz-cy)^2-(cx-az)^2-(ay-bx)^2$$

72 Solve the equations —

(*a*) $\sqrt{4x^2+20x+17-\sqrt{16x^2+11x+10}}=2(x+2)$.

(*b*) $\dfrac{4x+3}{9}+\dfrac{13x}{108}=\dfrac{8x+19}{18}$. [*Ans.* (*a*) -3; (*b*) 6

73 Simplify the expression $\dfrac{1}{(4x^3-3x)^2}-\left\{\dfrac{3\dfrac{\sqrt{1-x^2}}{x}-\dfrac{(1-x^2)^{\frac{3}{2}}}{x^3}}{1-3\left(\dfrac{1-x^2}{x^2}\right)}\right\}^2$.

[*Ans* 1

74 Multiply $a^{2n}-a^nx^n+x^{2n}$ by a^n+x^n, and find the greatest common measure of $x^2+\frac{7}{6}x+\frac{1}{3}$ and $x^2+\frac{2}{3}x+\frac{1}{12}$. [*Ans* $a^{3n}+x^{3n}$; $x+\frac{1}{2}$

75 Divide $x^{2n}-y^{2n}$ by $x^{2n-1}+y^{2n-1}$ [*App*, *Ex* 13]

76. Solve the equations —

(*a*) $x-k+\sqrt{k^2+x^2}=m$.

(*b*). $\begin{cases} a^xa^{y+1}=a^7 \\ a^{2y}a^{3x+5}=a^{20}. \end{cases}$ (*c*). $\begin{cases} \dfrac{4}{5}+\dfrac{10}{y}=2 \\ \dfrac{3}{x}+\dfrac{2}{y}=\dfrac{19}{20}. \end{cases}$ [*Ans* (*a*) $\dfrac{m(m+2k)}{2(m+k)}$.

(*b*) $x=y=3$

(*c*) 4, 10.

77 Two armies number 11000 and 7000 men respectively; before they fight, each is reinforced by 1000 men: in favour of which army is the increase? [*Ans* The latter

78 Simplify $\left\{\frac{1}{a}+\frac{2x^2}{a(b-x)}\right\}\left\{\frac{a}{x}-\frac{2ax}{x(b+x)}\right\}$ [*Ans* 1

79 Extract the square root of—

$x^{\frac{8}{6}}-2a^{-\frac{3}{6}}x^{\frac{11}{6}}+2a^{\frac{4}{6}}x^{\frac{4}{6}}+a^{-\frac{6}{6}}x^{\frac{14}{6}}-2a^{\frac{1}{6}}x^{\frac{7}{6}}+a^{\frac{8}{6}}$ [*Ans* $a^{-\frac{3}{6}}x^{\frac{7}{6}}-x^{\frac{4}{6}}-a^{\frac{4}{6}}$.

80 A boat goes up stream 30 miles and down stream 44 miles in 10 hours it also goes up stream 40 miles and down stream 55 miles in 13 hours Find the rate of the stream and of the boat [§ *246*, *Ex* *7*]

81 What do you mean by a negative quantity ?

Prove that $a-(b-c)=a-b+c$

82 Simplify $\frac{1}{abx}+\frac{1}{a(a-b)(x-a)}+\frac{1}{b(b-a)(x-b)}$ [*Mis Ex VI, Ex 61*]

83 A man receives $\frac{x}{y}$ths of 10 Rs and afterwards $\frac{y}{x}$ths of 10 Rs He then gives away 20 Rs Shew that he cannot lose by the transaction [*App*, *Ex* *14*]

84 What is an *equation* ? Prove that a simple equation has only one root

85 Solve the equations —

(1) $\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}=3$ [*Ans* 5

(2) $\frac{4\cdot05}{9x}-\frac{3}{8-2x}=\frac{1\cdot8}{x}-\frac{3\cdot6}{2\cdot4-6x}$ [*Ans.* ·3.

86 *A* challenged *B* to ride a bicycle race of 1040 yards He first gave *B* 120 yards start, but lost by 5 seconds, he then gave *B* 5 seconds start and won by 120 feet How long does each take to ride the distance ? [§ *246*, *Ex* *7* (*1*)]

87 Divide x^n-a^n by $x-a$ [§ *136*, *Ex* *7*]

88 Find the highest common factor of $x^3-7x^2-80x+576$ and $3x^2-14x-80$, and the lowest common multiple of these two expressions and $3x^2+17x-90$ [*Mis Ex V*, *Ex* *27*]

89 Solve the equations

(i) $65x+\frac{585x-975}{6}=\frac{1\cdot56}{2}-\frac{\cdot39x-78}{9}$ [*Ans.* 5

(ii) $\left.\begin{aligned}\frac{x-2}{2}-\frac{x+y}{14}&=\frac{x-y-1}{8}-\frac{y+12}{4}\\ \frac{x+7}{3}+\frac{y-5}{10}&=1-x-\frac{5(y+1)}{7}\end{aligned}\right\}$ [*Ans* 8, −15

90 The distance from a place *P* to another place *Q* is $3\frac{1}{2}$ miles two persons *A* and *B* start together from *P* to go to *Q*, the former by carriage which travels at the rate of 6 miles an hour, the latter walking at the rate of 3 miles an hour If *A* remains at *Q* for 15 minutes and then returns by the carriage to *P*, find where he will meet *B* [*See* § 233]

[*Ans* $2\frac{5}{8}$ miles from *P*.

91 Divide $(a-b)^2c^2+(a-b)c^3-(c^2-a^2)b^2+(c-a)b^3$ by

$(a-b)c^2-(c-a)b^2$ [§ 133. Ex 22]

92 Find the value of $\dfrac{x^2-y^2+x}{y^2-x^2+y}$, when $x=\dfrac{a-b}{a+b}$ and $y=\dfrac{a+b}{a-b}$

[*Mis Ex VI., Ex. 36*].

93 Solve the equations —

(*a*) $x^2+y^2=a^2,\ xy=b^2$. [*See* § 245, *Ex* 4],

[*Ans* $\frac{1}{2}\{\sqrt{(a^2+2b^2)}\pm\sqrt{(a^2-2b^2)}\}$

(*b*) $\dfrac{x}{\cdot 5}-\dfrac{1}{\cdot 05}+\dfrac{x}{\cdot 005}-\dfrac{1}{\cdot 0005}=0$ [*Ans* 10

94 Reverse the digits of a number and it will become five-sixths of what it was before; also the difference between the two digits is one. Find the number Also find that number of three digits which is the same when reversed, and the sum of whose digits is 16 and the difference 2 [*See* § 231] [*Ans.* 54, 646.

95 Find the co-efficient of x^4 in the product of

$x^4-ax^3+bx^2-cx+d$ and x^2+px+q. [*Mis Ex III., Ex. 34*].

96 Find the value of

(i) $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$. [§ 174., *Ex. 63*]

(ii) $\left(\dfrac{x}{x-1}\right)^2+\left(\dfrac{x}{x+1}\right)^2$ when $x=\sqrt{\dfrac{n-1}{n+1}}$ [*Ans* $n(n-1)$.

97. Solve the following equations :—

(i) $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$; (ii) $\dfrac{ax-1}{\sqrt{ax}+1}=4+\dfrac{\sqrt{ax}-1}{2}$;

(iii) $(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}=(2)^{\frac{1}{3}}$, [*See* § 225, *Ex* 42]

[*Ans.* (i) $\dfrac{1}{1-a}$, (ii) $\dfrac{81}{a}$; (iii) 1.

98 Find to 4 terms the square root of $1-x-x^2$.

[*Ans.* $1-\frac{1}{2}x-\frac{5}{8}x^2-\frac{5}{16}x^3$.

99 Two sums of money are together equal to £54. 12*s*, and there are as many pounds in the one as shillings in the other What are the sums ?

[*Ans.* £52, 52*s*

100 A boy buys a certain number of oranges at 3 for 2*d*, and one-third of that number at 2 for 1*d*; at what price must he sell them to get 20 per cent profit? If his profit be 5*s*. 4*d*, find the number brought [§ 236, *Ex* (*ii*)]

101. Extract the square root of $\dfrac{(a^2+b^2)^2}{a^4+b^4-2a^2b^2}+4\dfrac{a}{a+b}\times\dfrac{b}{a-b}$

[§ 198, *Ex. 7.*]

102 Solve the equation $16\left(\frac{a-x}{a+x}\right)^3=\frac{a+x}{a-x}$. [*Ans.* $\frac{1}{3}a$.

103 Express $(x+3a)(x+5a)(x+7a)(x+9a)$ as the difference of two square quantities. [*127, Ex* (*V*)]

104 Simplify $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$

[*Mis Ex VI., Ex 62*]

105 Solve the following equations —

(1) $\frac{x-2}{x-3}+\frac{x-3}{x-4}=\frac{x-1}{x-2}+\frac{x-4}{x-5}$ [*See* § *227, Ex 62*], [*Ans* $3\frac{1}{2}$

(2) $x-2y+z=0,\ 9x-8y-3z=0,\ 2x+3y+5z=36$ [*See* § *244*],

[*Ans* 1, 3, 5

106 An officer can form his men into a hollow square 5 deep, and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former, find the number of men

[§ *236, Ex* (*viii*)]

107 If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that (1) $\frac{a}{d}=\frac{a^3}{b^3}$,

(2) $(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2)$. [*Mis Ex VIII*, 3 (1)].

108 If $x=b+c,\ y=c-a,\ z=a-b$, prove that

$$x^2+y^2+z^2-2xy-2xz+2yz=4b^2$$ [§ 115, *Ex.* 3]

109 Divide

$(ax+by)^3+(ax-by)^3-(ay-bx)^3+(ay+bx)^3$ by $(a+b)^2x^2-3ab(x^2-y^2)$

[*Ans* $2(a+b)x$

110 Extract the square root of

(1) $(ab+ac+bc)^2-4abc(a+c)$ [§ *197, Ex* (*ii*), 3],

(2). $x^4+2(y+z)x^3+(3y^2+2yz+3z^2)x^2+2(y^3+y^2z+yz^2+z^3)x$ $+y^4+2y^2z^2+z^4$ [*Ans* $x^2+(y+z)x+y^2+z^2$.

111 Solve the equation

$(x+7)(y-3)+7=(y+3)(x-1)+5,\ 5x-11y+35=0$ [*Ans* 4, 5.

112 The dimensions of a rectangular court are such that if the length were increased by 3 yards, and the breadth diminished by the same, its area would be diminished by 18 square yards, and if its length were increased by 3 yards, and its breadth increased by the same, its area would be increased by 60 square yards, find its dimensions

[*Ans* Length 10 yds.; breadth 7 yds

113. If $\frac{x}{(b-c)(b+c-2a)}=\frac{y}{(c-a)(c+a-2b)}=\frac{z}{(a-b)(a+b-2c)}$, find the value of $x+y+z$ [*Ans* 0.

114 A man rides one-third of the distance from A to B at the rate of a miles per hour and the remainder at the rate of $2b$ miles per hour. If he had travelled at a uniform rate of $3c$ miles per hour he could have ridden from A to B and back again in the same time. Prove that

$$\frac{2}{c}=\frac{1}{a}+\frac{1}{b}. \quad [\S\ 236,\ Ex.\ (vi)\ ]$$

115 Simplify the expression

$(16x^5-20x^3+5x)^2+(1-x^2)\{16(1-x^2)^2-20(1-x^2)+5\}^2.$ [*Ans.* 1.

116 Extract the square root of $9x^2-24x+19-\frac{4}{x}+\frac{1}{4x^2}$

117 Solve the equation—

$$\frac{1}{x-a}-\frac{1}{x-a+c}=\frac{1}{x-b-c}-\frac{1}{x-b}$$

[*Solution.* Reduce each side and divide by c, thus

$\frac{1}{(x-a)(x-a+c)}=\frac{1}{(x-b)(x-b-c)}$; whence $(x-a)^2+c(x-a)=(x-b)^2-c(x-b)$, or $(x-b)^2-(x-a)^2=c(x-a+x-b)$, &c.]

118 Of the candidates in a certain examination 45 per cent passed If there had been 30 more candidates of whom 19 failed, the number of successful candidates would have been 44 8 per cent. How many candidates were there? [*Ans.* 1220.

1891

1 Divide $x+6a^{\frac{1}{5}}x^{\frac{4}{5}}+6a^{\frac{3}{5}}x^{\frac{2}{5}}+a+5a^{\frac{2}{5}}x^{\frac{3}{5}}+7a^{\frac{4}{5}}x^{\frac{1}{5}}$ by $x^{\frac{1}{5}}+a^{\frac{1}{5}}$.

[*Ans* $x^{\frac{4}{5}}+5a^{\frac{1}{5}}x^{\frac{3}{5}}+6a^{\frac{3}{5}}x^{\frac{1}{5}}+a^{\frac{4}{5}}$.

2 Solve the following equations—

(α) $x-\frac{3-x}{5}=3\frac{x-1}{2}+\frac{x+1}{5}-\frac{3}{10}$, [*Ans.* 2.

(β) $\frac{1}{(a-b)(x-a)}-\frac{1}{(c-d)(x-c)}=\frac{1}{(a-b)(x-b)}-\frac{1}{(c-d)(x-d)}$.

[*Ans.* $\frac{ab-cd}{(a+b)-(c+d)}$.

3 A tradesman sells two articles together for 46 rupees, making 10 per cent profit on one and 20 per cent on the other. If he had sold each article at 15 per cent. profit, the result would have been the same. At what price does he sell each article? [*Ans* ₹22, ₹24

4 Prove the rule for finding the greatest common measure of two numbers, a and b

Find the greatest common measure of $20a^4-3a^3b+b^4$ and $64a^4-3ab^3+5b^4$ [*Ans* $4a^2-3ab+b^2$.

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=$, each of these ratios will be

$$=\left\{\frac{pa^n+qc^n+re^n+}{pb^n+qd^n+rf^n+}\right\}^{\frac{1}{n}}$$

If $a+b$ $b+c=c+d$ $d+a$, prove that $a=c$ or $a+b+c+d=0$

[*Mis. Ex VIII, Ex. 12*]

1892

1 Solve the equations—

(i) $\frac{1}{5}(x-8)+\frac{4+x}{4}+\frac{x-1}{7}=7-\frac{23-x}{5}$ [*Ans* 8

(ii) $\frac{x+a}{x+b}=\frac{x+3a}{x+a+b}$. [*See* Addendum]

2 The express leaves Bristol at 3 P M and reaches London at 6 P M , the ordinary train leaves London at 1-30 P M and arrives at Bristol at 6 P M. If both trains travel uniformly, find the time when they will meet [*Ans* At 4-12 P M

3. Find the value of—

$$\frac{4y}{5}\left(y-x\right)-35\left[\frac{3x-4y}{5}-\frac{1}{10}\left\{3x-\frac{5}{7}\left(7x-4y\right)\right\}\right]$$

when $x=-\frac{1}{2}$ and $y=2$ [*App Mis Ex 16*].

[94

4 Find the square root of

$$\frac{4a^2-12ab-6bc+4ac+9b^2+c^2}{4a^2+9c^2-12ac}$$ [*Ans* $\frac{2a-3b+c}{2a-3c}$

5 If x y y z, find the simplest value of

$$\frac{xyz(x+y+z)^3}{(xy+yz+zx)^3}$$ [*Ans* 1.

1893

1 Find the Highest Common Factor of

$3x^3-5x^2+5x-2$ and $2x^4-2x^3+3x^2-x+1$ [*Ans* x^2-x+1

2 Extract the square root of

$4x^6-12x^5+13x^4-22x^3+25x^2-8x+16$. [*Ans* $2x^3-3x^2+x-4$

3 Solve the equations

(1) $120-4[5x-2\{6x+7(x-8)\}]=16-4[3x-2\{x-6(x-1)\}]$.

[*Ans* $2\frac{1}{1}\frac{5}{1}$

(2) $\frac{x+b}{a-b}=\frac{x-b}{a+b}$. (3) $\frac{6}{x}+\frac{4}{y}=3$, $\frac{9}{x}-\frac{1}{y}=2\frac{3}{4}$ [*Ans* (2) $-a$; (3) 3, 4

4 Divide the number 834 into two parts such that 30 per cent. of one part exceeds 40 per cent of the other part by 6 [*Ans.* $485\frac{1}{7}$, $348\frac{6}{7}$

5 If $a : b :: c : d$, prove that $ma-nb : a+b :: mc-nd : c+d$.

What number must be added to each of the numbers 3, 5, 7, 10, to give four numbers in proportion ? [*Ans* 5.

1894

1. If $a : b=c : d$ prove that

$$a^2+ab+b^2 : a^2-ab+b^2=c^2+cd+d^2 : c^2-cd+d^2$$

2. Find the G C M of

$$x^4-2x^3-4x^2-55x \text{ and } x^5+8x^4+25x^3+52x^2-11x$$

[*Ans* $x(x^2+3x+11)$.

3 Solve the equations

(1) $\dfrac{x+5}{6}+\dfrac{1}{9}\left(\dfrac{x}{2}+\dfrac{2}{5}\right)-\dfrac{2}{3}(3+2x)=\dfrac{4x-14}{3}+\dfrac{x+10}{10}$. [*Ans.* 1.

(2) $\dfrac{3-x}{2}-\dfrac{1}{3}\left(\dfrac{3-2x}{4}\right)=\dfrac{2x+3}{7}+\left(\dfrac{1}{4}-\dfrac{1+3x}{2}\right)$. [*Ans* $\frac{41}{37}$

4 Simplify

$$\frac{(2k-3l)^2-k^2}{4k^2-(3l+k)^2}+\frac{4k^2-(3l-k)^2}{9(k^2-l^2)}+\frac{9l^2-k^2}{(2k+3l)^2-k^2}.$$ [*Ans* 1.

1895.

1 Multiply together

$$x^3-99x^2+x-29 \text{ and } x^5-17x^4+105x^3-19x^2+23x-41,$$

and arrange the product in descending powers of x

[*Ans.* $x^8-116x^7+1789x^6-10460x^5+2502x^4-5382x^3+4633x^2$
$-708x+1189$

2 Find the G C M (if any) of

$$x^5+3x^4+46x^3+89x^2+127x+164$$

and $$x^6+3x^5+46x^4+89x^3+132x^2+169x+205$$ [*Ans* x^2+x+41.

3. Solve the simultaneous equations

$$3x+20=4y-10,\ 4(x-1)=3(y-3).$$ [*Ans* 10, 15

4 If $a : b=c : d$, you are required to prove that

$$\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^2 : \left(c^{\frac{1}{2}}+d^{\frac{1}{2}}\right)^2=a-b : c-d.$$

5 Simplify the expression :—

$$\left\{\frac{1}{\dfrac{4}{(1+t^2)^{\frac{3}{2}}}-\dfrac{3}{(1+t^2)^{\frac{1}{2}}}}\right\}^2-\left(\frac{3t-t^3}{1-3t^2}\right)^2+(1+t^2)^2-2t^2-2$$ [*Ans.* t^4.

1896

1 Find the G C M of

$x^4+ax^3+a^2x^2+2a^4$ and $x^5+a^2x^3+a^3x^2+a^5$ [*Ans* x^2-ax+a^2.

2 Simplify

$$\frac{bc(x-a)}{(a-b)(a-c)}+\frac{ca(x-b)}{(b-a)(b-c)}+\frac{ab(x-c)}{(c-a)(c-b)}$$ [*Ans.* x

3 Prove that $2\{(b+c-2a)^4+(c+a-2b)^4+(a+b-2c)^4\}$
$=\{(b-c-2a)^2+(c+a-2b)^2+(a+b-2c)^2\}^2$ [See *App* Ex 5].

4 Solve the following equations —

(1) $\frac{x-a}{3b+5c}+\frac{x-3b}{5c+a}+\frac{x-5c}{a+3b}=3$ [*See* Addendum]

(2) $\begin{cases} x-y+z=0, \\ bcx+cay-abz=0, \\ ax+by+cz-(b-c)(c-a)(a-b)=0 \end{cases}$ [See § 244]

[*Ans.* $a(b-c)$, $b(c-a)$, $c(a-b)$.

5 A number consists of two digits of which the digit in the units' place is double of the other ; if the digits be inverted, the new number exceeds the original number by 18 ; find the number [*Ans* 24.

1897.

1. Distinguish between *term* and *expression*, *power* and *index*, *measure* and *multiple*, *equation* and *identity*

(*a*) Write down the values of the following —

$A\times 0$, $0\times A$, $\frac{A}{0}$, $\frac{0}{A}$ and $\frac{0}{0}$ [*See* § 18, *Note* 1, § 266]

2 Resolve the following expressions into factors .—

$x^2-7x+12$, x^3-8 and $a^3-b^3+c^3-3abc$

3 Find the G C. M of $x^4+3x-20$ and $3x^5-2x^4+81x-85$

[*Ans* x^2-3x+5

4. Shew that

$$\frac{a-b}{1+ab}+\frac{b-c}{1+bc}+\frac{c-a}{1+ca}=\frac{(a-b)(b-c)(c-a)}{(1+ab)(1+bc)(1+ca)}$$ [*See* § 180].

5 Solve the equations —

(1) $(x-a)(x-b)=(x-c)(x-d)$, [*Ans* $\frac{ab-cd}{(a+b)-(c+d)}$.

(2) $ax+by=c$, $a'x+b'y=c'$ [*See* § 239, *Ex* 4] $x=\frac{bc'-b'c}{ab'-a'b}$, $y=\frac{ac'-a'c}{a'b-ab'}$

6 If $3a+4b : 5a+6b = 3c+4d : 5c+6d$, then will $a : b = c : d$

1898.

1 Resolve the following expressions into elementary factors

(1) $81a^4+64b^4$; (2) $a^3(b-c)+b^3(c-a)+c^3(a-b)$;

(3) $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$.

[*Ans.* (1) $(9a^2+12ab+8b^2)(9a^2-12ab+8b^2)$.
(2) $-(b-c)(c-a)(a-b)(a+b+c)$.
(3) $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

2 If $2s=a+b-c$, prove that

$$2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=abc.$$

3 If $x+y+z=xyz$, prove

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

4 Solve the following equations :

(1) $\dfrac{x-a}{b+c+2a}+\dfrac{x-b}{c+a+2b}+\dfrac{x-c}{a+b+2c}+3=0.$ [*See* Addendum]

(2) $\begin{cases} x+2y+3z=20, \\ 2x+3y-5z=-7, \\ 4x-5y+7z=21 \end{cases}$ [*Ans* (1) $-(a+b+c)$; (2) 2, 3, 4

5 A farmer bought equal numbers of two kinds of sheep, one kind at Rs 6 each, the other at Rs 8 each ; if he had expended his money equally in the two kinds, he would have had three sheep more than he did How many of each kind did he buy ? [*Ans* 72

6 If $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$, prove that $\dfrac{a}{d}=\dfrac{pa^3+qb^3+rc^3}{pb^3+qc^3+rd^3}$.

1899.

1 Divide $(a^2-bc)^3+8b^3c^3$ by a^2+bc [*Ans* $a^4-4a^2bc+7b^2c^2$.

2 Resolve the following into simple factors

(1) a^3-b^3 ; [§ 124*b*]

(2) $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$. [§ 132, *Ex.* 7] $(a+b+c)(bc+ca+$

3 (*a*) Find the G C M of $x^4+x^2-12x+21$ and $x^4-15x+14$.
[*Ans* x^2+3x+7.

(*b*) Find the cube of $x-\dfrac{1}{x}$. [*Ans* $x^3-\dfrac{1}{x^3}-3\left(x-\dfrac{1}{x}\right)$.

4 Solve the following equations :

(1) $\dfrac{x}{2x-a}+\dfrac{x}{2x-b}=1$; (2) $\begin{cases} x-2y+z=0, \\ 5z-3x-4y=0, \\ 7x+8y+9z=98 \end{cases}$ [*Ans* (1) $\dfrac{ab}{a+b}$.
[*Ans* (2) 3, 4, 5.

5 There is a number of two digits whose difference is 2, and if it be diminished by $\frac{3}{2}$ times the sum of the digits, the digits will be inverted find it [Ans. 75

6 If $a : b = c : d$, prove that

$$pa^2+qc^2 : pb^2-qd^2 = ma^2-nc^2 : mb^2-nd^2$$

1900

1 Prove that $a^m \times a^n = a^{m+n}$, where m and n are positive integers

Simplify $\left(\frac{x^m}{x^n}\right)^{m+n} \times \left(\frac{x^n}{x^l}\right)^{n+l} \times \left(\frac{x^l}{x^m}\right)^{l+m}$ [Ans 1

2. (a) Resolve into factors

(1) x^3+64, [Ans $(x+4)(x^2-4x+16)$

(2) x^4+64, [Ans $(x^2-4x+8)(x^2+4x+8)$

(3) $a^2(b-c)+b^2(c-a)-c^2(a-b)$. [§ 132, Ex 1]

(b) If $x=a+b-2c$, $y=b+c-2a$, $z=c+a-2b$, find the value of $x^3+y^3+z^3-3xyz$ [Ans 0

3 Solve the following equations

(1) $\frac{2-x}{3}+\frac{3-x}{4}-\frac{4-x}{5}+\frac{5-x}{6}-\frac{3}{4}=0$ [Ans 4.

(2) $x-2(3y-2z)=0$, $2y+3(x-z)=0$, $5x+7y-9z=67$. [Ans 2, 3, 4.

4 A person bought a certain number of eggs, half of them at 2 a penny and half at 3 a penny He sold them again at the rate of 5 for $2d$ and lost a penny by the transaction What was the number of eggs? [Ans 60.

5 If $a : b = c : d$, prove that

$$\sqrt{(3a^2-4c^2)} : \sqrt[3]{(5a^3-6c^3)} = \sqrt{(3b^2-4d^2)} : \sqrt[3]{(5b^3-6d^3)}$$

1901

1 (a) Prove that $a^m \div a^n = a^{m-n}$, where m and n are positive integers, and m greater than n

(b) Divide $a^{5m}+b^{5n}$ by a^m+b^n

[Ans $a^{4m}-a^{3m}b^n+a^{2m}b^{2n}-a^m b^{3n}+b^{4n}$

2. (a) Resolve $a^4+a^2b^2+b^4$ into two factors [§ 124, Ex 24]

(b) Find the G C M of $x^3+6x^2+11x+6$ and $x^4-x^3-4x^2-4x$

[Ans. x^2+3x+2

3 Simplify

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}.$$ [§ 180, Ex 16] Ans. 1

4 (a) Distinguish between an *Equation* and an *Identity*, and give an example of each

(*b*) Solve the following equations :

(1) $\frac{1}{3}(x-3)+\frac{3}{4}(x-8)+\frac{1}{6}(x-4)=2\frac{7}{12}$; [*Ans.* 8.

(2) $2x+3y+4z=38,\ 3x-2y+5z=26,\ 4x+6y-3z=21.$

[*Ans.* 3, 4, 5.

5 A number consists of two digits, the digit in the units' place being four times that in the tens' place. If the digits be inverted, the new number increased by 2 equals three times the old number. Find the number. [*Ans* 28.

6. If $x:a=y:b=z:c$, prove that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{(x+y+z)^2}{(a+b+c)^2}$.

1902.

1. (*a*) Divide $x^2(y+z)+y^2(z+x)+z^2(x+y)+3xyz$ by $yz+zx+xy$

[*Ans.* $x+y+z$.

(*b*) If $s=a+b+c$, prove that

$$(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2$$

2. (*a*) Prove that $(x^p)^q=x^{pq}$, where p and q are positive integers

(*b*) Simplify $\left(\frac{x^p}{x^q}\right)^{p+q}\div\left(\frac{x^{p+q}}{x^{p-q}}\right)^{\frac{p^2}{q}}$ [*Ans.* $\frac{1}{x^{p^2+q^2}}$.

3. Find the L. C. M of

$$x^3-x^2-14x+24,\ x^3-2x^2-5x+6, \text{ and } x^2-4x+3$$

[*Ans.* $(x^2-4x+3)(x^2-4)(x+4)$

4 (*a*) Prove that in a simple equation there cannot be more than one value of the unknown quantity

(*b*) Solve the following equations :

(1) $\frac{bc(ax-1)}{b+c}+\frac{ca(bx-1)}{c+a}+\frac{ab(cx-1)}{a+b}=a+b+c$; [*See* Addendum].

(2) $12x+34y=8\frac{1}{12},\ 34x+12y=8\frac{4}{3}$ [*Ans* $\frac{1}{3}, \frac{1}{6}$.

5 A sum of money was divided equally among a certain number of persons , had there been six more, each would have received a shilling less, and had there been four fewer, each would have received a shilling more than he did. Find the sum of money and the number of men

[*See* § 246, *Ex.* 3] [*Ans* £6 , 24.

6 (*a*) If $a:b::b:c::c:d$, prove that $a:d::a^3:b^3$.

(*b*) If $x:ax+by+cz=y:bx+cy+az=z:cx+ay+bz$,

shew that each of these ratios $=\frac{1}{a+b+c}$, supposing $x+y+z$ is not $=0$

[*Solution.* Taking each ratio $=k$, it will be seen that

$$(x+y+z)\{1-k(a+b+c)\}=0\ ;\ \therefore\ 1-k(a+b+c)=0.\ [\S\ 261]$$

1903.

1 (*a*) Prove that $(ac+bd)^2-(ad+bc)^2=(a^2-b^2)(c^2-d^2)$

(*b*) Divide $8a^3-b^3-27c^3-18abc$ by $2a-b-3c$

[*Ans* $4a^2+b^2+9c^2-3bc+6ca+2ab$.

2 Simplify

$$\frac{(x+1)^2}{(x-y)(x-z)}+\frac{(y+1)^2}{(y-z)(y-x)}+\frac{(z+1)^2}{(z-x)(z-y)}$$ [*Ans* 1.

3 (*a*) Prove that $a^m\times a^n=a^{m+n}$, where m and n are positive integers

(*b*) Simplify

$$\left(\frac{x^l}{x^m}\right)^l\times\left(\frac{x^m}{x^l}\right)^m-\{(x^l)^l\times(x^m)^m\}\times\{(x^m)^l\times(x^l)^m\}$$ [*Ans* 1

4 Solve the following equations —

(1) $\frac{x}{x+a-b}+\frac{x}{x+b-c}=2$, [*Ans* $\frac{2(a-b)(b-c)}{c-a}$

(2) $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=23$, $\frac{x}{3}+\frac{y}{4}+\frac{z}{2}=28$, $\frac{x}{4}+\frac{y}{2}+\frac{z}{3}=27$ [*Ans* 12, 24, 36

5 A general wishing to draw up his regiment in the form of a hollow square found that he had 50 men over when it was 4 deep, but that he wanted 50 men to complete it when it was 5 deep, the number of men in the front being the same in both cases Find the number of men in the regiment [*Ans* 530

6 If $x\ (b+c)=y\ (c+a)=z\ (a+b)$, prove that

$a\ (y+z-x)=b\ (z+x-y)=c\ (x+y-z)$

1904

1 Simplify

(1) $(2x+3y)^3-3(2x+3y)^2(2x-3y)+3(2x+3y)(2x-3y)^2-(2x-3y)^3$

[*Ans* $(6y)^3=$&c

(2) $$\left(\frac{x^l}{x^m}\right)^{l^2+lm+m^2}\times\left(\frac{x^m}{x^n}\right)^{m^2+mn+n^2}\times\left(\frac{x^n}{x^l}\right)^{n^2+nl+l^2}$$ [*Ans* 1

2 Reduce $\frac{3x^5-5x^3+2}{2x^5-5x^2+3}$ to its lowest terms [*Ans* $\frac{3x^3+6x^2+4x+2}{2x^3+4x^2+6x+3}$

3. Shew that $(x-1)(x-2)(x-3)(x-4)+1$ is a perfect square

[*Ans* Given expn $=(x^2-5x+5)^2$

4 (*a*) Distinguish between an *equation* and an *identity*, and give an example of each

(*b*) Solve the following equations

(1) $\frac{x(a+b)+c}{x+d}+\frac{x(a-b)+d}{x+c}=2a$ [*Ans* $\frac{2acd-c^2-d^2}{b(c-d)+(c-ad)+(d-ac}$

(2) $x+y+z=0$, $ax+by+cz=0$,

$a^2x+b^2y+c^2z+(b-c)(c-a)(a-b)=0$. [*Ans* $b-c$, $c-a$, $a-b$

5. At 7-40 A. M. an ordinary train starts from P, and reaches Q at 11-40 A. M.; an express train, which starts from Q at 9 A. M., arrives, at P at 11 40 A. M.; if both trains travel at a uniform speed without stopping, find the time when they meet. [*Ans.* At 10 4 A. M.

6. If $a : b :: x : y$, then $a^2+b^2 : \dfrac{a^3}{a+b} :: x^2+y^2 : \dfrac{x^3}{x+y}$.

1905.

1. (1) Given $x+y=5$ and $xy=7$, find the value of $x^3+y^3+4(x-y)^2$. [*Ans.* 8

(2) If $x^2+y^2=1$, prove that $(3x-4x^3)^2+(4y^3-3y)^2=1$.

2. Divide $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a+b+c$. [§ *132*, *Ex.* 2]

3. Simplify $\dfrac{b-c}{a^2-(b-c)^2}+\dfrac{c-a}{b^2-(c-a)^2}+\dfrac{a-b}{c^2-(a-b)^2}$. [*Ans.* 0

4. Solve

(1) $\dfrac{x-bc}{b+c}+\dfrac{x-ca}{c+a}+\dfrac{x-ab}{a+b}=a+b+c$. [*See 1902, Q. 4*] [*Ans.* $bc+ca+ab$.

(2) $x-y+z=2$, $4x+6y+5z=31$, $5x-11y+13z=22$. [*Ans.* 1, 2, 3]

5. A company of men is formed into a hollow square 10 deep. If the company be increased by 1600 men, the whole number may also be formed into a hollow square 10 deep, so that the front in the latter formation shall contain twice the number of men contained in the front of the former. Find the original number of men. [*Ans.* 1200

6. (1) If $a : b :: c : d$, prove that $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

(2) If $(a+b+c)x=(b+c-a)y=(c+a-b)z=(a+b-c)w$, shew that

$$\frac{1}{x}=\frac{1}{y}+\frac{1}{z}+\frac{1}{w}.$$

MADRAS UNIVERSITY.

Selections from the University Questions [1857—1890].

1. Expand and simplify $\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2$. [*Ans.* $3(a^2+b^2+c^2)-2(bc+ca+ab)$

2. Solve $6x-a : 4x-b :: 3x+b : 2x+a$. [*Ans.* $\dfrac{a^2-b^2}{4a-b}$.

3. Simplify $\left\{\dfrac{(x-a)(y-b)\sqrt{(xy)}\;x^7}{\sqrt[3]{(by)}\;\sqrt[3]{(a+b)(x-y)^7}}\right\}^0 \times \dfrac{(x+y)(x-y)(x^2+y^2)}{x^4-y^4}$

$\times\sqrt{(a^2+2ab+b^2)}$ [*Ans.* $a+b$.

4. If $a : b :: c : d$, and if x be homogeneous with a, b, c and d, then

$$a^3+x^3 : b^3c^3 :: 1+\frac{x^3}{a^3} : d^3.$$

5 Two years after the flood, when Shem had lived a sixth of his life, he begat Arphaxad; who, when he was thirty five years old, begat Salah. When Salah had lived a twentieth of Shem's life, he begat Eber, who after thirty-four years begat Peleg Peleg begat Reu at the same age as Salah begat Eber Reu was two years older when he begat Serug Serug begat Nahor at the same age as Peleg begat Reu At a year younger Nahor begat Terah, who, when he had lived $\frac{7}{60}$ths of the years of Shem's life, begat Abram Abram was born 292 years after the flood and lived 175 years By how many years did Abram survive Shem, or Shem Abram? [*Ans* Shem survived Abram 35 years

6 What will be the value, when $a=b(2+\sqrt{5})$, of

$$\frac{4a+6b}{a+b}+\frac{6a-4b}{a-b}-\frac{4a^2+6b^2}{a^2-b^2}+\frac{4b^2-6a^2}{a^2+b^2}-\frac{20b^4}{a^4-b^4}? \qquad [\textit{Ans}\ 0$$

7 Two trains start at the same time from A and B for the junction C The train from A should run 24 miles an hour and reach the junction half an hour before that from B but the former is so retarded as only to run at an average rate of 22 miles an hour The two trains arrive at the junction at the same time How far are A and B respectively from C, and how long were the trains upon the road? [*Ans* 132 mi ; 108 mi. , 6 hrs

8 Resolve into four factors $a(b^2-c^2)-bc(c^2-b^2)-a^2(c-b)$

[*Ans* $(b-c)(c-a)(a-b)(a+b+c)$

9. Solve $\sqrt{y}-\sqrt{(y-2x)}=\sqrt{(48-2x)}$, $y(x-15)=36$ [*Ans* 16, 36.

10 Simplify (1) $\dfrac{ac-1}{(a-b)(1+ax)}-\dfrac{bc-1}{(b-a)(1+bx)}$. [*Ans* $\dfrac{c+x}{(1+ax)(1+bx)}$

(2) $\dfrac{2^{2n+1}-2^{n+2}+2}{2^{2n+1}-2^{n+1}}$. [*Ans* $\dfrac{2^n-1}{2^n}$.

11 A and B start from opposite ends of a straight course, each walking uniformly, A, who is the faster walker, at the rate of 4 miles an hour, and meet at the end of two hours If, when A reached the middle point of the course, they had interchanged their rates of walking, they would have met a quarter of a mile nearer the middle point Find B's rate of walking, and the length of the course. [*Ans* 3 mi per hr ; 14 mi.

12. Divide the difference of $(x^2-bx-b^2)(x+a-b)$ and $(x^2-ax-a^2)(x-a-b)$ by $a-b$ [*Ans* $3x^2-2(a-b)x+a^2-b^2$.

13 For what values of n is x^n-a^n divisible by $x-a$, and x^n-a^n by $x-a$? Write down the last three terms of the quotient in each case

14 Solve $\sqrt{(ax-b)}-\sqrt{(bx-a)}=(\sqrt{a}-\sqrt{b})\sqrt{(x-1)}$ [*Ans* 0.

15. Separate into four factors :—

$(a^2-b^2)^2-(c^2-d^2)^2-(a-b)^2(c-d)^2-(a-b)^2(c+d)^2$.

[*Ans* $(a+b-c-d)(a-b-c-d)(a-b-c-d)(a-b+c-d)$.

16. Simplify

$$\frac{a-b}{(x-a)(x+b)}+\frac{b-c}{(x+b)(x-c)}+\frac{c-d}{(x+c)(x+d)}+\frac{d-a}{(x+d)(x+a)}. \qquad [\textit{Ans}\ 0.$$

17 A letter carrier has a hours allowed to him for going from A to B and back again, including c hours for rest at B. But he finds that he can get b hours for rest by going d miles an hour faster each way. Find his ordinary speed, and the distance from A to B

[*Ans* $\frac{d(a-b)}{b-c}$ miles per hour : $\frac{d(a-b)(a-c)}{2(b-c)}$ miles

18 If $x=\frac{a-b}{m-c}$, $y=\frac{b-c}{m-a}$, $z=\frac{c-a}{m-b}$ find the value of

$x+y+z+xyz$. [*Ans* 0.

19 Find what term is wanting to make the following expression a complete square : $a^2x^4+64b^2-4(ax^2+8b)(a-b)x$. [*Ans.* $4(a+b)^2x^2$.

20 Solve $\sqrt{\left(\frac{x-a}{x-b}\right)}+\frac{a}{x}=\sqrt{\left(\frac{x-b}{x-a}\right)}+\frac{b}{x}$. [*Ans.* $\frac{ab}{a+b}$.

21 Resolve into factors $(2a+2b-ab)^2-(b^2-4a)(a^2-4b)$

[*Ans.* $4(a-b)^2(a+b+1)$.

22 Solve $\frac{(x+1)^5+(x-1)^5}{(x+1)^3+(x-1)^3}=10$ [*Ans* $\pm\sqrt{5}$.

23 Simplify $\frac{x^3+3x^2+5x+15}{x^3+2x^2+5x+10}+\frac{x^4+x^3+3x^2+x-2}{x^4+2x^3+3x^2+4x-4}$. [*Ans* 2

24 Shew that if $a+b+c=1$, $ab+bc+ca=\frac{1}{3}$, $abc=\frac{1}{27}$,

then $$\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}=\frac{27}{4}.$$

25 A mail coach runs between two places A and B, and back again A traveller, who starts walking from A 5 hours before the mail coach, is overtaken by it half way between A and B He then doubles his rate of walking and meets the mail coach on its return journey 3 miles from B The traveller then goes to B at the same rate and returns, and by the time he comes again midway between A and B, the mail coach reaches A. Find the distance between A and B and the rate at which the mail coach runs [*Ans.* 30 miles ; 6 miles per hour

26 Simplify

$$\frac{(a+b)^2}{(x-a)(x+a+b)}-\frac{a+2b+x}{2(x-a)}+\frac{(a+b)x}{x^2+bx-a^2-ab}+\frac{1}{2}.$$ [*Ans.* 0.

27. Transform $(x^2+y^2+z^2+2xy)^2-2(x+y)^2z^2$ into the sum of two perfect squares [*Ans.* $(x+y)^4+z^4$.

28 Is $(a+b+c)\{a^2-(b+c)a+(b+c)^2\}-3bc(b+c)$

$=(a+b+c)\{b^2-(a+c)b+(a+c)^2\}-3ac(a+c)$

an equation or an identity ? [*Ans* Identity.

29 Three equal vessels A, B, C are placed one above another, A being the highest. A is full, B half full, C empty. In the bottom of A is a hole which would empty it in 16 minutes, under the bottom of B one which would empty it in 4 minutes. How long will it be before C is full ? [*Ans* 8 minutes.

30 Simplify the fraction

$$\frac{(a^4-b^4)^2+2a^6b^2+5a^4b^4-2a^2b^6}{(a^2+ab+b^2)^2(a^2-ab+b^2)^2}$$ [*Ans* 1

31 Solve the equation —

$$\left.\begin{aligned}(a^2-b^2)x-(a^2-ab+c^2)y&=a(a-2b)-\frac{bc^2}{a-b},\\ \frac{x}{a}+\frac{y}{b}&=\frac{2a}{a^2-b^2}\end{aligned}\right\}$$ [*Ans* $x=\frac{a}{a+b}$, $y=\frac{b}{a-b}$

32 A, B, C, D are four railway stations From B to C is $2\frac{2}{3}$ miles more, and from C to D $5\frac{2}{3}$ miles less than from A to B A train starts from A, and travels at the rate of 14 miles an hour At B an accident happens to the engine, which causes a delay of 6 hours After this the train proceeds to C at half speed There another delay of $\frac{1}{2}$ an hour occurs, and then the train moves on to D at a speed further diminished by one mile an hour A man starts from A at the same time as the train, and travels straight across country to D, a distance of 58 miles Including stoppages he averages 3 miles an hour and reaches D just with the train What is the distance by rail from A to D? [*Ans* $102\frac{8}{11}$ miles

33 Shew that $x^3+6(y+z)x^2-12(y-z)^2x+8(y+z)^3$
$$=4(2y+3x+6z)y^2+(x+6y-2z)(x+2z)^2$$

34 Simplify $\dfrac{8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3}{3(2a+b+c)(a+2b+c)(a+b+2c)}$ [*Ans* 1

35 Solve the equation $\dfrac{(x-1)(x-2)(x-6)}{(x-3)^3}=1$ [*Ans* $2\frac{1}{2}$

36 A steamer sailed from a certain port with first, second and third class emigrants, numbering 100 The fares of the three classes were in the proportion 4 2 1, and the total amount received was £3780 When she had completed two thirds of her voyage, the steamer broke down, and a passing vessel was requested to take all her third class, and half of her second class passengers for the remainder of the voyage, for a proportionate part of their fares, which would have amounted to £420 This was refused for want of accommodation, but an offer was made to take, on like conditions, one quarter both of the first and second class passengers This was accepted, and £240 paid for the service. How many passengers were there in each class, and what were the respective fares?

[*Ans* 30, 20, 50, £72, £36, £18

37 Simplify

$$\frac{a^4+b^4+ab(a^2+b^2)}{(a+b)^2}-\frac{a^4+b^4-ab(a^2+b^2)}{(a-b)^2}-\frac{12a^2b^2}{(a+b)^2-(a-b)^2}.$$ [*Ans* ab

38 Find the square root of

$$\frac{x^4}{y^4}+\frac{y^4}{x^4}-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5$$

[*Ans* $\frac{x^2}{y^2}-\frac{x}{y}+1-\frac{y}{x}+\frac{y^2}{x^2}$

39 Two parties of workmen are placing sleepers for parallel lines of railway. The first set had placed 36 sleepers when the second began, and place 8 sleepers to the second set 7 The first, however, have to place 4 sleepers in the same space in which the second place 3 At what distance from the starting point will the one overtake the other, supposing there are 1764 of the more closely placed sleepers in a mile? [*Ans* $\frac{1}{4}$ mile.

40. Divide a by $a-2x$ to six terms [$1+\frac{2x}{a}+\frac{4x^2}{a^2}+\frac{8x^3}{a^3}+\frac{16x^4}{a^4}+\frac{32x^5}{a^5}$.

41 Reduce $\left(\frac{ap^2-aq^2+2bpq}{p^2+q^2}\right)^2+\left(\frac{bq^2-bp^2+2apq}{p^2+q^2}\right)^2$ [*Ans.* a^2+b^2

42 Simplify $\frac{xy+2x^2-3y^2+4yz+xz-z^2}{2x^2-9xz-5xy+4z^2-8yz-12y^2}$. [*Ans* $\frac{x-y+z}{x-4y-4z}$

43 A merchant engaged two writers, their pay being Rs 60 for the first month, with a fixed monthly increase afterwards They agreed to serve for one year, and each of them placed in the merchant's hands a deposit to be forfeited in proportion to the part of the year during which he might not serve One remained at his post $7\frac{1}{2}$ months, and received for salary and portion of deposit returned Rs 537 The other remained 10 months, and received in like manner Rs 728 5as. 4p What was the monthly increase, and what the amount of the deposit? [*Ans* Re 1 ; Rs. 100.

44 Find the value of $(ma-nb)(mb-nc)(mc-na)$
$+(na-mb)(nb-mc)(nc-ma)$, when $a-b=0$ [*Ans.* 0.

45 Shew that $\frac{a-b}{m+ab}+\frac{b-c}{m+bc}+\frac{c-a}{m+ca}=m\frac{(a-b)(b-c)(c-a)}{(m+ab)(m+bc)(m+ca)}$.

46 A letter-carrier has to go daily from P to Q in a prescribed time If he goes a mile an hour faster than his ordinary rate, he arrives at Q half an hour before the time. But if he goes a mile an hour slower, he arrives three quarters of an hour too late Find his ordinary rate, and the distance from P to Q [*App Mis Ex. 81*]

47 Solve the equation $\frac{x}{x-2}+\frac{9-x}{7-x}=\frac{x+1}{x-1}+\frac{8-x}{6-x}$. [*Ans* 4.

48 A set of bearers on a journey perform one third of the distance at a certain rate and then halt one hour to take their food The remainder of the journey is accomplished at only two-thirds of the former rate, and the bearers reach their destination in 7 hours after first starting Had they travelled at the former rate $4\frac{1}{2}$ miles further than they did before halting, they might have halted $22\frac{1}{2}$ minutes longer and yet reached the end of journey in the same time Find the length of the journey.

[*Ans* 27 miles.

49 Solve the equation : $\frac{(x-a)(x+b)}{x-a+b}=\frac{x(x-c)-b(x+c)}{x-b-c}$ [*Ans* $\frac{b(a+c)}{a-c}$.

50. Two men A and B are employed on a piece of work which has to be finished in 14 days In 3 days, they do one-fifth of the work, and then A's place is taken by C B and C work for one day and do one-twentieth of the whole work, and then B's place is taken by A. A and C

finish the work a day before the appointed time. Find the time in which the work could have been done (1) by each working separately, and (2) by all working together. [*Ans* 20, 60, 30 days; 10 days.

51. Prove that $(x-y)(x+1)(y+1)-x(y+1)^2+y(x+1)^2$
$$=(x-y)(x+y+2xy).$$

52. Simplify $\dfrac{(x+y)^2+(x-y)^2}{(x-y)^2-(x-y)^2}-\dfrac{x^4-y^4}{2xy(x-y)}$. [*Ans* $\dfrac{1}{x+y}$.

53. Prove that $\dfrac{(a+1)^2}{(a-b)(a-c)}+\dfrac{(b+1)^2}{(b-c)(b-a)}+\dfrac{(c+1)^2}{(c-a)(c-b)}=1$.

54. A number consists of two digits. When the number is divided by the sum of its digits, the quotient is 7. The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits. Find the number. [*Ans* 63.

55. Resolve into three factors $(x+1)(x+3)(x+5)(x+7)+15$.
[*Ans* $(x+2)(x+6)(x^2+8x+10)$.

56. Shew that
$$4(a^2+ab+b^2)^3-(a-b)^2(a+2b)^2(2a+b)^2=27a^2b^2(a+b)^2.$$

57. If $y=\dfrac{1+x}{1-x}$, prove that $\left(x-\dfrac{1}{x}\right)\left(y-\dfrac{1}{y}\right)=4\dfrac{xy+1}{x-y}$.

58. Solve the equation $\dfrac{a+c}{x-2b}-\dfrac{b+c}{x-2a}=\dfrac{a-c}{x+2b}-\dfrac{b-c}{x+2a}$. [*Ans* $\dfrac{-2ab}{c}$.

59. In a quater of a mile race, *A* gives *B* a start of 22 yards, and beats him by 2 seconds; and in a 300 yards race, he gives him a start of 2 seconds, and beats him by $10\frac{1}{3}$ yards. Find the rate of each.
[*Ans* 8 yds. and $7\frac{1}{3}$ yds. per sec.

60. If $x^2=ab+bc-ca$, shew that $(a^2+x^2)(b^2+x^2)(c^2+x^2)$ is a perfect square. [*See* Addendum]

61. Simplify the expression—
$$\frac{b^2+c^2-2a^2}{(a-b)(a-c)}+\frac{c^2+a^2-2b^2}{(b-c)(b-a)}+\frac{a^2+b^2-2c^2}{(c-a)(c-b)}$$ [*Ans* -3.

62. Solve the equation $\dfrac{(x+1)(x+9)}{(x+2)(x+4)}=\dfrac{(x+6)(x+10)}{(x+5)(x+7)}$. [*Ans* $-5\frac{1}{2}$.

63. Extract the square root of
$x^2(x^2+y^2+z^2)+2x(y+z)(yz-x^2)+y^2z^2$. [*Ans* $x^2-(y+z)x-yz$.

64. Simplify $\dfrac{a^2-(b-c)^2}{(a-b)(a-c)}+\dfrac{b^2-(c-a)^2}{(b-c)(b-a)}+\dfrac{c^2-(a-b)^2}{(c-a)(c-b)}$. [*Ans* 4.

65. Solve the equation $\dfrac{x-1}{x-2}+\dfrac{x-4}{x-5}=\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}$. [*Ans* $3\frac{1}{2}$.

1891.

1 Simplify $7(a-3b+c)-[2(2b+4c)(6c-3b)-3(a-4b)(a+3b)$
$+\{(5a-4b+3c)\times 4+\overline{a-47b+c}\}-7]$
[*Ans* $3a^2-12b^2-96c^2-3ab+4a-12b+5c$.

2 Divide $\frac{2}{3}-x$ by $\frac{1}{3}-x-x^2$ to *five* terms [*Ans.* $2+4x-8x^2+16x^3+32x^4$.

3 (i) Find the G. C. M. of $x^4-8x^3+28x^2-53x+42$
and $x^4+6x^3-42x^2+129x-154$ [*Ans* $x^3-5x^2+13x-14$.

(ii) Simplify $\frac{3x-12}{x^2-5x+6}+\frac{5x-3}{x^2-2x-3}-\frac{x+15}{x^2-5x-6}$

[*Ans* $\frac{7x^2-50x+18}{(x+1)(x-2)(x-6)}$.

4. Solve the equations

(i) $\frac{3x-2}{5}+\frac{4x-1}{7}-\frac{10x}{9}=5(x-9)+3-\frac{x}{3}$. [*Ans.* 9

(ii) $49x-57y=172$, $57x-49y=252$. [*Ans.* 7, 3.

(iii) $\frac{x}{a}+\frac{y}{b-a}=3m$, $\frac{x}{b}+\frac{y}{a-b}=7m$. [*Ans* $\frac{12abm}{a+b}$, $\frac{(a-b)(7b-5a)m}{a+b}$.

5 A composition of copper and tin containing 140 cubic inches weighs 42 lbs. 3 ozs. How many ounces of each are there if a cubic inch of copper weighs $5\frac{1}{4}$ ozs., and a cubic inch of tin $4\frac{1}{4}$ ozs.? [*Ans.* 420 ; 255

1892.

1. If a, b, c be three quantities whose sum is zero, shew that
$a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$.

2 Break up into factors $a^2b-b^2c+c^2a-(ab^2+bc^2+ca^2)$
[*Ans.* $-(b-c)(c-a)(a-b)(a+b+c)$.

3. Simplify $\frac{(y-z)(y+z)^2+(z-x)(z+x)^2+(x-y)(x+y)^2}{(y+z)(y-z)^2+(z+x)(z-x)^2+(x+y)(x-y)^2}$. [*Ans.* -1.

4. Extract the square root of $\frac{4x^2}{9a^2}+\frac{4x^2}{b^2}+\frac{9a^2}{4x^2}+\frac{9a^2x^2}{b^4}+\frac{9a^2}{b^2}+2$

[*Ans.* $\frac{2x}{3a}+\frac{3ax}{b^2}+\frac{3a}{2x}$.

5. Prove the rule for finding the G. C. M. of two quantities, and find that of $7x^4-2x^3-9x-2$ and $5x^4-6x^2-6x-11$. [*Ans.* x^2+x+1.

6. Solve :—

(1) $2x-\frac{y+3}{4}=7+\frac{3y-2x}{5}$, $4y+\frac{x-2}{3}=26\frac{1}{2}-\frac{2y+1}{2}$. [*Ans.* 5, 5

(2) $\frac{x+16}{5}+\frac{11}{x}=\frac{4x-17\frac{1}{2}}{3}$. [*Ans* 9 or $-$[illegible].

7 In a half mile race A gives B 22 yards' start and wins by 6 seconds In a three quarter mile race he gives him 20 seconds' start but is beaten by 29 yds 1 ft In what time can each of them run a mile ?
[*Ans.* $13\frac{4}{7}$ min, $13\frac{5}{7}$ min

1893

1 (1) From $a(b+c)^2+b(c+a)^2+c(a+b)^2$ subtract
$(a+b)(a-c)(b-c)+(b+c)(a-b)(a-c)-(a+c)(a-b)(b-c)$ [*Ans* $12abc$.

(2) Shew that $(4x^2-5x+7)^2-(5x^2+14x+2)^2$
is divisible by x^2+x+1, and find the quotient [*Ans* $9(5-19x-x^2)$

(3) Divide $x^4+5ax^3+(25a-b-29)x^2-5(4a+b-4)x+4b$
by x^2+5x-4 [*Ans* $x^2+5(a-1)x-b$

2 (1) Find the four factors of $(1-y)^2-2(1+y^2)x^2+(1-y)^2x^4$
[*Ans* $(1+x)(1-x)(1+x+y-xy)(1-x+y+xy)$

(2) Simplify $\dfrac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)}-\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{ab}}{\dfrac{1}{b}-\dfrac{1}{a}}$ [*Ans* $a-b$

3 Find the G C M of $6x^5-4x^4-11x^3-3x^2-3x-1$
and $4x^4+2x^3-18x^2+3x-5$ [*Ans* $2x^3-4x^2+x-1$.

4 Extract the square root of
$x^{4m+2}-6x^{3m+1}-10x^{2m+1}y^{m-2}+9x^{2m}-30x^my^{m-2}+25y^{2m-4}$.
[*Ans* $x^{2m+1}+3x^m-5y^{m-2}$.

5 Solve (1) $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}$, $\frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}$ [§ *241, Ex. 41*]

(2) $\dfrac{3}{5-x}+\dfrac{2}{4-x}=\dfrac{8}{x+2}$ [*Ans* 2 or $4\frac{6}{13}$

6 A boy bought a number of oranges for Rs 2. Had he bought 8 more for the same money, he would have paid 4 *p* less for each How many did he buy ?
[*Ans.* 24

1894

1 Simplify the expressions —

(1) $\left[\sqrt[3]{\dfrac{x^2}{y^4}}\times\sqrt{\dfrac{y^3}{x^4}}\right]^{12}\times x^{22}$ [*Ans* y^2.

(2) $2(z^3+x^3)-[(x+y)(xy-x^2-y^2)-\{2(x+y+z)$
$(yz+zx+xy-x^2-y^2-z^2)-(x-y)(x^2+xy+y^2)\}]$ [*Ans* $6xyz$

2 Divide the product of $ab(x^2+1)+(a^2+b^2)x$ and x^3+1 by that of $x+1$ and $ax+b$
[*Ans* $(bx+a)(x^2-x+1)$.

3 Extract the square root of
$x^4(x-a)^2+4a^2x^4+a^4(2x+a)^2-2a^3x^2(x+a)$ [*Ans.* $x^3-ax^2+2a^2x+a^3$.

4. Find the L. C. M. of $x-a$, x^2-a^2, x^3-a^3, $(x^3+a^3)^2$; and the H. C. F. of x^5+11x^3-54 and $2x^5-11x^2-9$
[*Ans.* $(x^3-a^3)(x^3+a^3)^2$; x^2+2x+3.

5. Express $\frac{(a-b)(b-c)}{(c-d)(d-a)}-\frac{(b-c)(c-d)}{(d-a)(a-b)}+\frac{(c-d)(d-a)}{(a-b)(b-c)}-\frac{(d-a)(a-b)}{(b-c)(c-d)}$ as a fraction whose numerator and denominator consists of four factors each.
[*Ans.* $\frac{(a-b+c-d)(a-b-c+d)(b-c+d-a)(b-c-d+a)}{(a-b)(b-c)(c-d)(d-a)}$.

6. Solve the equations :—

(1) $\frac{(x+1)(x+2)}{x+3}=\frac{(x+4)(x+5)}{x+9}$.

(2) $\left.\begin{array}{l}3x+4y+2z=19,\\ 7x-3y=15,\\ 7z-4y=-1.\end{array}\right\}$

(3) $\sqrt{(x^2+3x+15)}+2x=9$

[*Ans.* $-2\frac{1}{3}$; 3, 2, 1; 2 or 11.

7. If 3 be added to the numerator and denominator of a certain fraction, the fraction becomes $\frac{2}{3}$, if 5 be subtracted from the numerator and denominator, it becomes $\frac{1}{2}$. Find the fraction. [*Ans.* $\frac{13}{21}$

8. A party of travellers coming to a hotel find that there are a too few bedrooms for each to have one. If they sleep two and two in a room, there are b empty rooms. How many rooms are left empty if they sleep three in a room? [*Ans.* $\frac{1}{3}(a+4b)$

1895.

1. Shew that $(4x^2-8x-1)^2-(2x^2-5x+7)^2$ is divisible by $2x^2-3x-8$, and express the quotient as the product of two factors.
[*Ans.* $(2x-3)(3x-2)$

2. (1) Simplify $\left\{1-\frac{4}{x-1}+\frac{12}{x-3}\right\}\left\{1+\frac{4}{x+1}-\frac{12}{x+3}\right\}$ [*Ans.* 1

(2) Prove that

$$\frac{a(x-b)(x-c)}{bc(a-b)(a-c)}+\frac{b(x-c)(x-a)}{ca(b-c)(b-a)}+\frac{c(x-a)(x-b)}{ab(c-a)(c-b)}=\frac{x^2}{abc}.$$

3. Find the H. C. F. of $x^3+2x^2-3x+20$ and $3x^4-34x^2+51x-20$
[*Ans.* $x+4$.

4. Extract the square root of

$$x^6+2x^4+4x^3+x^2+4x-\frac{4}{x^2}-\frac{8}{x^3}+\frac{4}{x^6}.$$ [*Ans.* $x^3+x+2-\frac{2}{x^3}$.

5. Solve the equations :—

(1) $\frac{1}{x+1}+\frac{2}{2x+3}=\frac{6}{3x+5}$

(2) $\left.\begin{array}{l}\frac{x-y}{a}+\frac{x+y}{b}=c\\ \frac{x-y}{b}-\frac{x+y}{a}=c\end{array}\right\}$

(3) $141x^2-88x-45=0$

[*Ans.* (1) $-\frac{7}{5}$; (2) $\frac{a^2bc}{a^2+b^2}, \frac{-ab^2c}{a^2+b^2}$; (3) $-\frac{1}{3}$ or $\frac{45}{47}$.

6 The gross income of a certain person was Rs 4 more in the second of two particular years than in the first, but as he paid income-tax at the rate of $4p$ in the rupee in the first year and at the rate of $5p$ in the rupee in the second year, his net income in the second year was Rs $6\frac{1}{2}$ less than his net income in the first What was his gross income in each year? [*Ans* Rs 1996 1st year, Rs 2000 second year

1896

1 (1) Simplify $\frac{1}{x^2+3x+2}+\frac{x-1}{2x^2+5x+2}-\frac{x}{2x^2+3x+1}$ [*Ans* 0

(2) Prove that $\frac{(a+2b-3c)^2}{(b+2c-3a)(c+2a-3b)}+\frac{(b+2c-3a)^2}{(c+2a-3b)(a+2b-3c)}$

$$+\frac{(c+2a-3b)^2}{(a+2b-3c)(b+2c-3a)}=3$$

2 Find the H C F of
$6x^4+2x^3+19x^2+8x+21$ and $4x^4-2x^3+10x^2+x+15$ [*Ans* $2x^2+2x+3$

3 Find the L C M of
x^3+a^3, x^3-a^3, $x^4+a^2x^2+a^4$ and x^2-ax+a^2 [*Ans* x^6-a^6.

4 Extract the square root of $4x^2+4x-11-\frac{10}{x}+\frac{7}{x^2}+\frac{6}{x^3}+\frac{1}{x^4}$

[*Ans* $2x+1-\frac{3}{x}-\frac{1}{x^2}$

5 Solve the equations —

(1) $\frac{(x-3)(x-4)}{x-7}=\frac{(x+1)(x+3)}{x+4}$ (2) $\left.\begin{aligned}&\frac{1}{x}+\frac{3}{y}-\frac{2}{z}=6,\\ &\frac{3}{x}+\frac{1}{z}=5,\ \frac{2}{y}+\frac{5}{z}=16\end{aligned}\right\}$

(3) $129x^2-34x-80=0$

[*Ans* (1) $-\frac{23}{3}$, (2) 1, $\frac{1}{3}$, $\frac{1}{2}$, (3) $-\frac{2}{3}$ or $\frac{40}{43}$.

6 One person starts from a place A to walk to a place B and back again at the same time as another person starts from B to walk to A, and back again They meet first at a distance of 2 miles from A and afterwards at a distance of 4 miles from A Find the distance between A and B [*Ans* 5 miles

1897

1 Perform the multiplications —

(i) $(3a^4-4a^3x-5x^4)(3a^4+4ax^3+5x^4)$, (ii) $(x+a)^3(x-a)^5$

[*Ans* (i) $9a^8-12a^7x+12a^5x^3-20a^3x^5-20ax^7-25x^8$,

(ii) $x^8-2ax^7-2a^2x^6+6a^3x^5-6a^5x^3+2a^6x^2+2a^7x-a^8$.

2 Divide (i) $x^5-y^5+\frac{y^{10}}{x^5}$ by $x-y+\frac{y^2}{x}$,

(ii) $(b-c)(x-a)^3+(c-a)(x-b)^3+(a-b)(x-c)^3$ by $(b-c)(c-a)(a-b)$

[*Ans* (i) $x^4+x^3y-xy^3-y^4-\frac{y^5}{x}+\frac{y^7}{x^3}+\frac{y^8}{x^4}$, (ii) $3x-a-b-c$.

3. Resolve into factors —

(i) $16x^8-1$, (ii) $4(ac+bd)^2-(a^2-b^2+c^2-d^2)^2$;

(iii) $(x-1)(x-2)-2(y-1)(x-2)+(y-1)(y-2)$

[*Ans* (i) $(2x^2-1)(2x^2+1)(2x^2+2x+1)(2x^2-2x+1)$;

(ii) $(a-b+c+d)(a+b+c-d)(a+b+c+d)(b+c+d-a)$.

(iii) $(x-y)(x-y-1)$

4. Simplify (i) $\dfrac{(b+c-2a)^3-(c+a-2b)^3}{(c+a-2b)^3-(a+b-2c)^3}$, [*Ans* $\dfrac{a-b}{b-c}$.

(ii) $\dfrac{a+b}{a-b+\dfrac{b^2}{a+b}}-\dfrac{a-b}{a+b+\dfrac{b^2}{a-b}}$. [*Ans.* $\dfrac{4b}{a}$.

5 If $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$, prove that $\dfrac{1}{2}\left(\dfrac{y^2z^2}{x^2}+\dfrac{z^2x^2}{y^2}+\dfrac{x^2y^2}{z^2}\right)=(x+y+z)^2$

6 Solve the equations (i) $\frac{1}{3}(x-4)+\frac{3}{7}(2x-7)-\frac{7}{9}(1+5x)=4(1-x)$.

(ii) $3x-4y+5z+26=0=3y-4z-5x-13$
$=3z-4x+5y-5$ (iii) $\dfrac{4}{2x+3}-\dfrac{18}{7x+12}=5$

7 When the price of sugar rises 50 per cent and the price of tea 10 per cent, the increase in the cost of 3 lbs of tea and 4 lbs of sugar, which together originally cost Rs 3 8 as, is 12 as What is the original price of tea ? [*Ans* 13 as 4 p per lb

1888.

1 Divide (*a*) $(x^2-1)^4-3(x^2-1)^2+1$ by x^4-3x^2+1;

(*b*) $a^3(1-x)+ab(a-b)(x+y)+b^3(1+y)$ by $a(1-x)+b(1+y)$

[*Ans* x^4-x^2-1, a^2-ab+b^2.

2 Resolve into factors —

(1) $(a^2+b^2)^2-(a^2-b^2)^2-(a^2+b^2-c^2)^2$

[*Ans* $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$

(2) $a^3(a+1)+b^3(b+1)-ab(a-b)^2$

[*Ans* $\{a(a+1)+b(b+1)\}(a^2-ab+b^2)$.

3 Find the H. C. F of $3x^4-2x^3+2x^2+8$ and $x^3-7x^2+12x-10$

[*Ans.* x^2-2x+2

4. (*a*) Simplify $\dfrac{1}{x^3-x^2+x-1}+\dfrac{3}{2x^2-x-1}+\dfrac{1-3x}{2x^3+x^2+2x+1}$.

[*Ans* $\dfrac{3}{(x-1)(x^2+1)}$

(*b*) Prove that $\dfrac{(a-5b)(3a+b)^3+(5a-b)(a+3b)^3}{a+b}=32(a-b)^3$.

5. Extract the square root of $b^2(a+4b)^2+3(3a^2-2ab+b^2)(a^2+3b^2)$

[*Ans* $3a^2-ab+5b^2$.

6 Solve the equations :—

(1) $\frac{2x-7}{2x-8}-\frac{3x-8}{3x-9}=\frac{x-3}{x^2-7x+12}$. (2) $x-\frac{ay}{a-b}=b=\frac{ax}{a-b}-y$

(2) $\frac{x-3}{x-2}-\frac{x-4}{x-1}-\frac{1}{4}=0$ [Ans (1) 9, (2) $a-b$, $b-a$ (3) $\frac{1}{12}(43\pm\sqrt{193})$

7 The length and breadth of a room are such that if the former were increased and the latter diminished by 3 yards the area of the room would be diminished by 18 sq yds, while if both were increased by 3 yards, the area would be increased by 60 sq yds Find the length and breadth of the room [Ans 10 yds, 7 yds.

BOMBAY UNIVERSITY.

Selections from the University Questions [1859—1890].

1 Find by inspection the G. C. M of
$(x-1)^2(x-2)(x-3)$ and $(x-1)^3(x-4)(x-5)$ [Ans $(x-1)^2$.

2 Required the product of $\sqrt[3]{a}$ and $\sqrt[4]{a^3}$ [Ans $a^{\frac{13}{12}}$

3. There is a certain number whose three digits are in descending order of magnitude and differ from each other in succession by the same amount If the number be divided by the sum of its digits, the quotient will be 48 ; and if from the number 198 be subtracted, the digits of the difference will be the same as those of the original number but in the reverse order ; find the number [Ans 432.

4 Prove that $\frac{x-y}{\sqrt{(x-y)}-y}=\frac{\sqrt{(x^2-x^2 \ -x\ ^2-y^2)}}{\sqrt{(x^2-y^2)}-y\sqrt{(x-y)}}$

5 Solve $\frac{3x-1}{\sqrt{3x}+1}=1+\frac{\sqrt{3x}-2}{2}$ [Ans $\frac{4}{3}$.

6 Two boats start at the same time from Bassein and Tanna, the distance between which is 18 miles ; at a distance of 1 mile from Tanna the Callian creek falls into the Tanna creek, causing a current of $2\frac{1}{2}$ miles an hour towards Tanna and of 2 miles an hour towards Bassein. The Tanna boat is rowed at the rate of $3\frac{1}{2}$ miles an hour, and the Bassein boat at 3 miles an hour, where will they meet ? [$3\frac{3}{5}$ miles from Bassein

7 Solve the equations
$$\frac{4x^2+2xy+288-6y^2}{2x-13-2y}=2x-3y-131,\quad 5x-4y=22$$ [Ans 26, 27

8 A and B start simultaneously from Poona to go to Kirkee A would reach Kirkee half an hour before B, but missing his way, goes a mile and back again needlessly, during which he walks at twice his former pace and he reaches Kirkee 6 min before B ; C starts 20 min after A and B, and walking at the rate of $2\frac{3}{4}$ miles an hour arrives at Kirkee 10 min after B Find the rates of walking of A and B and the distance from Poona to Kirkee. [Ans $2\frac{1}{2}$ and 2 miles per hour, 5 miles.

9. Solve $\frac{75-x}{3(x+1)}+\frac{80x+21}{5(3x+2)}=\frac{23}{x+1}+5$. [*Ans.* 3.

10 *A* and *B* agreed to reap a field for Rs. 20; if they had worked together every day the field would have been reaped in 15 days, but at the end of 7 days *A* left off working for 4 days, and it consequently took $16\frac{1}{2}$ days to reap the field. In how many days could *A* alone, and in how many days could *B* alone have reaped the field? And what share of the Rs. 20 ought each to receive for the work he actually did?
[*Ans.* 40 and 24 days; Rs. 6. 4 as., Rs. 13. 12 as.

11 If $2s=a+b+c$, prove that

$$s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b)=(s-a)(s-b)(s-c)+abc$$

12 Divide the product of $y^3-12y+16$ and $y^3-12y-16$ by y^2-16, and $(x^2-yz)^3+8y^3z^3$ by x^2+yz. [*Ans.* $(y^2-4)^2$; $x^4-4x^2yz+7y^2z^2$.

13 Solve $\sqrt{\{(x-a)^2+2ab+b^2\}}=x-a+b$ [*Ans.* $2a$.

14. The sum of the three digits of which a number consists is 9; the first digit is one-eighth of the number consisting of the last two, and the last is likewise one eighth of the number consisting of the first two; find the number. [*Ans.* 324.

15. Shew that $\frac{x^2}{a^2}+\left(\frac{z-x}{b}\right)^2$ and $\frac{z^2}{a^2+b^2}+\frac{a^2+b^2}{a^2b^2}\left(x-\frac{za^2}{a^2+b^2}\right)^2$ are identical expressions such that the one can be deduced from the other.

16 Simplify

$$\left\{\sqrt{\left(\frac{a+x}{x}\right)}-\sqrt{\left(\frac{x}{a+x}\right)}\right\}^2-\left\{\sqrt{\frac{x}{a}}-\sqrt{\frac{a}{x}}\right\}^2+\frac{x^2}{a(a+x)}$$ [*Ans.* 1.

17. Solve the equation

$$\frac{a}{x+a}+\frac{b}{x+b}=\frac{a-c}{x+a-c}+\frac{b+c}{x+b+c}$$ [See Addendum]

18. If the telegraph posts by the side of a railway be 60 yards apart, shew that twice the number passed by a train in a minute gives roughly the number of miles per hour at which the train is moving. If eleven posts be passed in a minute, in what time would the distance traversed, estimated by this rule, be one mile in error? [*Ans.* 2 hrs.

19 Divide $a^2x^8+(2ac-b^2)x^4+c^2$ by ax^4+c-bx^2. [*Ans.* ax^4+c+bx^2.

20 If $\left(a+\frac{1}{a}\right)^2=3$, prove that $a^3+\frac{1}{a^3}=0$

21. Extract the cube root of

$$x^3+\frac{8}{x^3}-12x^2-\frac{48}{x^2}+54x+\frac{108}{x}-112$$ [*Ans.* $x+\frac{2}{x}-4$.

22 A person has a number of rupees which he tries to arrange in the form of a square. On the first attempt he has 116 over. When he increases the side of the square by three rupees, he wants 25 to complete the square. How many rupees has he? [*Ans.* 600.

23 Determine the time, between ten and twelve o'clock, at which the hour and minute hands of a common clock are exactly together

[*Ans* $5\frac{5}{11}$ min. to 11, just at 12 o'clock

24 Divide $(4x^3-3a^2x)^2+(4y^3-3a^2y)^2-a^6$ by $x^2+y^2-a^2$.

[*Ans* $16(x^4-x^2y^2+y^4)-8a^2(x^2+y^2)+a^4$

25 If $(a+b)(b+c)(c+d)(d+a)=(a+b+c-d)(bcd-cda+dab+abc)$, then prove that $ac=bd$

26 A person walks from A to B, a distance of $7\frac{1}{2}$ miles, in 2 hours $17\frac{1}{2}$ minutes and returns in 2 hours 20 minutes, his rates of walking up hill, down hill and on a level road being 3, $3\frac{3}{4}$ and $3\frac{1}{2}$ miles per hour respectively Find the length of the level road between A and B [*Ans* $4\frac{5}{8}$ miles

27 I bought a horse and a carriage for £90, I sold the horse at a gain of 12 per cent and the carriage at a loss of 4 per cent, and gained on the whole 6 per cent Find the prime cost of the carriage [*Ans* £33 15*s*.

28 If $a+b=c+d$, prove that either of them is equal to

$$\frac{abcd}{ab+cd}\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right\}$$

29 If $x+\frac{1}{y}=1$ and $y+\frac{1}{z}=1$, prove that $z+\frac{1}{x}=1$ and $xyz+1=0$

30 Simplify $\frac{(b+c)(x^2+a^2)}{(c-a)(a-b)}+\frac{(c+a)(x^2+b^2)}{(a-b)(b-c)}+\frac{(a+b)(x^2+c^2)}{(b-c)(c-a)}$ [*Ans* 0

31 Extract the square root of

$(a-b)^2\{(a-b)^2-2(a^2+b^2)\}+2(a^4+b^4)$, [§ *198*, *Ex* *46*]

32 Given the relation $\frac{1-2bx+b^2}{1-b^4}=\frac{1-b^2}{1+2by+b^2}$,

prove that $\frac{x-y}{1-xy}=\frac{2b}{1+b^2}$ [*App. Mis Ex. 9*]

33 Divide $1+a+a^2+a^3+a^4+a^6+a^7+a^8+a^9+a^{15}$ by $1-a^5+a^6$.

[*Ans* $1+a+a^2+a^3+a^4+a^5+a^6+a^7+a^8$.

34 Simplify

$$\frac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3}.$$ [*Ans* -1.

35 Given $ax+by=m$, $bx-ay=n$, $a^2+b^2=1$, shew that

$$x^2+y^2=m^2+n^2$$

36 Solve $2^x=8^{y+1}$, $9^y=3^{x-9}$ [*Ans* 21, 6

37. Find the value of the expression $x(y+2)+\frac{x}{y}+\frac{y}{x}$ in terms of a, when $x=\frac{y}{y+1}$ and $y=\frac{a-2}{2}$ [*Ans* a.

38 Simplify $\dfrac{\left(p+\frac{1}{q}\right)^m\left(p-\frac{1}{q}\right)^n}{\left(q+\frac{1}{p}\right)^m\cdot\left(q-\frac{1}{p}\right)^n}$ [*Ans.* $\left(\frac{p}{q}\right)^{m+n}$.

39 If $a+b=1$, prove that $(a^2-b^2)^2=a^3+b^3-ab$.

40. Solve the following equation —

$$\frac{a+x}{a^2+ax+x^2}+\frac{a-x}{a^2-ax+x^2}=\frac{2a^4}{x(a^4+a^2x^2+x^4)}.$$ [*Ans* a.

41 Shew that if a number of two digits is four times the sum of its digits, the number formed by interchanging the digits is seven times their sum

42 Find k when $(x-a)(x-3a)(x+a)(x+3a)+k$ is a perfect square [*Ans* $16a^4$.

43 Find the value of—

$$\frac{x^3-3abx-2b^3}{x^2-ab}+\frac{x^2-4ab}{x-2a},\text{ when } x=a+b$$ [*Ans.* 0.

44 If $m=a^x$, $n=a^y$, $a^2=(m^yn^x)^z$, shew that $xyz=1$.

[*App*, *Mis Ex* 169]

45 Simplify—

$$\frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}$$ [*Ans* 1.

46 Two vessels contain mixtures of wine and water, in one there is twice as much wine as water, and in the other, three times as much water as wine Find how much must be drawn off from each, to fill a third vessel which holds 15 gallons, in order that its contents may be half wine and half water [*See* § 236, *Ex* (iii)] [*Ans* 9 gals, 6 gals

1891

1 Simplify by using factors —

(i) $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2}\div\dfrac{x^2-5xy+4y^2}{x^2+xy-2y^2}$. [*Ans* $\dfrac{x-3y}{x+3y}$.

(ii) $\dfrac{(x^2-xy+y^2)^3+(x^2+xy+y^2)^3}{2(x^2+y^2)}$ [*Ans* $x^4+5x^2y^2+y^4$.

2 (i) If $a^2-b^2=b^2-c^2=c^2-a^2$, shew that

$$\frac{ab-c^2}{a-b}+\frac{bc-a^2}{b-c}+\frac{ca-b^2}{c-a}=0$$

(ii) Simplify $\dfrac{\left(p^2-\frac{1}{q^2}\right)^p\left(p-\frac{1}{q}\right)^{q-p}}{\left(q^2-\frac{1}{p^2}\right)^q\left(q+\frac{1}{p}\right)^{p-q}}$. [*Ans* $\left(\frac{p}{q}\right)^{p+q}$.

3 Find the G C M and L C M of
$x^3-2x^2-19x+20$, $x^3+2x^2-23x-60$, $x^4+7x^3-4x^2-52x+48$
[*Ans* $(x+4)$; $(x-1)(x-2)(x+3)(x+4)(x-5)(x+6)$.

4 Solve $$\cfrac{6}{7-\cfrac{6}{7-\cfrac{6}{7-x}}}=1$$ [*Ans* 1

5 What value of a will make the product of $3-8a$ and $3a+4$ equal to the product of $6a+11$ and $3-4a$? [*Ans* 7.

6 The gross income of a certain man was £40 more in the second of two particular years than in the first, but in consequence of the income tax rising from $4d$ in the pound to $6d$ in the pound in the second year, his net income after paying the tax was unaltered Find his income in each year [*Ans* £4680, £4720

7 The sum of the ages of a man and his wife are six times the sum of the ages of their children Two years ago the sum of their ages was ten times the sum of the ages of the children, and six years hence the sum of their ages will be three times the sum of the ages of the children How many children have they? [§ *246*, *Ex* *35*]

1892.

1 Simplify —

(a) $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}+\dfrac{2x^3}{a^4+a^2x^2+x^4}$ [*Ans.* $\dfrac{2(a+x)}{a^2+ax+x^2}$.

(b) $\dfrac{6x^2y^2}{m+n}-\left[\dfrac{3(m-n)x}{7(r+s)}-\left\{\dfrac{4(r-s)}{21xy^2}-\dfrac{r^2-s^2}{4(m^2-n^2)}\right\}\right]$. [*Ans* [illegible].

2 If $a=y+z-2x$, $b=z+x-2y$, $c=x+y-2z$, find $b^2+c^2-a^2+2bc$ in terms of x, y, z [*Ans* 0

3 (a) Find the factors of $(a+b)^2+(a+c)^2-(b+d)^2-(c+d)^2$

(b) Find the L C M of x^4+7x^2+16, x^3+3x+4, x^3+3x-4.

[*Ans* $2(a-d)(a+b+c+d)$, $(x-1)(x+1)(x^2-x+4)(x^2+x+4)$.

4. Find the square root of
$x^4+x^2yz+\frac{1}{4}y^2z^2-2x^2z^2-yz^3+z^4$ [*Ans* $x^2-z^2+\frac{1}{2}yz$

Find the cube root of $\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}-3\left(\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}\right)+5$ [*Ans* $\dfrac{a}{b}-\dfrac{b}{a}-1$.

5 Solve (i) $(a+b-x)(a-b+x)+(a+x)(b+x)-a^2=0$ [*Ans* $\dfrac{b(b-a)}{a+3b}$.

(ii) $3x-2y+2=5x-3y+1\frac{5}{6}=6x-y-4\frac{1}{2}$ [*Ans* $1\frac{1}{3}$, $2\frac{1}{3}$.

6 The denominator of a fraction exceeds the numerator by 4, and if 5 be taken from each, the sum of the reciprocal of the new fraction and four times the original fraction is 5 Find the original fraction [*Ans* $\frac{8}{12}$

1893.

1. Simplify :—

$$\left(1-\frac{1}{1+x}\right)\left(x+\frac{1}{2+x}\right)\times\frac{\frac{1}{x^2}-x}{1+\frac{1}{x}}\div\left(1+x+\frac{1}{x}\right). \qquad \left[Ans.\ \frac{x(1-x)}{2+x}.\right.$$

(ii) $\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}-\frac{x}{x^2-1}+\frac{3}{x(x^2-1)}$. [*Ans.* $\frac{2(x^2+1)}{x(x^2-1)}$.

2. If $b^2=ac$, $x=\frac{1}{2}(a+b)$, and $y=\frac{1}{2}(b+c)$, prove that $\frac{a}{x}+\frac{c}{y}=2$.

3. Find the H. C. F. of $x^3+6x^2+11x+6$ and $x^3+9x^2+27x+27$, and the L. C. M. of xy, $x-y$ and y^3-x^2y. [*Ans.* $x+3$, $xy(y^2-x^2)$.

4. (i) Find the square root of $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-1\frac{3}{4}$, [*Ans.* $\frac{x}{y}-\frac{y}{x}-\frac{1}{2}$.

(ii) and the cube root of $x^3+\frac{8}{x^3}-12x^2-\frac{48}{x^2}+54x+\frac{108}{x}-112$.

[*Ans.* $x-4+\frac{2}{x}$.

5. Solve (i) $\frac{1-x^2}{1-4x^2}-\frac{x}{2x-1}+\frac{1}{4}=0$ [*Ans.* $-1\frac{1}{4}$.

(ii) $\frac{2}{x}-\frac{3}{2y}=\frac{41}{35}$, $\frac{2\frac{1}{2}}{2x}+\frac{3\frac{1}{2}}{y}=-\frac{73}{70}$. [*Ans.* $3\frac{1}{2}$, $-2\frac{1}{2}$.

6. A labourer is engaged for 30 days, on condition that he receives 2*s.* 6*d.* for each day he works, and loses 1*s.* for each day he is idle; he receives £2 7*s.* in all; for how many days did he work? [*Ans.* 22.

1895.

1. Divide

(i) $\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}-2\right)^2$ by $\frac{y}{x}-\frac{x}{y}$ [*Ans.* $\left(\frac{y}{x}-\frac{x}{y}\right)^3$.

(ii) $a_1a_2x^2-b_1b_2y^2+c_1c_2z^2+(a_1b_2+a_2b_1)xy+(a_1c_2+a_2c_1)xz$
$-(b_1c_2+b_2c_1)yz$ by $a_2x+b_2y+c_2z$. [*Ans.* $a_1x+b_1y+c_1z$.

2. Find the H. C. F. of $6x^4-2x^3+9x^2+9x-4$ and $9x^4+80x^2-9$, and find such value of x as will make both these expressions vanish.

[*Ans.* $3x-1$; $\frac{1}{3}$.

3. Resolve into elementary factors the following expressions,

(i) x^4+324 [*Ans.* (i) $(x^2-6x+18)(x^2+6x+18)$.

(ii) $8x^3-5x+3$. [*Ans.* (ii) $(x+1)(8x^2-8x+3)$.

(iii) $56x^2+5xy-99y^2$ [*Ans.* (iii) $(7x-9y)(8x+11y)$.

4. Prove that $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$ and hence or otherwise shew that

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3}=\frac{(a+b)(b+c)(c+a)}{abc}.$$

5 Find the square root of

$$a^3\left(\frac{a^3}{9}-\frac{10}{3b^3}\right)-b^3\left(2-\frac{9b^3}{a^6}\right)-\frac{30}{a^3}+\frac{25}{b^6}$$ [Ans $\frac{a^3}{3}-\frac{3b^3}{a^3}-\frac{5}{b^3}$.

6 Solve —

(i) $\frac{2x-11}{x-5}-\frac{9x-9}{3x-4}=\frac{4x-13}{x-3}-\frac{15x-47}{3x-10}$ [Ans $-\frac{5}{6}$

(ii) $\left.\begin{array}{l} b(a-b)x=a(a-b)y, \\ \frac{a-bx}{a^2}-\frac{b-ay}{b^2}=\frac{x}{a}+\frac{y}{b}, \end{array}\right\}$ [Ans $\frac{a}{a-b}, \frac{b}{a-b}$.

7 A father's age was triple that of his son 5 years ago, while 5 years hence the father will be twice as old as his son; determine their respective ages. [Ans 35; 15.

1896

1 If a number is equal to the sum of two perfect squares, shew by an algebraical relation that the square of the number is itself the sum of two other perfect squares [Ans $(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$.

Express $(34)^2$ as the sum of two perfect squares [Ans 16^2+30^2.

2 Find an expression containing no higher power of x than the first, which added to $x^4+6x^3+13x^2+6x-1$, will make it a complete square
[Ans $6x+3$.

3 Find the cube root of

$8x^9-12x^8+6x^7-37x^6+36x^5-9x^4+54x^3-27x^2-27$ [Ans $2x^3-x^2-3$.

4 (a) Find the Least Common Multiple of

$4x^2-6yz-(9y^2+z^2)$, $9y^2-4xz-(4x^2-z^2)$, and $z^2-12xy-(4x^2-9y^2)$
[Ans $(2x-3y+z)(2x-3y-z)(2x-3y+z)(2x-3y-z)$.

(b) Simplify $\frac{3x^2-(4a+2b)x+a^2+2ab}{x^3-(2a-b)x^2-(a^2-2ab)x-a^2b}$ [Ans $\frac{3x-(a+2b)}{(x-a)(x-b)}$.

5. Solve (i) $\frac{x^2-4x-4}{x-1}-\frac{x^2-3x-1}{x-2}=2\frac{x^2-5x+5}{x-3}$. [Ans 1.

(ii) $(a-b)x+by=ax+(a+b)y=a^3-b^3$ [$a(a-b)$, $b(a-b)$

6 A certain number of two digits is equal to seven times the sum of the digits If the digit in the units' place be decreased by 2 and that in the tens' place by 1, and the number thus found be divided by the sum of its digits, the quotient is 10 Find the number [Ans 42

1897

1 Define simple and compound algebraical expressions; and give an example of a homogeneous expression of 5 dimensions containing four terms

Multiply together the expressions $1+ax+\frac{1}{2}a(a-1)x^2+\frac{1}{6}a(a-1)(a-2)x^3$ and $1+bx+\frac{1}{2}b(b-1)x^2+\frac{1}{6}b(b-1)(b-2)x^3$ as far as the term involving x^3, and resolve into factors the coefficient of x^3 in the product

[Ans Term involving $x^2=\frac{1}{2}(a-b)(a-b-1)x^2$.

2 (*a*) Find what quantity not involving higher powers of x beyond the second should be added to $x^8-3x^7-5x^5+2x^4+5x^3+4x^2+1$ to make it exactly divisible by x^3+2x-1. [*Ans.* $-5x$.

(*b*) Resolve into factors $x^4-11x^2y^2+y^4$ and $(a+b+c)^3-a^3-b^3-c^3$.
[*Ans* $(x^2-3xy-y^2)(x^2+3xy-y^2)$, $3(b+c)(c+a)(a+b)$.

3 Find the H C F of $6x^5+35x^4+59x^3+19x^2-17x-6$ and $6x^5-5x^4-41x^3+71x^2-37x+6$ [*Ans* $2x^2+5x-3$.

4 Extract the square root of
$(a+b)(a+b+c)(a+b+2c)(a+b+3c)+c^4$. [*Ans* $(a+b)^2+3c(a+b)+c^2$.

5 Prove that the value of a fraction is not altered if its numerator and denominator are multiplied by the same quantity

Reduce to a single term

$$\frac{1}{\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)}+\frac{1}{\left(1-\frac{c}{b}\right)\left(1-\frac{a}{b}\right)}+\frac{1}{\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right)}$$ [*Ans* 1

6 Solve the equations

(i) $\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}$. [*Ans* 4.

(ii) $\begin{cases}2ab(x-y)=xy(a-b),\\ 2ab(x+y)+xy(a+b+2ab)=0\end{cases}$ [*Ans* $-\frac{2b}{b+1}, -\frac{2a}{a+1}$.

7 A market womans sells 1000 oranges, some at a gain of 25 per cent. and the rest at a gain of 15 per cent, and thereby gains 18 per cent. on the whole How many of each sort does she sell? [*Ans* 300, 700.

1900

1 (*a*) Divide $x^4-3(a+1)x^3+2(3a+1)x^2+3(a+1)(a^2-1)x+(a^2-1)^2$ by $x^2-(a+3)x-a^2+1$ [*Ans* $x^2-2ax-(a^2-1)$.

(*b*) Resolve into factors $12x^2+x-35$, $(2b-a)^3+(2a-b)^3-(a+b)^3$
[*Ans* $(3x-5)(4x+7)$; $3(2a-b)(a-2b)(a+b)$.

2 Prove that the product of the Highest Common Factor and the Lowest Common Multiple of two expressions is equal to the product of the expressions themselves

Find the H. C F of
$x^4+5x^3-36x^2+50x+48$ and $x^4+x^3-12x^2-2x+80$ [*Ans* x^2-5x+8.

3 Simplify (i) $\frac{a^2}{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{c}\right)}+\frac{b^2}{\left(\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}-\frac{1}{a}\right)}+\frac{c^2}{\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}-\frac{1}{b}\right)}$;
[*Ans* $abc(a+b+c)$.

(ii) $\frac{\sqrt[3]{(64x^6-48x^4+12x^2-1)}-\sqrt{(16x^4-64x^3+24x^2+80x+25)}}{4x^2-12x-7}$.

[*Ans* $\frac{4}{2x-7}$ or $\frac{4x-6}{2x-7}$.

4 Solve the equations—

(i) $\frac{21}{4}\left(\frac{2x}{3}-\frac{5}{18}\right)+\frac{7x-3\frac{3}{4}}{12}=2\frac{10}{144}-\frac{14-15x}{3}$, [Ans $\frac{5}{6}$.

(ii) $(1+p)(x-py)=2p^2\left(\frac{x}{1+p}+\frac{y}{1-p}\right)=\frac{2p^2}{1-p}$ [Ans $\frac{p}{1-p}, \frac{1}{1+p}$.

5 Two passengers have together 7 maunds of luggage, and for the excess above the weight allowed free one of them is charged Rs 3 and the other Rs 5 If all the luggage had belonged to one passenger he would have been charged Rs 11 What amount of luggage is each passenger allowed free of charge? [Ans $1\frac{1}{2}$ mds

1901.

1 (a) Divide $2x^4-6ax^3+(4a^2+ab-2b^2)x^2+3ab^2x-a^2b^2$ by $x^2-(2a-b)x-ab$ [Ans $2x^2-2(a+b)x+ab$

(b) Resolve into factors $x^2+6x-187$ and $x^4-5x^3+9x^2-7x+2$ [Ans $(x+11)(x-17)$, $(x-1)^3(x-2)$

2 Simplify (1) $\frac{x^3+2x^2-29x-30}{x^3-3x^2-34x+120}$ [Ans $\frac{x+1}{x+4}$

(2) $\frac{3a+2b+2c}{(a-b)(a-c)}+\frac{3b+2c+2a}{(b-c)(b-a)}+\frac{3c+2a+2b}{(c-a)(c-b)}$ [Ans 0

3 Find the square root of $x^6-\frac{3}{2}x^4+\frac{2}{3}x^3+\frac{9}{16}x^2-\frac{1}{2}x+\frac{1}{9}$, and the cube root of $8+36x+42x^2-9x^3-21x^4+9x^5-x^6$

[Ans $x^3-\frac{3}{4}x+\frac{1}{3}$, $2+3x-x^2$.

4 Upon what axioms does the process of solving a simple equation depend?

Solve (i) $\frac{x}{5}-\{3x+6-\frac{4}{5}(x+10)\}=2\frac{1}{4}-11(9-\frac{5}{12}x)$, [Ans 15

(ii) $\frac{2x+3y}{5a+b}=\frac{ab}{a^2-b^2}=\frac{ax+by}{a^2+b^2}$. [Ans $\frac{ab}{a+b}, \frac{ab}{a-b}$

5 Two cyclists ride from A to B, a distance of 55 miles, and the first arrives 30 minutes before the second They then ride from B to A, the first giving the second a start of 4 miles, and yet arriving 6 minutes before him Find the rate of each cyclist in miles per hour [Ans 11, 10

1902

1 (a) Subtract $\frac{x+5}{x^2+5x-6}$ from $\frac{x+6}{x^2+3x-10}$, and divide the difference by $1+\frac{2(x^2+4x-8)}{x^2+11x+30}$ [Ans $\frac{3x^2+19x+14}{(x-1)(x-2)(x+5)(x+6)}$; $\frac{1}{(x-1)(x-2)}$

(b) Resolve into factors—

(1) $4x^2-9y^2-6x-9y$ [Ans. $(2x+3y)(2x-3y-3)$

(2) $x^4-5x^3+x^2+21x-18$ [Ans $(x-1)(x+2)(x-3)^2$.

2 State and prove the rule for finding the lowest common multiple of two algebraical expressions.

Find the L. C. M. of $4x^3-20x^2+17x-4$, $2x^3-15x^2+31x-12$ and $4x^3-16x^2+13x-3$ [*Ans.* $2(x-1)^2(x-3)(x-4)$

3 Find the square root of—

$a^2x^6+6abx^4-2acx^3+9b^2x^2-6bcx+c^2$, [*Ans* $ax^3+3bx-c$.

and the cube root of—

$x^6+\frac{27}{x^6}-6\left(x^4+\frac{9}{x^4}\right)+21\left(x^2+\frac{3}{x^2}\right)-44$ [*Ans.* $x^2+\frac{3}{x^2}-2$.

4 Solve the following equations:—

(1) $\frac{1.05x+10}{50}+\frac{1.35x-2}{40}-\frac{1.5x-18}{10}+\frac{1.5x-3}{15}=1.854$. [*Ans* 4

(2) $\frac{(a-b)x+(a+b)y}{a^2-b^2}=\frac{ab}{a-b}=\frac{ab(x-y)-(a^2y-b^2x)}{2ab^2}$. [*Ans* $\frac{a^2b}{a-b}$, $\frac{ab^2}{a+b}$.

5 A man travels part of a journey on a bicycle, and then for the last 72 miles takes a train which travels four times as fast as he did on his bicycle, and arrives at his destination in $3\frac{1}{2}$ hours from the start. If he had travelled the whole way in the train he would have saved $1\frac{1}{2}$ hours. Find the length of the journey in miles. [*Ans* 96 miles

PUNJAB UNIVERSITY.

Selections from the University Questions [1887—1890]

1 Extract the square root of $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$. [*Ans* $x-2-\frac{1}{x}$.

2 Two towns L and M are 30 miles apart. A sets off from L to M and B from M to L at the same moment. A reaches M 16 hours, and B reaches L 36 hours after they have met on the road. Find the time taken by each to perform the journey. [*Ans* 40 hrs. ; 60 hrs.

3 If $alx+bmy+cnz=apx+bqy+crz=ax^2+by^2+cz^2=0$, prove that

$x(mr-qn)+y(np-rl)+z(lq-pm)=0$ [See Addendum].

4 Extract the square root of $1+\frac{41}{16}x-\frac{3+3x}{2}\sqrt{x}+x^2$. [*Ans* $1-\frac{3}{4}\sqrt{x}+x$

5 Simplify $\left\{\frac{(x^m)^{\frac{1}{r}}(x^n)^{\frac{1}{p}}}{\sqrt[n]{x^p}\,\sqrt[m]{x^r}}\right\}^{pr}$. [*Ans.* $x^{\frac{(mp+nr)(mn-rp)}{mn}}$

6 Prove that $\frac{2}{b-c}+\frac{2}{c-a}+\frac{2}{a-b}+\frac{(b-c)^2+(c-a)^2+(a-b)^2}{(b-c)(c-a)(a-b)}=0$.

[§ 183, Ex. 8].

7 What quantity of corn at Rs 5 per maund must a tradesman mix with 560 maunds, at Rs 6 per maund, in order to gain 20 per cent. by selling the whole at 2 annas 6 pies per seer? [*Ans.* 2128 mds

8 If $y^2 - z^2 \quad yz \quad b - c \quad x$ and $z^2 - x^2 \quad zx \cdot c - a \quad y$,
then prove that $x^2 - y^2 \quad xy \quad a - b \quad z$ [*See* Addendum]

9. Divide $x^5 + y^5$ by a number which is greater than x by y
[*Ans* $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

10 Resolve $a^2x^3 - \frac{8a^2}{y^3} - x^3 + \frac{8}{y^3}$ into four factors.

[*Ans* $(a+1)(a-1)\left(x - \frac{2}{y}\right)\left(x^2 + \frac{2x}{y} + \frac{4}{y^2}\right)$.

11 What is the solution of $(x+2a)^2 + y^2 = 0$? [*See* § 264] [*Ans* $-2a$, 0

12 A train travelled a certain distance at a uniform rate Had the speed been 6 miles an hour more, the journey would have occupied 4 hours less, and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more Find the distance [*See* § 233]. [*Ans* 720 miles.

13 Multiply $(b+c)^2 - a^2$ by $a^2 - b^2 - c^2 + 2bc$, and divide the product by $b^2 - (c-a)^2$ [*Ans* $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$, $c^2 + 2ca + a^2 - b^2$.

14 Solve the equations (*a*) $1 - \dfrac{1 - \dfrac{3}{3-x}}{3-x} = \dfrac{x}{x-3}$ [*Ans* $4\frac{1}{2}$.

(*b*) $xy + 3y = 20$, $5y - 4 = 2xy$ [*Ans* 2, 4

15 If y is a mean proportional between x and z, shew that $xy + yz$ is a mean proportional between $x^2 + y^2$ and $y^2 + z^2$

16 *A* and *B* travel together 120 miles by rail *A* takes a return ticket for which he has to pay one fare and a half Coming back they find that *A* has travelled cheaper than *B* by 4 annas 2 pies for every 100 miles Find the fare per mile [*Ans* 2 pies.

1891

1 Simplify $(x-y)(x-y-z)(x+2y-2z) + y(y-z)(3x-2y-2z)$, and express the result as the product of factors [*Ans* $x(x-z)(x-2z)$

2 Simplify $\left(\frac{x^2-x+1}{12}\right)^3 - 27\left\{\frac{(x+1)(x-2)(2x-1)}{432}\right\}^2$, and extract the square root of the result [*Ans* $\frac{x^2(x-1)^2}{256}$, $\frac{x(x-1)}{16}$.

3 Find the relation between b and c, so that $x^3 + bx + c$ and $x^3 + cx + b$ may have a common divisor [*Mis Ex* V, *Ex* 34]

4 If h be the highest common divisor and l the lowest common multiple of two quantities x and y, and if $h + l = x + y$, prove that $h^3 + l^3 = x^3 + y^3$.

4 Solve the equations —

(1) $\frac{5\ 2x}{13} - \frac{1-2x}{5}\left(\frac{1}{5} - 1\right) = \cdot 1x - \frac{5x-2}{4} + \ 028$ [*Ans.* 4.

(2) $2x + 4y = 1\ 2$, $3\cdot 4x - \ 02y = \cdot 01$. [*Ans.* $\cdot 02$, 2 9.

5 A person bought an article and sold it at a profit of 6 per cent. Had he bought it at 4 per cent less, and sold at 1 Re. 3 as more, his profit would have been 12 per cent. For how much did he buy it? [*Ans.* Rs. 78 2 as.

1892.

1. Divide

$$\frac{x^4}{3}-\frac{11x^3}{12}+\frac{41x^2}{8}-\frac{23x}{4}+6 \text{ by } \frac{2x^2}{3}-\frac{5x}{6}+1. \qquad \textit{Ans. } \frac{x^2}{2}-\frac{3x}{4}+6.$$

2 Find the simplest expression for

$$\frac{p+q}{(q-r)(r-p)}+\frac{q+r}{(r-p)(p-q)}+\frac{r+p}{(p-q)(q-r)}. \qquad [\textit{Ans. } 0.$$

3 Solve :—

(i) $\sqrt{a^2+x^2}+\sqrt{a^2-x^2}=d.$ [*Ans* $\{\frac{1}{4}d^2(4a^2-d^2)\}^{\frac{1}{2}}$.

(ii) $\frac{x+2}{7}+\frac{y-x}{4}=2x-8,\ \frac{2y-3x}{3}+2y=3x+4.$ [*Ans.* 5, 9

4. Define ratio, and shew that when c is greater than d, and c, d and y are all positive, $c+y : d+y$ is less than $c : d$. Also shew that if $c : d=x : y$, then will $\frac{cd}{xy}=\frac{c^2-d^2}{x^2+y^2}$.

5 A and B start together from the same point on a walking match round a circular course. After half an hour, A has walked three complete circuits, and B four and a half. Assuming that each walks with uniform speed, find when B next overtakes A. [*Ans.* 10 min.

1893

1 If $a+b=x$, $a-b=y$, express $16(a^4+a^2b^2+b^4)$ as the product of two expressions involving x and y. [*Ans.* $(3x^2+y^2)(x^2+3y^2)$

2 Prove the Remainder Theorem [§ 275]

Exhibit in the form of products :—

(i) $xy(x-y)+yz(y-z)+zx(z-x)$ [*See* § 132, *Ex.* 1]

(ii) $(x-y)^5+(y-z)^5+(z-x)^5$. [*See* § 132, *Ex.* 5]

3. Divide $x^6-1-5(x-1)$ by $(x-1)^2$ [*Ans.* x^3+2x^2+3x+4.

4 Solve the equations :—

(i) $\frac{a}{bx-p}=\frac{b}{ax-p}$. (ii) $\frac{x}{4}+\frac{y}{5}+1=\frac{x}{5}+\frac{y}{4}=23$. [*Ans* $\frac{p}{a+b}$; 40, 60.

5 The sum of two fractions, which are reciprocals of each other, is $2\frac{1}{6}$. Find their difference. [*Ans* $\frac{5}{6}$

1894.

1. Exhibit $(x^2+y^2)(a^2+b^2)$ as the sum of two squares.

[*Ans.* $(ax+by)^2+(ay-bx)^2$.

2. Find the H. C. F. of $3x^3-23x^2+43x-8$ and $x^4-5x^3-6x^2+35x-7$. [*Ans.* x^2-5x+1.

3 Simplify $\left\{\dfrac{\frac{x}{y}+2}{\frac{x}{y}+1}+\dfrac{x}{y}\right\}-\left\{\dfrac{x}{y}+2-\dfrac{\frac{x}{y}}{\frac{x}{y}+1}\right\}$ [Ans 1

4 Solve

(i) $\dfrac{\sqrt{(x+a)}+\sqrt{(x-b)}}{\sqrt{(x+a)}-\sqrt{(x-b)}}=\dfrac{a+b}{a-b}$ [Ans $\dfrac{ab}{a-b}$

(ii) $\dfrac{a}{x}+\dfrac{b}{y}=c,\ \dfrac{b}{x}+\dfrac{a}{y}=d$ [Ans $\dfrac{a^2-b^2}{ac-bd}, \dfrac{a^2-b^2}{ad-bc}$

5 A man who went out between 5 and 6 and returned between 6 and 7, found that the hands of his watch had exactly changed places When did he go out? [Ans $32\frac{4}{13}$ min past 5

1895

1 Divide $a+b+c+3(b^{\frac{1}{3}}+c^{\frac{1}{3}})(c^{\frac{1}{3}}+a^{\frac{1}{3}})(a^{\frac{1}{3}}+b^{\frac{1}{3}})$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$

[Ans $(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})^2$

2 Find the square root of

$a^2(x^6+2x^3+1)+2ab(x^5+x^4+x^2+x)+b^2(x^4+2x^3+x^2)$

[Ans $a(x^3+1)+b(x^2+x)$.

3 Find the condition that $x^3+(p+q)x+a$ may be divisible by $x+p+q$, and find the H C D of $x^5+11x-12$ and x^5+11x^3+54

[Ans $(p+q)^2(p+q+1)=a$, x^2-2x+3

4 Solve the equation $(a+x)(b+x)-a(b+c)=\dfrac{a^2c}{b}+x^2$ [Ans $\dfrac{ac}{b}$

5 Two men leave two places A and B, distant d miles from each other, and travel a and b miles a day respectively in the same straight line AB What is their distance apart at the end of t days, and after what time will they come together? [Ans $\{d-(a+b)t\}$ miles, $\dfrac{d}{a+b}$ hours

1896

1 Reduce to their simplest forms

(i) $\dfrac{\sqrt{x}}{\sqrt{x}-\sqrt{a}}-\dfrac{\sqrt{a}}{\sqrt{x}+\sqrt{a}}-\dfrac{x-a}{x+a}$ [Ans $\dfrac{4ax}{x^2-a^2}$

(ii) $\dfrac{(a+b\sqrt{c})^3+(a-b\sqrt{c})^3-(2b\sqrt{ac})^2}{(a+b\sqrt{c})^2+(a-b\sqrt{c})^2}$. [Ans a

2 Find the H C F of $20x^4+x^2-1$ and $25x^4+5x^3-x-1$ [Ans $5x^2-1$

3 If $a : b$ be the duplicate ratio of $a+x : b+x$, find x [Ans ab

4 Solve (i) $\dfrac{1+\frac{4bx}{a}}{bx}=\dfrac{\frac{4a}{bx}+1}{a}$ (ii) $\dfrac{x-a}{y+b}=\dfrac{y-b}{x+a}=\dfrac{c}{d}$

[Ans (i) $\dfrac{a}{b}$, (ii) $\dfrac{2bcd+a(c^2+d^2)}{d^2-c^2}$, $\dfrac{2acd+b(c^2+d^2)}{d^2-c^2}$.

5 Two numbers each consisting of the same two digits are in the ratio of 4 7. Find the numbers [*Ans* 12, 21

1897.

1 Simplify the following expression :—

$$1+\frac{a}{x-a}+\frac{bx}{(x-a)(x-b)}+\frac{cx^2}{(x-a)(x-b)(x-c)} \qquad \left[\frac{x^2}{(x-a)(x-b)(x-c)}\right.$$

2 Find the factors of $a^3-2abc-ab^2-ac^2$. [*Ans.* $a(a+b+c)(a-b-c)$.

3. Extract the square root of

$(bc+ca+ab+a^2)(bc+ca+ab+b^2)(bc+ca+ab+c^2)$

[*Ans* $(b+c)(c+a)(a+b)$

4 Solve the following equations —

(i) $15x+1\ 2-\ 875+\ 375-\ 0625x=0$ [*Ans* 2

(ii) $cy+bz=bc,\ az+cx=ca,\ bx+ay=ab$ [*Ans* $\frac{1}{2}a,\ \frac{1}{2}b,\ \frac{1}{2}c$

5 A man rows to a place 48 miles distant and back in 14 hours He finds that he can row 4 miles with the stream in the same time as 3 miles against the stream Find the rate of the stream [*Ans* 1 mile per hr

1898

1. Simplify $\left\{\dfrac{b+\dfrac{a-b}{1+ab}}{1-\dfrac{(a-b)b}{1+ab}}-\dfrac{a-\dfrac{a-b}{1-ab}}{1-\dfrac{a(a-b)}{1-ab}}\right\}-\left(\dfrac{a}{b}-\dfrac{b}{a}\right)$ [*Ans* $\dfrac{ab}{a+b}$.

2 Solve (i) $\dfrac{3x+\frac{1}{2}}{3}+\dfrac{x+1}{5}=\dfrac{5}{6}$. [*Ans* $\frac{7}{18}$

(ii) $\dfrac{x}{a+b}+\dfrac{y}{a-b}=2,\ b\left(\dfrac{1}{x}+\dfrac{1}{y}\right)=a\left(\dfrac{1}{y}-\dfrac{1}{x}\right)$ [*Ans* $a+b,\ a-b$

3 Express $X^3+Y^3+Z^3-3XYZ$ in terms of a, b, and c, being given $X=b+c-a$, $Y=c+a-b$, $Z=a+b-c$ [*Ans* $4(a^3+b^3+c^3-3abc)$

4 Find the H C F. of $x^3-3a^2x-2a^3$ and $x^3-ax^2-4a^3$, and the L C M of $2x^3-7x-2$ and $2x^2-x-6$ [$x-2a$, $(2x^2+4x+1)(2x^2-x-6)$.

5 Find three numbers which are to one another as 2 3 . 5, and such that the sum of the greatest and the least exceeds the other by 24.

[*Ans* 12, 18, 30.

1899.

1 Divide $x^{12}-x^{-12}+6(x^8-x^{-8})+9(x^4-x^{-4})$ by $x^6-x^{-6}+3(x^2-x^{-2})$ [*Ans* $x^6+x^{-6}+3(x^2+x^{-2})$

2 Find the value of $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$, when $x=\dfrac{4ab}{a+b}$. [§ 183, Ex 1]

3 Find an expression containing no higher power of x than the first, which added to $x^4+6x^3+13x^2+6x+1$ will make it a complete square

[*Ans.* $6x+3$

4 Simplify $\frac{1}{2} \cdot \frac{\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}-1} \cdot \frac{1+\sqrt{\left(\frac{x-1}{x+1}\right)}}{1-\left(\frac{x-1}{x+1}\right)}$

$$+\frac{1}{4} \cdot \frac{\sqrt{(x+1)}-\sqrt{(x-1)}}{x-\sqrt{(x^2-1)}} \cdot \frac{\sqrt{(x-1)}}{\sqrt{\left(\frac{x+1}{x-1}\right)}+1}$$ [*Ans* $\frac{1}{2}x$

5 Find two numbers in the ratio $1\frac{1}{2} : 2\frac{2}{3}$, such that when increased by 15, they shall be in the ratio $1\frac{2}{3} : 2\frac{1}{2}$ [*Ans* 27, 48

1900

1 If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, prove that

$$bx+cy+az=0=cx+ay+bz$$

2 Find a homogeneous and symmetrical expression of the second degree in x and y which shall be equal to 3, when x and y are equal to unity, and shall be equal to 11, when $x=2$, $y=1$. [*Ans* $5x^2-7xy+5y^2$

3. Find the square root of—

$$x+\frac{1}{x}+\sqrt{2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)+\frac{5}{2}$$ [*Ans* $\sqrt{x}+\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{2}}$.

4 Solve the equations—

(i) $69x-\frac{49}{y}=182\frac{1}{2}$, $49x-\frac{69}{y}=112\frac{1}{2}$, [*Ans* 3, 2.

(ii) $2y+z=11$, $2z+x=12$, $2x+y=13$ [*Ans* $4\frac{2}{3}$, $3\frac{2}{3}$, $3\frac{2}{3}$.

5 Three numbers are in the ratios of 2 : 3 : 5, and the sum of their cubes is 4320 Find them [*Ans* 6, 9, 15

If four positive numbers are in continued proportion, shew that the difference between the extremes is at least three times as great as the difference between the means

ALLAHABAD UNIVERSITY.

Selections from the University Questions [1889—1890]

1 Remove the brackets from the expression

$$(x-a)(x-b)(x-c)-[bc(x-a)+\{(a+b+c)x-a(b+c)\}x]$$

[*Ans* $x^3-2ax^2-2bx^2-2cx^2+2abx+2acx$.

2 Simplify

(a) $42\left\{\frac{4x-3y}{6}-\frac{3x-4y}{7}\right\}-56\left\{\frac{3x-2y}{7}-\frac{2x-3y}{8}\right\}$. [*Ans.* $-2y$.

3 Solve the following equations —

$(a+b)x-(a-b)y=3ab$, $(a+b)y-(a-b)x=ab$ [*Ans.* $\frac{1}{2}(2a+b)$, $\frac{1}{2}(2a-b)$.

4 A who travels $3\frac{1}{2}$ miles an hour starts $2\frac{1}{2}$ hours before B who goes the same road at $4\frac{1}{2}$ miles an hour where will he overtake A?
[*Ans* At $39\frac{3}{8}$ miles from starting place

5 If $s=a+b+c$, prove that

$$(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c-a)^2(a+b)^2.$$

6 Extract the square root of $\frac{a^2}{b^2}+\frac{b^2}{a^2}+3+\frac{2a}{b}+\frac{2b}{a}$. [*Ans* $\frac{a}{b}+1+\frac{b}{a}$.

7. Simplify $\left(y-\frac{a^2-xy}{y-x}\right)\left(x+\frac{a^2-xy}{y-x}\right)+\left(\frac{a^2-xy}{y-x}\right)^2$. [*Ans* a^2.

8 A person walked out a certain distance at the rate of $3\frac{1}{2}$ miles an hour, and then ran part of the way back at the rate of 7 miles an hour, walking the remaining distance in 7 minutes. He was out 35 minutes. How far did he run? [*See* § 233, *Ex* 9] [*Ans.* $\frac{49}{60}$ mile.

1891

1. Define the following :—"term," "dimension of a term," "homogeneous terms"

2 Express in their simplest forms—

(i) $\left(1-\frac{2xy}{x^2+y^2}\right)\div\left(\frac{x^3-y^3}{x-y}-3xy\right)$, [*Ans* $\frac{1}{x^2+y^2}$.

(ii) $(x-y+z)(x+y-z)-(x+y+z)(x-y-z)-4yz$ [*Ans.* 0

3 State and prove the two lemmas on which the proof of the rule for finding the G. C. M. depends [§§ *153 and 154*.]

Find the G. C. M. of

$x^3-2ax^2-5a^2x-12a^3$ and $x^3-7ax^2+13a^2x-4a^3$. [*Ans.* $x-4a$

4 Solve—

(i) $\frac{x-\frac{1}{2}}{x-1}-\frac{3}{5}\left(\frac{1}{x-1}-\frac{1}{3}\right)=\frac{23}{10(x-1)}$. [*Ans.* 3

(ii) $\left.\begin{array}{l}(a+b)x+(a-b)y=2a\\(a-b)x+(a+b)y=2b\end{array}\right\}$. [*See* § 241, *Ex.* 18]

5 A farmer bought equal numbers of two kinds of sheep, one at £3 each, the other at £4 each Had he expended his money equally in the two kinds, he would have had two more sheep than he did How many did he buy?
[*Ans* 96

6. Find the square root of

$x^6-12x^5+60x^4-160x^3+240x^2-192x+64$. [*Ans.* $x^3-6x^2+12x-8$

1892

1. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$. [§ *183 Ex 1*]

2 (*a*) Find the H. C. F. of $x^3-x^2-8x+12$ and $3x^2-2x-8$
[*Ans.* $x-2$.

(b) Extract the square root of $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$ [Ans $x-2-\frac{1}{x}$

3 Simplify :—

$$\frac{a(a+1)+1}{(a-b)(a-c)}+\frac{b(b+1)+1}{(b-a)(b-c)}+\frac{c(c+1)+1}{(c-a)(c-b)}$$ [§ 180, Ex 10]

4 Solve the following equations —

(i) $\frac{(x+a)(x+b)}{x+a+b}=\frac{(x+c)(x+d)}{x+c+d}$ [Ans $\frac{ab(c+d)-cd(a+b)}{cd-ab}$.

(ii) $\frac{m}{x}+\frac{n}{y}=a,\ \frac{n}{x}+\frac{m}{y}=b$ [Ans $\frac{m^2-n^2}{am-bn},\ \frac{m^2-n^2}{bm-an}$.

5 If a b c d, prove that $\frac{2a+3b}{4a+5b}=\frac{2c+3d}{4c+5d}$

1893

1 Simplify —

(a) $\left(\frac{x-y}{x+y}-\frac{x^3-y^3}{x^3+y^3}\right)\left(\frac{x+y}{x-y}+\frac{x^3+y^3}{x^3-y^3}\right)$ [Ans $\frac{-4xy(x^2+y^2)}{x^4+x^2y^2+y^4}$

(b) $\frac{2}{b-c}+\frac{b-c}{(c-a)(a-b)}+\frac{2}{c-a}+\frac{c-a}{(a-b)(b-c)}+\frac{2}{a-b}+\frac{a-b}{(b-c)(c-a)}$ [0

2 Find H C D of

$2x^4-2x^3+x^2+3x-6$ and $4x^4-2x^3+3x-9$ [Ans $2x^2-3$

3 Solve —

(a) $5+\cfrac{2}{3-\cfrac{1}{4-x}}=\frac{29}{5}$ (b) $\frac{x-a}{b-a}+\frac{x-c}{b-c}=2$ [Ans (a) 2 , (b) b

(c) $\frac{x-a}{c-a}+\frac{y-b}{c-b}=1,\ \frac{x+a}{c}+\frac{y-a}{a-b}=\frac{a}{c}$ [Ans $c,\ b$

4 If I subtract from the double of my present age the treble of my age six years ago, the result is my present age What is my present age ?
[Ans 9 years

5 What is Involution ? Find the square root of

$1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$ [Ans $1-2x+3x^2-4x^3$

6 If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, each of these ratios $=\left(\frac{pa^3+qc^3+re^3}{pb^3+qd^3+rf^3}\right)^{\frac{1}{3}}$

1894

1 (a) Divide $(x+y)^3-8z^3$ by $x+y-2z$
[Ans $(x+y)^2+2z(x+y)+4z^2=$&c

(b) Shew that $(x+2)(x+3)(x+4)(x+5)+1$ is a perfect square
[Ans $\{(x^2+7x)+11\}^2$

2. Resolve into elementary factors
$39x^2-7x-22$, x^4+2x^2+9, $a^3+b^3+c^3-3abc$ and $(a+b-3c)^2-a-b+3c$
[*Ans.* $(3x+2)(13x-11)$; $(x^2+2x+3)(x^2-2x+3)$, § 130; $(a+b-3c)(a+b-3c-1)$

3. Simplify
$$\left(\frac{ax}{x^2-y^2}-\frac{b}{y-x}-\frac{a}{x+y}\right)-\left(\frac{ax}{a^2-b^2}-\frac{y}{b-a}-\frac{x}{a+b}\right).$$ [*Ans* $\frac{a^2-b^2}{x^2-y^2}$.

4. Solve :—

(i) $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{x+a+b}+\frac{1}{x}$, [*Ans* $\frac{1}{3}(a+b)$

(ii) $\frac{a+b}{x}-5b=\frac{a-b}{y}-a$, $\frac{a}{x}-2a=\frac{b}{y}-3b$. [*Ans* $\frac{1}{2}$, $\frac{1}{3}$

5. *A* says to *B* : Two-fifths of my salary is $\frac{4}{25}$ of yours, and the difference between our salaries is Rs 600. What is *A*'s salary?
[*Ans* Rs 400

6. If $a : b = c : d$, prove that $a : a+c = a+b : a+b+c+d$

1895.

1. Resolve into factors :—
(i) x^3+4x^2+4x (ii) x^3+x^2-x-1 (iii) $a^2b^2-a^2-b^2+1$.
[*Ans* $x(x+2)(x+2)$, $(x+1)^2(x-1)$, $(a+1)(a-1)(b+1)(b-1)$.

2. Simplify $\frac{x}{(x-y)(x-z)}+\frac{y}{(y-z)(y-x)}+\frac{z}{(z-x)(z-y)}$ [*Ans* 0

3. Solve : (i) $\frac{1}{\sqrt{x}-1}+\frac{1}{\sqrt{x}-2}=\frac{3}{\sqrt{x}-3}$. [*Ans.* 3.

(ii) $(a+b)x+(a-b)y=2ac$, $(b+c)x+(b-c)y=2bc$ [*Ans* Each $=c$

4. If $a : b = b : c$, shew that $a^2+ab+b^2 : b^2+bc+c^2 = a : c$

5. Two sums of money are together equal to £54 12*s*, and there are as many pounds in the one as shillings in the other. What are the sums?
[*Ans.* £52; 52*s*.

1896

1. (*a*) Factorise (i) $x^{12}-a^{12}$. (ii) x^4+2x^2+9. (iii) $8x^2+6x-27$
[*Ans* $(x-a)(x+a)(x^2+a^2)(x^2-ax+a^2)(x^2+ax+a^2)(x^4-a^2x^2+a^4)$, $(x^2+2x+3)(x^2-2x+3)$, $(4x+9)(2x-3)$

(*b*) Find the H. C. F. of x^9-1 and $x^{10}-1$ [*Ans.* $x-1$

2. Simplify $\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}$ [*Ans* $a+b+c$

3. Solve the following equations :—

(*a*) $\frac{7x+1}{x-1}=\frac{35}{9}\left(\frac{x+4}{x+2}\right)+\frac{28}{9}$. [*Ans* -107

(*b*) $\frac{a_1}{x}+\frac{b_1}{y}=c_1$, $\frac{a_2}{x}+\frac{b_2}{y}=c_2$. [*Ans* $\frac{a_1b_2-a_2b_1}{c_1b_2-c_2b_1}$, $\frac{b_1a_2-b_2a_1}{c_1a_2-c_2a_1}$.

4 If $a : b = c : d$, and $p : q = r : s$, prove that
$$ap+cr : bq+ds = \sqrt{(acpr)} : \sqrt{(bdqs)}$$

5 Two towns X and Y, on a railway, are 64 miles apart. Coals at X cost 18*s* per ton and at Y 16*s* per ton; they cost two pence per ton per mile to carry on the line. Find the distance from X of the place at which it is immaterial to the consumer whether he buys coals from X or from Y.
[*Ans* 26 miles

1897

1 Shew that $(ay-bx)^2+(bz-cy)^2+(cx-az)^2+(ax+by+cz)^2$ is divisible by $a^2+b^2+c^2$ and $x^2+y^2+z^2$. [See *App* Ex 1]

2 Find the H. C. D. of $4x^4-9x^2+6x-1$ and $6x^3-7x^2+1$
[*Ans* $(x-1)(2x-1)$

3 Simplify the expressions :—

(i) $\dfrac{1}{a^2-3b^2+2ab}+\dfrac{1}{b^2-3a^2+2ab}-\dfrac{2}{3a^2+10ab+3b^2}$ [*Ans* 0

(ii) $\dfrac{a^4-x^4+ax(a^2+x^2)+a^2x^2}{a^5-x^5} \div \dfrac{a^2+x^2+ax}{a^3-x^3}$ [*Ans.* 1.

4 A merchant buys goods at 24 guineas the cwt, and by retailing them at 5*s* 3*d* the lb makes 10 per cent more profit than if he had sold the whole for £240. What weight did he buy? [*Ans* 1000 lbs.

5 If $a : b :: b : c :: c : d$, prove that $a^2 : d^2 :: a^3 : c^3$

1898

1 Find the H. C. F. of $2x^5-11x^2-9$ and $4x^5+11x^4+81$
[*Ans* x^2-2x+3.

2 Simplify (*a*) $\dfrac{m-n}{(x-m)(x-n)}+\dfrac{n-p}{(x-n)(x-p)}+\dfrac{p-m}{(x-p)(x-m)}$. [*Ans* 0

(*b*) $\dfrac{a^2}{(x-a)^n}+\dfrac{2a}{(x-a)^{n-1}}+\dfrac{1}{(x-a)^{n-2}}$ [*Ans* $\dfrac{x^2}{(x-a)^n}$.

3 Solve (*a*) $2x-\dfrac{3}{y}=4,\ 3x+\dfrac{2}{y}=5$ [*Ans.* $1\frac{5}{8}$, $2\frac{1}{2}$.

(*b*) $\dfrac{1}{x+5}+\dfrac{1}{x+10}=\dfrac{2}{x}$ [*Ans* $-6\frac{2}{3}$

4 If $a : b = c : d$, shew that $a(a+b+c+d)=(a+b)(a+c)$

5 The number of months in the age of a man, on his birth day in th year 1875, was exactly half of the number denoting the year in which h was born. In what year was he born? [*Ans.* 180

1899

1. Find the difference between $(1-x)^3+(1+x)^2y+(1+x)y^2+y^3$ and $3x(x+1)+y(y+1)+2xy+1$, and shew by what expression this difference must be multiplied that the product may be y^4-x^4.

[*Ans* $x^3+x^2y+xy^2+y^3$; $y-x$.

2. Find the H. C. D. of $x^5-4x^3-x^2+2x+2$ and x^3-x^2-2x+2, and find such a value of x as will make both the expressions vanish [$x-1$; 1

3. Reduce the following expressions into factors :

(i) x^4-10x^2+9 ; [*Ans.* $(x^2+2x-3)(x^2-2x-3)$,

(ii) $a^2(a+b-c)^2-c^2(b+c-a)^2$. [*Ans* $\{(a-c)^2+b(a+c)\}(a-c)(a+b+c)$.

4. Solve the equations —

(i) $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}$. [*Ans.* $\dfrac{ab(a+b)}{a^2+b^2}$.

(ii) $\dfrac{1}{x}+\dfrac{1}{y}=\frac{5}{6}$, $3x+2y=2xy$. [*Ans.* 2, 3

5 Simplify $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$. [*Ans.* $\dfrac{1}{abc}$.

6. If $\dfrac{x}{a}=\dfrac{y}{b}$, prove that $\dfrac{x^2+a^2}{x+a}+\dfrac{y^2+b^2}{y+b}=\dfrac{(x+y)^2+(a+b)^2}{(x+y)+(a+b)}$.

1900

1. Divide $(1-a^2)(1-b^2)(1-c^2)-(a+bc)(b+ca)(c+ab)$ by $1-a^2-b^2-c^2-2abc$; [*Ans* $1+abc$

and extract the square root of $1+(x+1)(x+2)(x+3)(x+4)$ [x^2+5x+5

2. Simplify $\dfrac{(x-a)^2}{(a-b)(a-c)}+\dfrac{(x-b)^2}{(b-a)(b-c)}+\dfrac{(x-c)^2}{(c-a)(c-b)}$. [*Ans.* 1.

3 The expression $ax-by$ is equal to 10 when $y=2$ and $y=3$ and it is equal to 25 when $x=3$ and $y=2$, a and b being constants ; find a and b.

[*Ans.* 11, -4

Solve $\dfrac{(x+a)(x+b)}{(x+c)(x+d)}=\dfrac{x-c-d}{x-a-b}$. [*Ans.* $\dfrac{cd(c+d)-ab(a+b)}{(a^2+ab+b^2)-(c^2+cd+d^2)}$

4. $a : b$, $c : d$, $e : f$, &c are m equal ratios : prove that each of them is equal to $\sqrt[n]{\left(\dfrac{pa^n+qc^n+re^n+\dots}{pb^n+qd^n+rf^n+\dots}\right)}$ and to $\sqrt[m]{\left(\dfrac{ace\dots}{bdf\dots}\right)}$, where n, p, q, r,... are any quantities whatever.

5 A's present age is to B's present age as 8 : 7 ; 27 years ago their ages were as 5 : 4 Find their present ages. [*Ans.* 72 yrs, 63 yrs

1901

1 (a) Find the H. C. F. of x^3-2x^2+1 and $2x^3+x^2-4x-7$. [*Ans* $x-1$

(b) Extract the square root of

$$(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$$ [*Ans* a^2+b^2

2. Simplify (i) $(a-b+c)^3+(a+b-c)^3+6a\{a^2-(b-c)^2\}$ [*Ans* $8a^3$

(ii) $\dfrac{1}{1+x+x^2}-\dfrac{1}{1-x+x^2}+\dfrac{2x}{1-x^2+x^4}$ [*Ans* $\dfrac{4x^3}{1+x^4+x^8}$

3 Solve the equations :—

(i) $\dfrac{3x-1}{4}-2(6-x)=\dfrac{5x-4}{7}-\dfrac{x-2}{3}$. [*Ans* 5

(ii) $\dfrac{2}{x-1}-\dfrac{3}{y+1}=2,\quad \dfrac{3}{x-1}+\dfrac{2}{y+1}=\dfrac{13}{16}$. [*Ans* $-2\frac{1}{3}$, $\frac{1}{4}$.

4 A number has three digits which increase by 1 from left to right. The quotient of the number divided by the sum of the digits is 26. What is the number? [*Ans* 234

5 If $\dfrac{x-y}{x+y}=a,\ \dfrac{y-z}{y+z}=b,\ \dfrac{z-x}{z+x}=c$, shew that

$$(1-a)(1-b)(1-c)=(1+a)(1+b)(1+c)$$

1902

1 Simplify (i) $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}$; [*Ans* $\dfrac{a^2+b^2}{a}$.

and (ii) $\dfrac{a^3+b^3+c^3-3abc}{(a-b)^2+(b-c)^2+(c-a)^2}$ [*Ans* $\dfrac{a+b+c}{2}$.

2 Extract the square root of $16x^3(x-2)-8x(1-3x)+1$ [*Ans* $4x^2-4x+1$

3. Solve $\dfrac{4x-17}{x-4}-\dfrac{5x-36}{x-7}=\dfrac{2x+7}{x-3}-\dfrac{3x+19}{x-6}$ [*Ans* -5

4 Find x and y from the two equations

$$a(x+y)+b(x-y)=2a,\quad y(a+b)-x(a-b)=2b$$ [*Ans* 1, 1.

5 I wished to give a certain number of old men 1 anna 8 pies each, and I found that I had not money enough in my purse by 11 annas; so I gave them 1 anna 5 pies each, and then I had money enough and 3 annas 3 pies to spare. Find the number of old men. [*Ans* 57.

6 If $a : b=c : d$, prove that

$$a^2b-3ac^2 : b^3-3ad^2=a^2+5c^2 : b^2+5d^2.$$

OPINIONS.

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